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GLOBALIZING ESTIMATES FOR THE PERIODIC KPI EQUATION

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1. Introduction

Consider the initial value problems

$$(1) \qquad \begin{aligned} u_t + uu_x + u_{xxx} &= \pm D^{-1} u_{yy} \\ u(x, y, 0) &= g(x, y) \end{aligned}$$

where D^{-1} is defined by the formula $D^{-1} f(x, y) = \int_0^x f(s, y)\, ds$. The $+$ and $-$ equations are called the KPI and $KPII$ equations respectively. They were first introduced by Kadomtsev and Petviashvili in [2]. The well-posedness theories of these two equations differ in their present state and perhaps intrinsically. For $KPII$, Bourgain [1] has shown global well-posedness in H^s for $s \geq 0$ on the torus Π^2 and $\mathbf{R}^2$. The method of his proof is a fix-point argument using norms defined via Fourier transform. Bourgain's method does not apply to the KPI equation. For KPI the present theory is expressed in terms of certain anisotropic Sobolev spaces V_m motivated by the linearized equation and/or their natural appearance in the conserved densities of the KPI flow. For $m = 0, 1, 2, \ldots$ define

$$(2) \qquad V_m = \left\{ u \in L^2(\Pi^2) \colon \int_0^1 u(x, y)\, dx = 0,\ \|u\|_{V_m} \leq \infty \right\}$$

where

$$(3) \qquad \|u\|_{V_m} = \left\{ \sum_{i=0}^{m} \sum_{j=0}^{i} \|\partial_x^{i-2j} \partial_y^j u\|_{L^2}^2 \right\}^{\frac{1}{2}}.$$

Negative exponents of ∂_x are interpreted via D^{-1}. The compatibility of the zero x-mean assumption is explained in Bourgain's paper. Ukai [7] showed *KPI* is locally well-posed in H^3 on the torus and has local results on other domains. A short proof is given by Saut in [5]. Schwarz [6] showed global well-posedness in V_3 on the torus provided the inital data g is small enough in L^2. It is shown in this note that the form of the conserved densities of the *KPI* flow imply:

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THEOREM. *If $g \in V_3$ then there exists a constant C depending only on $\|g\|_{V_3}$ such that*

$$\|u(t)\|_{V_3} \le C \tag{4}$$

where $u(t)$ is the solution of KPI at time t.

This theorem globalizes the local solutions of Ukai, since H^3 is contained in V_3.

Multiplying the *KPI* equation $u_t + uu_x + u_{xxx} = D^{-1}u_{yy}$ by u, integrating over Π^2 and recognizing perfect derivatives reveals

$$\partial_t \int_{\Pi^2} u^2(t)\,dx\,dy = 0. \tag{5}$$

This gives $\|u(t)\|_{V_0} = \|u(t)\|_{L^2} = \|g\|_{L^2} \le C$. Rewrite *KPI* as

$$u_t = D^{-1}u_{yy} - u_{xxx} - uu_x \tag{6}$$

and multiply by $(D^{-2}u_{yy} - u_{xx} - \frac{1}{2}u^2)$. Integrating over Π^2 leads to

$$\partial_t \int_{\Pi^2} \left((u_x(t))^2 + (D^{-1}u_y(t))^2 - \frac{1}{3}u^3(t)\right) dx\,dy = 0. \tag{7}$$

Therefore, if it could be shown for some $\gamma \ge 0$ and $0 \le \delta < 2$ that

$$\|u\|_{L^3}^3 \le C\|u\|_{L^2}^{\gamma}(\|u_x\|_{L^2}^2 + \|D^{-1}u_y\|_{L^2}^2)^{\frac{\delta}{2}} \tag{8}$$

then it would follow that

$$\int_{\Pi^2} \left((u_x(t))^2 + (D^{-1}u_y(t))^2\right) dx\,dy \le C. \tag{9}$$

Together with the L^2 conservation this would imply

$$\|u(t)\|_{V_1} \le C. \tag{10}$$

In fact, the L^3 estimate is true for $\gamma = \delta = \frac{3}{2}$ as will be shown shortly. A similar argument applied to the next two (nontrivial) conservation laws for the *KPI* flow will show $\|u(t)\|_{V_2}$ and $\|u(t)\|_{V_3}$ remain bounded for all time. A stronger result for the KdV equation was proven by Lax in [3]. The rest of the paper is organized as follows: Two a priori Sobolev-type estimates for L^p norms in terms of V_m norms are proven. These estimates contain the L^3 estimate above. The conserved densities for the *KP* equations are presented and then $\|u(t)\|_{V_2}$ and $\|u(t)\|_{V_3}$ are proven to be bounded for all time. Finally some remarks concerning the limitations of this approach and higher order regularity results are made.

2. A priori estimates

The L^p norm is compared to the V_m norms in the following two estimates.

ESTIMATE 1. *The following estimate holds for* $2 \leq p < 6$:

$$\|u\|_{L^p} \leq C\|u\|_{V_0}^{-1/2+3/p}\|u\|_{V_1}^{3/2-3/p}. \tag{11}$$

Proof. Let $q \in [1,2)$ and set $w(m,n) = \max(|m|, \frac{|n|}{|m|})$. The definition of V_0 allows us to assume $|m| \neq 0$ so that $w^2(m,n) \leq m^2 + \frac{n^2}{m^2}$ and $w \geq 1$. For all $R > 0$ define $T_R = \{(m,n) \in \mathbf{Z}^2\colon\ w(m,n) \leq R\}$ and $S_R = \mathbf{Z}^2 - T_R$. Let $A^2 = \sum_{(m,n)\in\mathbf{Z}^2} |a_{mn}|^2$ and $B^2 = \sum_{(m,n)\in\mathbf{Z}^2} w^2(m,n)|a_{mn}|^2$. Hölder's inequality and $|T_R| \leq CR^3$ imply

$$\sum_{T_R} |a_{mn}|^q \leq CR^{\frac{3(2-q)}{2}} A^q. \tag{12}$$

Hölder gives

$$\sum_{S_R} |a_{mn}|^q \leq \left(\sum_{S_R} (w(m,n))^{\frac{2q}{q-2}}\right)^{\frac{2-q}{2}} B^q. \tag{13}$$

The cardinality of the level set $\{(m,n)\colon\ w(m,n) = t\}$ is Ct^2.

$$\sum_{S_R} (w(m,n))^{\frac{2q}{q-2}} = \sum_{t=R}^{\infty} t^{\frac{2q}{q-2}} t^2 \tag{14}$$

$$= \int_R^\infty t^{\frac{4q-4}{q-2}}\, dt \tag{15}$$

$$= CR^{\frac{5q-6}{q-2}},$$

provided $\frac{6}{5} < q < 2$.

Combining these estimates gives

$$\|a_{mn}\|_{l^q}^q = \sum_{S_R} |a_{mn}|^q + \sum_{T_R} |a_{mn}|^q \tag{16}$$

$$\leq CR^{\frac{6-5q}{2}} B^q + CR^{\frac{3(2-q)}{2}} A^q. \tag{17}$$

Minimizing over R leads to selecting $R = \frac{B}{A}$ which yields

$$\|a_{mn}\|_{l^q} \leq C\|a_{mn}\|_{l^2}^{\frac{5q-6}{2q}} \|w(m,n)a_{mn}\|_{l^2}^{\frac{6-3q}{2q}}. \tag{18}$$

Hausdorff-Young, Parseval and $w(m,n)^2 \le m^2 + \frac{n^2}{m^2}$ imply

$$\|u\|_{L^{q'}(\Pi^2)} \le C\|u\|_{V_0}^{\frac{6-q'}{2q'}} \|u\|_{V_1}^{\frac{3q'-6}{2q'}} \tag{19}$$

for $2 \le q' < 6$. Using $\frac{1}{q} + \frac{1}{q'} = 1$ and renaming $q' = p$ establishes Estimate 1.

Remark. Refining the preceding a bit using the Littlewood-Paley square function theorem yields (11) for $p = 6$ as well.

Redefining w as $w(m,n) = \max(m^2, \frac{n^2}{m^2})$ and mimicking the proof of Estimate 1 establishes:

ESTIMATE 2. *The following estimate holds for* $2 \le p \le \infty$:

$$\|u\|_{L^p} \le C\|u\|_{V_0}^{\frac{8}{3p+2}} \|u\|_{V_2}^{\frac{3p-6}{3p+2}}. \tag{20}$$

3. Conserved densities

A linear change of variables converts the KP equations to the form

$$u_t - 6uu_x - u_{xxx} = -3\alpha^2 D^{-1} u_{yy}. \tag{21}$$

where $\alpha^2 = \pm 1$. The minus now corresponds to *KPI*. In this context the conservation laws for the KP equations may be described as follows, see the appendix in [4]. Define $L = (\partial_x + \alpha D^{-1}\partial_y)$. The recursion

$$v_0 = u \tag{22}$$

$$v_n = Lv_{n-1} + \sum_{m=0}^{n-2} v_{n-2-m} v_m \tag{23}$$

defines a sequence of expressions $v_n[u]$. The (nontrivial) conservation laws for the KP flow for $n = 0, 1, 2, \ldots$ are

$$\partial_t \int_{\Pi^2} v_{2n}[u]\, dx\, dy = 0. \tag{24}$$

The choice $n = 1$ leads to the L^2 conservation and $n = 2$ gives the conservation law involving $\|u\|_{V_1}$ and $\|u\|_{L^3}^3$. Different constants arise due to the alternate form of *KPI* used here. Calculating v_6 directly from the recursion shows that

$$\int_{\Pi^2} \big[(u_{xx})^2 - 10\alpha^2(u_y)^2 + 5\alpha^4(D^{-2}u_{yy})^2 + 5u^4 - 6u(u_x)^2 \tag{25}$$

$$+ 6\alpha^2 u^2(D^{-2}u_{yy}) + 4\alpha^2 u(D^{-1}u_y)^2\big]\, dx\, dy \tag{26}$$

is a conserved quantity. Since $\alpha^2 = -1$ the first three terms of the integrand are equivalent to $\|u\|_{V_2}^2$. Estimates 1 and 2 can be used to control the last four terms:

$$\|u\|_{L^4}^4 \leq C\|u\|_{V_0}^1\|u\|_{V_1}^3 \leq C \tag{27}$$

$$\int u(u_x)^2 \leq \|u\|_{V_0}\|u_x\|_{L^4}^2 \leq C\|u\|_{V_1}^{\frac{1}{2}}\|u\|_{V_2}^{\frac{3}{2}} \leq C\|u\|_{V_2}^{2-\eta} \tag{28}$$

$$\int u^2(D^{-2}u_{yy}) \leq \|u\|_{L^4}^2\|D^{-2}u_{yy}\|_{L^2} \leq C\|u\|_{V_2} \tag{29}$$

$$\int u(D^{-1}u_y)^2 \leq \|u\|_{L^2}\|D^{-1}u_y\|_{L^4}^2 \leq C\|u\|_{V_1}^{\frac{1}{2}}\|u\|_{V_2}^{\frac{3}{2}} \leq C\|u\|_{V_2}^{2-\eta} \tag{30}$$

for some $\eta \geq 0$. Using these estimates and the conservation law gives

$$\|u(t)\|_{V_2} \leq C \tag{31}$$

where C depends upon $\|g\|_{V_0}$, $\|g\|_{V_1}$, $\|g\|_{V_2}$.

The recursion, recognition of perfect derivatives and integrations by parts may be used to show that the terms appearing in $\int_{\Pi^2} v_8[u]\,dx\,dy$ are (up to constant multiples) $\|u\|_{V_3}^2$ and

$$(K^3u)(u)(Ku),\ (K^2u)^2(u),\ (K^2u)(u^3),\ (K^2u)(Ku)^2, \tag{32}$$

$$(Ku)^2(u^2),\ (u^5),\ (K(u^2))^2,\ (K^3u)(K(u^2)),\ (K(u^2))(u)(Ku), \tag{33}$$

where $K = L$ or $K = (-\partial_x + \alpha D^{-1}\partial_y)$. Terms above not containing $K(u^2)$ may be estimated using Hölder's inequality and Estimates 1 and 2 in a manner analogous to the V_6 terms. Cauchy-Schwarz and the following may be used to estimate the remaining terms.

LEMMA 1. *The following estimate holds*

$$\|K(u^2)\|_{L^2} \leq C\left[\|u\|_{L^4}\|u_x\|_{L^4} + \|u\|_{L^{2p}}\|u_y\|_{L^{2p'}}\right] \tag{34}$$

where $p^{-1} + p'^{-1} = 1$.

Consequently, choosing p' so that $2 < 2p' < 6$, using Estimates 1, 2 and $\|u(t)\|_{V_m} \leq C$ for $m = 0, 1, 2$ gives, for some $\eta > 0$,

$$\|K(u^2)\|_{L^2} \leq C + C\|u(t)\|_{V_3}^{1-\eta}. \tag{35}$$

Proof of Lemma 1.

$$\begin{aligned} \|K(u^2)\|_{L^2}^2 &\leq C\left[\int_{\Pi^2}(u^2)(u_x^2)\,dx\,dy + \alpha\int_{\Pi^2}(D^{-1}(uu_y))^2\,dx\,dy\right] \\ &\leq C\|u\|_{L^4}^2\|u_x\|_{L^4}^2 + C\int_{\Pi^2}(D^{-1}(uu_y))^2\,dx\,dy \end{aligned} \tag{36}$$

$$(37)\quad \int_{\Pi^2} (D^{-1}(uu_y))^2\,dx\,dy = \int_{\Pi^2} \left(\int_0^x uu_y\,ds\right)^2 dx\,dy$$
$$\leq \int_{\Pi^2} \left(\int_0^1 u^2\,ds\right)\left(\int_0^1 {u_y}^2\,ds\right) dx\,dy$$
$$\leq \left(\int_\Pi \left(\int_0^1 u^2\,ds\right)^p dy\right)^{\frac{1}{p}} \left(\int_\Pi \left(\int_0^1 {u_y}^2\,ds\right)^{p'} dy\right)^{\frac{1}{p'}}$$

where Π has been identified with $[0, 1)$. Then Minkowski's inequality implies

$$(38)\qquad \int_{\Pi^2} (D^{-1}(uu_y))^2\,dx\,dy \leq \|u\|^2_{L^{2p}} \|u_y\|^2_{L^{2p'}},$$

which, upon combining terms, completes the proof of the lemma.

The estimates of the $v_8[u]$ terms combine to imply

$$(39)\qquad \|u(t)\|_{V_3} \leq C$$

where C depends upon $\|g\|_{V_3}$. This completes the proof of the theorem.

4. Remarks

It seems likely that, by analogy with the result proven for KdV in [3], the conservations law $\int_{\Pi^2} v_{2n}[u]\,dx\,dy$ will imply $\|u(t)\|_{V_{n-1}} \leq C$ showing that regularity of the initial data is preserved by the *KPI* flow. The difficulties encountered while handling the $v_8[u]$ terms, however (in particular the need to handle the $K(u^2)$ terms separately), forecast problems with an inductive argument like that presented by Lax. Part of the problem here is that L does not satisfy the product rule due to the presence of D^{-1}.

The local well-posedness theory of Ukai establishes uniqueness using the standard Gronwall argument. This requires an L^∞ estimate on u_x. Then Estimate 2 explains why the local theory takes place in V_3. This dependence on higher conservation laws prevents the study of perturbations of *KPI*, and also precludes studying the evolution of data g with fractional smoothness. Perhaps a fix-point argument as used in [1] for *KPII* can overcome both of these difficulties at once.

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