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POLYNOMIAL PARAMETRIZATION AND ETALE EXOTICITY

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1. Introduction

This paper relates two properties of varieties or rather of constructible sets. The first is the manner in which we can parametrize our sets, and we will be interested in polynomial parametrizations. The second is the fact that the set is etale exotic (see [4], [5], [6]).

In the second section we will prove a theorem on polynomial parametrizations of complex spheres (Theorem 1). An n-dimensional complex sphere is the hypersurface in $\mathbb{C}^{n+1}$ which is defined by the equation

$$X_1^2 + \cdots + X_{n+1}^2 = 1$$

We will see that the complex sphere has a polynomial parametrization whenever it is even dimensional. We do not know the answer in the odd dimensional case. In the one dimensional case the answer is negative as can be seen easily. The manner in which these polynomial parametrizations were derived is via varieties which are generalizations of the etale exotic surface S_2 [6] and of the Winkelmann's quadratic which is a 4 dimensional variety that is embedded in $\mathbb{C}^5$ [8]. This already hints that there is a connection between the possibility of polynomially parametrizing a variety of a certain type and the fact that the variety is etale exotic.

In the third section we will generalize the definition of an etale exotic surface given in [6] to higher dimensions, and motivated by the 2 dimensional result on the exoticity of S_2 we will prove that Winkelmann's quadratic is a 4 dimensional etale exotic variety (Theorem 2). The method of proof of that fact is a generalization of the "grading by weights" that was used to prove the exoticity of S_2 [5], [6]. The grading that is chosen in the course of the proof of Theorem 2 simply separates the 5 variables that parametrize Winkelmann's variety into 2 pairs of symmetric variables in a natural way plus an extra variable.

Finally, we will mention that our motivation to study etale exotic varieties emerges from a certain approach to the Jacobian Conjecture that relates it to the excluding of the existence of asymptotic values for maps which are built out of Jacobian pairs. In

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that connection we mention the nice counterexample of S. Pinchuk [7] that solves the Real Jacobian Conjecture. See also [4], [5], [6].

Recently, we found that these varieties were of interest also because of other reasons (see [9]).

2. Polynomial parametrizations of complex spheres

Definition 1. Let V be a variety over a field K. A parametrization of V will be called polynomial if it is realized by polynomials over K, i.e., if

$$X_1 = P_1(U_1, \dots, U_m)$$

$$\vdots$$

$$X_n = P_n(U_1, \dots, U_m)$$

where $P_j(U_1, \dots, U_m) \in K[U_1, \dots, U_m]$, $1 \le j \le n$.

Remark 1. As usual such a parametrization defines an injective (polynomial in this case) map from K^m into V. Not all of V is necessarily covered, in fact quite often the parametrization covers a Zariski dense constructible subset of V.

Example 1. The doubly ruled surface S_2: $XZ = Y(Y+1)$ which is etale exotic has the polynomial parametrization

$$X = V, \quad Y = VU, \quad Z = VU^2 + U$$

which originates in the structure of the ring $I(X^{-1}, X^2Y - X)$ [5], [6].

Definition 2. The complex n-sphere S^n is the complex variety

$$S^n = V_{\mathbb{C}}(X_1^2 + \cdots + X_{n+1}^2 - 1) = \{(a_1, \dots, a_{n+1}) \in \mathbb{C}^{n+1} \mid a_1^2 + \cdots + a_{n+1}^2 = 1\}.$$

Remark 2. It is well known that S^n has a rational parametrization for any n, it is of genus 0. An elementary way of proving this is via the streographic projection, as follows. We connect the point $(0, \dots, 0, 1)$ on S^n to the point $(U_1, \dots, U_n, 0)$ on the hyperplane $U_{n+1} = 0$. This line is parametrized by $(TU_1, \dots, TU_n, 1-T)$, $T \in \mathbb{C}$. To find the value of T that corresponds to the line's intersection with S^n we substitute

$$X_1 = TU_1, \dots, X_n = TU_n, X_{n+1} = 1 - T$$

into the defining equation of S^n,$X_1^2 + \cdots + X_{n+1}^2 = 1$ and obtain

$$(U_1^2 + \cdots + U_n^2 + 1)T^2 - 2T = 0.$$

This gives the solution we want, $T = 2/(U_1^2 + \cdots + U_n^2 + 1)$. Hence we get the desired parametrization of S^n:

$$X_1 = TU_1 = 2U_1/(U_1^2 + \cdots + U_n^2 + 1)$$

$$\vdots$$

$$X_n = TU_n = 2U_n/(U_1^2 + \cdots + U_n^2 + 1)$$

$$X_{n+1} = 1 - T = (U_1^2 + \cdots + U_n^2 - 1)/(U_1^2 + \cdots + U_n^2 + 1).$$

Here we will be interested in polynomial parametrizations of S^n and not merely in rational parametrizations. To motivate the construction of the polynomial parametrizations that will be given in the proof of Theorem 1 we will now solve the cases of S^1, S^2 and of S^4.

S^1 has no polynomial parametrization.

Proof 1. The most naive way to prove this is as follows. Let us assume that S^1: $X^2 + Y^2 = 1$ has a polynomial parametrization over $\mathbb{C}$, say

$$X = a_m T^m + \cdots + a_1 T + a_0$$

$$Y = b_m T^m + \cdots + b_1 T + b_0$$

with $a_m \neq 0$. Then we obtain the following identity

$$(a_m T^m + \cdots + a_1 T + a_0)^2 + (b_m T^m + \cdots + b_1 T + b_0)^2 = 1.$$

On pluging in $T = 0$ we get $a_0^2 + b_0^2 = 1$. On the other hand if we equate coefficients on both sides we obtain $a_0^2 + b_0^2 = 0$ which will lead to the desired contradiction. To see that $a_0^2 + b_0^2 = 0$ we consider the m highest coefficients on the left hand side. All must be 0 and so we have

$$a_m^2 + b_m^2 = 0$$

$$a_m a_{m-1} + b_m b_{m-1} = 0$$

$$2a_m a_{m-2} + a_{m-1}^2 + 2b_m b_{m-2} + b_{m-1}^2 = 0$$

$$\cdots$$

$$a_m a_0 + b_m b_0 = 0.$$

From the first of these equations we get $b_m = ia_m$ or $b_m = -ia_m$, say, $b_m = ia_m$. From the second equation,

$$a_m a_{m-1} + (ia_m)b_{m-1} = 0$$

or $b_{m-1} = ia_{m-1}$ (for $a_m \neq 0$). From the third equation,

$$2a_m a_{m-2} + a_{m-1}^2 + 2(ia_m)b_{m-2} + (ia_{m-1})^2 = 0$$

or $b_{m-2} = ia_{m-2}$ (for $a_m \neq 0$). A simple inductive argument shows that

$$b_j = ia_j, \qquad j = m, \dots, 0$$

or

$$b_j = -ia_j, \qquad j = m, \dots, 0.$$

In particular $b_0 = ia_0$ (or $b_0 = -ia_0$) and so $a_0^2 + b_0^2 = 0$. $\square$

Proof 2. A much simpler proof that hints of the other two cases to come goes as follows. Consider S^1: $X_1^2 + X_2^2 = 1$. The following regular linear transformation T maps S^1 onto the curve $Z_1 W_1 = 1$:

$$T\colon X_1 = 1/2(Z_1 + W_1)$$

$$X_2 = i/2(Z_1 - W_1).$$

Now any polynomial parametrization of S^1 induces such a parametrization on $Z_1 W_1 = 1$. However, the last curve clearly has no polynomial parametrizations (because of degree considerations). $\square$

S^2 has the polynomial parametrization

$$X = V - (VU^2 + U)$$

$$Y = -i(V + (VU^2 + U))$$

$$Z = 1 + 2UV.$$

This parametrization does not cover the whole of S^2. The straight line

$$\{(-T, -iT, -1) \in \mathbb{C}^3 \mid T \in \mathbb{C}\}$$

is not covered.

Proof. The etale exotic surface S_2: $XZ = Y(Y+1)$ has the polynomial parametrization

$$X = V$$

$$Y = VU$$

$$Z = VU^2 + U$$

which covers the whole of S_2 except for the line

$$L = \{(0, -1, Z) \in \mathbb{C}^3 \mid z \in \mathbb{C}\}.$$

See [5], [6]. The linear transformation

$$X = 1/2(X_1 + iX_2)$$

$$T\colon\ Z = 1/2(-X_1 + iX_2)$$

$$Y = 1/2(X_3 - 1)$$

has the inverse transformation

$$X_1 = X - Z$$

$$T^{-1}\colon\ X_2 = -i(X + Z)$$

$$X_3 = 1 + 2Y.$$

If we use T on S_2 we get

$$\begin{aligned} 0 &= XZ - Y(Y+1) \\ &= (1/2(X_1 + iX_2))(1/2(-X_1 + iX_2)) - (1/2(X_3 - 1))(1/2(X_3 - 1) + 1) \\ &= (-1/4)(X_1^2 + X_2^2 + X_3^2 - 1) \end{aligned}$$

or $X_1^2 + X_2^2 + X_3^2 = 1$ and we have S_2 linearly transformed to S^2. We use T^{-1} and the polynomial parametrization of S_2 in order to get a polynomial parametrization of S^2, namely

$$X_1 = X - Z = V - (VU^2 + U)$$

$$X_2 = -i(X + Z) = -i(V + (VU^2 + U))$$

$$X_3 = 1 + 2Y = 1 + 2VU.$$

Since the original parametrization of S_2 did not cover the line L and since the image of L under T is the line

$$\begin{aligned} T(L) &= \{(0-Z, -i(0+Z), 1+2(-1)) \in \mathbb{C}^3 \mid Z \in \mathbb{C}\} \\ &= \{(-Z, -iZ, -1) \in \mathbb{C}^3 \mid Z \in \mathbb{C}\}, \end{aligned}$$

this last line is not covered by the polynomial parametrization we found for S^2. The claim is now proved. □

We end this introductory discussion by treating the 4 dimensional case:

S^4 has a polynomial parametrization which covers the whole of the variety except for a copy of $\mathbb{C}^2$.

We are not giving the precise formulas of the parametrization and the defining equations for the uncovered $\mathbb{C}^2$ because later we will treat the general even dimensional case in full detail in Theorem 1.

The motivation for the following construction originates in a beatiful talk by Hanspeter Kraft in August 1993 at the Technion, Haifa,Israel. The title of the talk was "Cancellation in algebra and geometry" and it was given in the course of a workshop on "Affine Algebraic Geometry". Among other things, Hanspeter Kraft described a work of Winkelmann concerning a derivative acting in $\mathbb{C}^5$ and exotic manifolds. We refer the reader to the paper of J. Winkelmann [8]. The derivative mentioned was

$$d = W_1 \partial/\partial W_3 + W_2 \partial/\partial W_4 + (1 + D_W)\partial/\partial W_5$$

where $D_W = \begin{vmatrix} W_1 & W_2 \\ W_3 & W_4 \end{vmatrix}$. A $\mathbb{C}^+$-action is given by

$$t \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 + tW_1 \\ W_4 + tW_2 \\ W_5 + t(1 - D_W) \end{pmatrix}$$

Generators of the invariants of the action are given by

$$Z_1 = W_1,\ Z_2 = W_2,\ Z_3 = W_3(1 + D_W) - W_1 W_5,$$

$$Z_4 = W_4(1 + D_W) - W_2 W_5, Z_5 = D_W.$$

There is a relation among these generators given by

$$Z_1 Z_4 - Z_2 Z_3 = Z_5(1 + Z_5).$$

This is the defining equation for an affine quadratic in $\mathbb{C}^5$ that we'll denote by Q. We let

$$Q' = Q - \{Z_1 = Z_2 = 0,\ Z_5 = -1\}$$

which is not affine. Winkelmann [8, Lemma 3] proved the following:

THEOREM. *$\pi\colon \mathbb{C}^5 \to Q'$ is a principal $\mathbb{C}^+$-bundle.*

Remark 3. Q' is diffeomorphic with $\mathbb{C}^4$ (we will see that later).

We can cover Q' as follows: $Q' = U_1 \cup U_2 \cup U_3$ where $U_1 = Q_{Z_1\neq 0}, U_2 = Q_{Z_2\neq 0}$ and $U_3 = Q_{Z_3\neq 0}$ and here each U_j is isomorphic with $\mathbb{C}^* \times \mathbb{C}^3$. Another fact that was proven by Winkelmann [Lemma 4 in 8] is the following

PROPOSITION. *There exist infinitely many non-equivalent* 4*-bundles P_ϕ over Q' and all are affine varieties.*

P_ϕ is diffeomorphic with $\mathbb{C}^5$, however, it is not known if P_ϕ is isomorphic with $\mathbb{C}^5$.

We turn to Winkelmann's quadratic in $\mathbb{C}^5$ that relates the generators of the invariants of the above $\mathbb{C}^+$-action. In fact it will be more convenient to look at the following affine quadratic in $\mathbb{C}^5$:

$$Z_1Z_4 + Z_2Z_3 = Z_5(1 + Z_5)$$

One can, no doubt, see resemblance of that quadratic to the etale exotic surface S_2. Moreover, the image of $\mathbb{C}^5$ under π is Q minus a plane $\mathbb{C}^2$, $\{Z_1 = Z_2 = 0, Z_5 = -1\}$, which reminds one of the fact that the polynomial parametrization of S_2 covered S_2 minus the line $L = \{X = 0,\ Y = -1\}$. Thus we are led to make an analogy between

$$Z_1Z_4 + Z_2Z_3 = Z_5(1 + Z_5)$$

and

$$XZ = Y(1 + Y)$$

where Z_1, Z_2 will correspond to X and Z_3, Z_4 will correspond to Z and Z_5 will correspond to Y. We recall the polynomial parametrization of S_2:

$$X = V$$

$$Y = VU$$

$$Z = VU^2 + U$$

This immediately leads us to assign the following guess for a parametrization of Q':

$$Z_1 = V_1$$

$$Z_4 = V_1U_1^2 + U_1 + \cdots$$

$$Z_2 = V_2$$

$$Z_3 = V_2U_2^2 + U_2 + \cdots$$

$$Z_5 = V_1U_1 + V_2U_2.$$

Plugging that into the defining equation of Q we get

$$\begin{aligned} Z_1Z_4 + Z_2Z_3 &= V_1(V_1U_1^2 + U_1 + \cdots) + V_2(V_2U_2^2 + U_2 + \cdots) \\ &= Z_5(1 + Z_5) = (V_1U_1 + V_2U_2)(1 + V_1U_1 + V_2U_2); \end{aligned}$$

comparing both sides of this equation one immediately gets the desired polynomial parametrization of Q', namely

$$Z_1 = V_1$$

$$Z_4 = V_1U_1^2 + U_1 + U_1V_2U_2$$

$$Z_2 = V_2$$

$$Z_3 = V_2U_2^2 + U_2 + V_1U_1U_2$$

$$Z_5 = V_1U_1 + V_2U_2.$$

Now it is a matter of a regular linear transformation of Q onto S^4:

$$Z_1 = 1/2(X_1 + iX_2)$$

$$Z_4 = 1/2(-X_1 + iX_2)$$

$$Z_2 = 1/2(X_3 + iX_4)$$

$$Z_3 = 1/2(-X_3 + iX_4)$$

$$Z_5 = 1/2(X_5 - 1)$$

so that

$$Z_1Z_4 + Z_2Z_3 = -1/4(X_1^2 + X_2^2 + X_3^2 + X_4^2)$$

and

$$Z_5(1+Z_5) = 1/2(X_5-1)1/2(X_5+1) = 1/4(X_5^2-1)$$

which, indeed, shows that Q is linearly equivalent to S^4: $X_1^2+X_2^2+X_3^2+X_4^2=1$. And hence we obtain a polynomial parametrization of S^4 minus a $\mathbb{C}^2$.

From here it is rather clear how to go about proving such a result for any even dimensional sphere S^{2n}.

THEOREM 1. *Let n be a positive integer. Let S^{2n} be the $2n$-sphere*

$$X_1^2+Y_1^2+\cdots+X_n^2+Y_n^2+X_{n+1}^2=1.$$

Then S^{2n} has the polynomial parametrization

$$X_j = V_j - U_j\left(1+\sum_{k=1}^n V_kU_k\right)$$

$$Y_j = -i\left(V_j + U_j\left(1+\sum_{k=1}^n V_kU_k\right)\right)$$

for $1\le j\le n$ and

$$X_{n+1} = 1+2\sum_{k=1}^n V_kU_k.$$

This parametrization does not cover the whole of S^{2n}. The following $\mathbb{C}^n$ is not covered:

$$\{(-W_1,-iW_1,\dots,-W_n,-iW_n,-1)\in\mathbb{C}^{2n+1}\mid W_1,\dots,W_n\in\mathbb{C}\}$$

Proof. One quick way of proving the theorem is by a direct check of all the details. We prefer, however, to give a proof along the lines of the Winkelmann's manifold in $\mathbb{C}^5$ described before. Thus we consider the following affine quadratic in $\mathbb{C}^{2n+1}$:

$$Q_{2n}\colon Z_1W_1+\cdots+Z_nW_n = Y(1+Y)$$

An immediate generalization of the polynomial parametrization of the Winkelmann's variety gives the following polynomial parametrization for our quadratic Q_{2n}:

$$Z_j = V_j,\quad W_j = U_j\left(1+\sum_{k=1}^n V_kU_k\right),\quad 1\le j\le n$$

$$Y = \sum_{k=1}^n V_kU_k.$$

Now we make the obvious regular linear transformation

$$Z_j = 1/2(X_j + iY_j)$$

$$W_j = 1/2(-X_j + iY_j)$$

for $1 \le j \le n$ and

$$Y = 1/2(X_{n+1} - 1).$$

This transforms our affine quadratic Q_{2n} onto S^{2n}: $X_1^2+Y_1^2+\cdots+X_n^2+Y_n^2+X_{n+1}^2 = 1$. The inverse transformation is given by

$$X_j = Z_j - W_j$$

$$Y_j = -i(Z_j + W_j)$$

for $1 \le j \le n$ and

$$X_{n+1} = 1 + 2Y.$$

Thus we get the desired polynomial parametrization of S^{2n}:

$$X_j = Z_j - W_j = V_j - U_j(1 + \sum_{k=1}^{n} V_k U_k)$$

$$Y_j = -i(Z_j + W_j) = -i(V_j + U_j(1 + \sum_{k=1}^{n} V_k U_k))$$

for $1 \le j \le n$ and

$$X_{n+1} = 1 + 2Y = 1 + 2\sum_{k=1}^{n} V_k U_k.$$

Finally, since the original parametrization covers Q_{2n} minus the $\mathbb{C}^n$

$$L_n = \{Z_1 = \cdots = Z_n = 0,\ Y = -1\}$$

it follows that the polynomial parametrization of S^{2n} covers S^{2n} minus the $\mathbb{C}^n$ which is the image of L_n:

$$X_j = 0 - W_j,\ Y_j = -i(0 + W_j),\ 1 \le j \le n, \quad X_{n+1} = 1 + 2(-1) = -1$$

which is the $\mathbb{C}^n$

$$\{(-W_1, -iW_1, \ldots, -W_n, -iW_n, -1) \in \mathbb{C}^{n+1} \mid W_1, \ldots, W_n \in \mathbb{C}\}$$

The theorem is now proved. $\square$

3. Etale exotic varieties of higher dimensions

Etale exotic surfaces are important because of their connection with the Jacobian conjecture [4], [5], [6]. In fact one possible way to prove the conjecture is to try to exclude the existence of asymptotic values of maps whose coordinates form Jacobian pairs. Now any asymptotic value of a regular etale map could be realized along a rational curve that extends to the point at infinity. In fact the two components of such an etale map satisfy a so called asymptotic identity with respect to the rational parametrization of the asymptotic curve. This means that the composition of each of the polynomials in the pair with a certain rational map $\mathbb{C}^2 \to \mathbb{C}^2$ gives rise to another polynomial which is called the dual polynomial of the original one. This behaviour is singular in the sense that usually the composition of a polynomial with a rational map results in a rational function (and not in a polynomial). This singular behaviour originates in certain cancellation properties that are simultanously shared by the polynomials in the Jacobian pair and result in the cancellation of the expected poles.

As a conclusion, one can use the above in order to disprove the Jacobian conjecture. In fact all that is needed is to be able to construct a Jacobian pair whose polynomials satisfy asymptotic identities with respect to the same rational map. Such a Jacobian pair will serve as a counterexample to the conjecture. This is the way the Real Jacobian conjecture was disproved by S. Pinchuk [7]. His nice example is composed of a real Jacobian pair both of which polynomials satisfy an asymptotic identity with respect to the rational map $(X^{-1}, YX^2 - X)$.

Thus we are naturally led to explore the structure of the ring of all the polynomials which satisfy an asymptotic identity with respect to $(X^{-1}, YX^2 - X)$ (or in general with respect to the underlying rational map $R(X, Y)$). This ring is denoted by $I(X^{-1}, YX^2 - X)$ (or in general by $I(R(X, Y))$. For a wide family of rational maps it is possible to show that this ring is in fact a polynomial ring. For example in the Pinchuk's case we have

$$I(X^{-1}, YX^2 - X) = K[V, VU, VU^2 + U]$$

where $K = R$ the real field. As opposed to this it can be shown that there is no (complex) Jacobian pair for the case $K = \mathbb{C}$, i.e., in the ring $\mathbb{C}[V, VU, VU^2 + U]$ [5], [6], [9]. Surprisingly, one recognizes the polynomial parametrization of S_2 in the set of generators of the ring $I(X^{-1}, YX^2 - X)$. The algebraic fact that there is no Jacobian pair in that ring over the complex field is the equivalent of the geometric fact that S_2 is an etale exotic surface.

We now recall the definition of an etale exotic surface from [6].

Definition 3. An etale exotic surface S over $\mathbb{C}$ is a surface which has the following properties:

(a) There is a diffeomorphism ϕ: $\mathbb{C}^2 \to S$ which is realized by a birational map ϕ.

(b) There is no regular etale map $S \to \mathbb{C}^2$ (just into).

Examples of such surfaces in $\mathbb{C}^{N+1}$ are given by

$$S_N\colon\ X_1 = V,\ X_2 = VU,\ X_3 = VU^2 + U, \dots, X_{N+1} = VU^N + U^{N-1}, \quad N \geq 2$$

See [6]. In fact S_N is not Zariski closed, it is only a constructible set. Its affine closure is given by

$$X_1X_3 - X_2(X_2+1) = 0, \quad X_1X_{j+2} - X_2X_{j+1} = 0, \quad 2 \leq j \leq N-1$$

In the previous section we saw that the closure of S_2, namely, $XZ - Y(Y+1) = 0$, is linearly equivalent to the 2-sphere $X^2 + Y^2 + Z^2 = 1$. In terms of the previous section the diffeomorphism $\phi\colon \mathbb{C}^2 \to S$ is merely a polynomial parametrization of the constructible set S. These are the relations between the existence of polynomial parametrizations and being etale exotic in the 2 dimensional case.

The main purpose of this section is to generalize the notion of being etale exotic, to higher dimensions and then to give an example of a variety that has this type of exoticity. The generalization we have in mind will bear the same kind of relation with respect to polynomial parametrization as in the 2 dimensional case.

Definition 4. A constructible set V of dimension n over $\mathbb{C}$ is called etale exotic if it has the following properties:

(a) There is a diffeomorphism $\phi\colon \mathbb{C}^n \to V$ which is realized by a rational map ϕ.
(b) There is no regular etale map $V \to \mathbb{C}^n$ (just into).

Remark 4. As in the 2 dimensional case the diffeomorphism $\phi\colon \mathbb{C}^n \to V$ is a polynomial parametrization of the constructible set V.

We will now give an example of an etale exotic variety of dimension higher than 2. Guided by our intuition from the previous section we consider Winkelmann's quadratic in $\mathbb{C}^5$ as a natural candidate for such an example.

THEOREM 2. *The constructible set Q_4' which has the polynomial parametrization*

$$Q_4'\colon\ Z_1 = V_1, Z_2 = V_2, Y = V_1U_1 + V_2U_2,$$

$$W_1 = U_1(1 + V_1U_1 + V_2U_2), W_2 = U_2(1 + V_1U_1 + V_2U_2)$$

is an etale exotic constructible set of dimension 4.

Remark 5. (1) As noted in the previous section the affine closure of Q'_4 is the Winkelmann's quadratic

$$Z_1W_1 + Z_2W_2 = Y(1+Y).$$

(2) The proof of Theorem 2 will be purely algebraic. It will merely be an extension of the "grading by weights" technique that was used in [5], [6] to handle the 2 dimensional case.

(3) The weights in the proof will naturally split the 5 variables into the two pairs (Z_1, W_1) and (Z_2, W_2) and the extra variable Y.

(4) We still do not know of geometric proofs for theorems such as Theorem 2, not even in the two dimensional case. There are classifications of ruled surfaces even up to biregular classes [1], [2], [3]. However, there does not seem to be any attempt to geometrically classify these surfaces into regular etale classes—a classification we seek here.

Proof. In order to prove that Q'_4 is etale exotic we need to show the following: If $P_j(Z_1, Z_2, Y, W_1, W_2) \in \mathbb{C}[Z_1, Z_2, Y, W_1, W_2]$, $1 \le j \le 4$, then

$$\partial(P_1, P_2, P_3, P_4)/\partial(U_1, V_1, U_2, V_2) \notin \mathbb{C}^*$$

where the Jacobian is evaluated after we substitute into (Z_1, Z_2, Y, W_1, W_2) the expressions that parametrize Q'_4. We call

$$Z_1 = V_1, Z_2 = V_2, Y = V_1U_1 + V_2U_2,$$

$$W_1 = U_1(1 + V_1U_1 + V_2U_2), W_2 = U_2(1 + V_1U_1 + V_2U_2)$$

the generators, for they generate the polynomial ring we are dealing with, namely

$$I = \mathbb{C}[V_1, V_2, V_1U_1 + V_2U_2, U_1(1 + V_1U_1 + V_2U_2), U_2(1 + V_1U_1 + V_2U_2)].$$

We assign the weights

$$\deg V_1 = 1, \deg U_1 = -1, \deg V_2 = \sqrt{2}, \deg U_2 = -\sqrt{2}.$$

With these weights the generators become homogeneous of degrees $1, \sqrt{2}, 0, -1, -\sqrt{2}$ respectively. We will let $T_1 = V_1U_1$, $T_2 = V_2U_2$. Note that the set of degrees in our grading is exactly the set

$$\{n + m\sqrt{2} \mid n, m \in Z\}.$$

Also, since $1, \sqrt{2}$ are independent over Z, the effect of this grading is to separate the pairs (U_1, V_1) and (U_2, V_2). If $k = n + m\sqrt{2}$, $n, m \in Z$ then we will denote the homogeneous part (with respect to our grading) of degree k of a polynomial $P \in I$ by P_k or by $P_{n,m}$. If $k_j = n_j + m_j\sqrt{2}$, $1 \leq j \leq 4$, then clearly

$$\deg\{\partial(P_{k_1}, Q_{k_2}, R_{k_3}, S_{k_4})/\partial(U_1, V_1, U_2, V_2)\} = k_1 + k_2 + k_3 + k_4$$

where $P, Q, R, S \in I$. So if $P, Q, R, S \in I$ satisfy the identity

$$\partial(P, Q, R, S)/\partial(U_1, V_1, U_2, V_2) = 1$$

then

$$\sum_{k_1+k_2+k_3+k_4=0} \partial(P_{k_1}, Q_{k_2}, R_{k_3}, S_{k_4})/\partial(U_1, V_1, U_2, V_2) = 1 \tag{1}$$

where the summation goes through all the ordered 4-tuples (k_1, k_2, k_3, k_4) such that $k_1 + k_2 + k_3 + k_4 = 0$.

We want to classify the homogeneous polynomials in our grading. Recall that for each pair $(n, m) \in Z^2$ we think of the degree $n + m\sqrt{2}$. So if $n, m \geq 0$ then one can easily check that we have the following structures of homogeneous polynomials in I:

$$P_{n,m} = V_1^n V_2^m f_1(T_1) f_2(T_2)$$

$$P_{n,-m} = V_1^n V_2^{-m} T_2^m (1 + T_1 + T_2)^m f_1(T_1) f_2(T_2)$$

$$P_{-n,m} = V_1^{-n} V_2^m T_1^n (1 + T_1 + T_2)^n f_1(T_1) f_2(T_2)$$

$$P_{-n,-m} = V_1^{-n} V_2^{-m} T_1^n T_2^m (1 + T_1 + T_2)^{n+m} f_1(T_1) f_2(T_2)$$

for some $f_1(T_1) \in \mathbb{C}[T_1]$, $f_2(T_2) \in \mathbb{C}[T_2]$. Finally, if $k_j = n_j + m_j\sqrt{2}$, $1 \leq j \leq 4$, then $k_1+k_2+k_3+k_4 = 0$ is equivalent to $n_1+n_2+n_3+n_4 = m_1+m_2+m_3+m_4 = 0$. With all that at hand we can check case by case that for any $(k_1, k_2, k_3, k_4) \in (Z + Z\sqrt{2})^4$ such that $k_1 + k_2 + k_3 + k_4 = 0$ we have

$$(1 + T_1 + T_2) \mid \partial(P_{k_1}, Q_{k_2}, R_{k_3}, S_{k_4})/\partial(U_1, V_1, U_2, V_2). \tag{2}$$

This, however, is in conflict with equation (1) which proves the theorem. $\square$

To check (2) in the proof above it is, in fact, more convenient to work with the variables (T_1, V_1, T_2, V_2) instead of (U_1, V_1, U_2, V_2). Since

$$\begin{aligned}
&\partial(P, Q, R, S)/\partial(U_1, V_1, U_2, V_2) \\
&\quad = \partial(P, Q, R, S)/\partial(T_1, V_1, T_2, V_2)\partial(T_1, V_1, T_2, V_2)/\partial(U_1, V_1, U_2, V_2) \\
&\quad = V_1 V_2 \partial(P, Q, R, S)/\partial(T_1, V_1, T_2, V_2),
\end{aligned}$$

instead of equation (1) we have

$$V_1 V_2 \sum_{k_1+k_2+k_3+k_4=0} \partial(P_{k_1}, Q_{k_2}, R_{k_3}, S_{k_4})/\partial(T_1, V_1, T_2, V_2) = 1. \tag{3}$$

Let us consider the case where

$$n_1 + n_2 + n_3 - n_4 = m_1 + m_2 + m_3 - m_4 = 0, \quad n_j, m_j \geq 0, \quad 1 \leq j \leq 4.$$

We use the notation

$$P_1 = P_{n_1,m_1} = V_1^{n_1} V_2^{m_1} f_1(T_1) f_2(T_2)$$

$$P_2 = P_{n_2,m_2} = V_1^{n_2} V_2^{m_2} g_1(T_1) g_2(T_2)$$

$$P_3 = P_{n_3,m_3} = V_1^{n_3} V_2^{m_3} h_1(T_1) h_2(T_2)$$

$$P_4 = P_{-n_4,-m_4} = V_1^{-n_4} V_2^{-m_4} T_1^{n_4} T_2^{m_4} (1 + T_1 + T_2)^{n_4+m_4} l_1(T_1) l_2(T_2).$$

We note that

$$\partial(P_1, P_2, P_3, P_4)/\partial(T_1, V_1, T_2, V_2)$$
$$= \begin{vmatrix} V_1^{n_1} V_2^{m_1} f_1' f_2 & n_1 V_1^{n_1-1} V_2^{m_1} f_1 f_2 & V_1^{n_1} V_2^{m_1} f_1 f_2' & m_1 V_1^{n_1} V_2^{m_1-1} f_1 f_2 \\ V_1^{n_2} V_2^{m_2} g_1' g_2 & n_2 V_1^{n_2-1} V_2^{m_2} g_1 g_2 & V_1^{n_2} V_2^{m_2} g_1 g_2' & m_2 V_1^{n_2} V_2^{m_2-1} g_1 g_2 \\ V_1^{n_3} V_2^{m_3} h_1' h_2 & n_3 V_1^{n_3-1} V_2^{m_3} h_1 h_2 & V_1^{n_3} V_2^{m_3} h_1 h_2' & m_3 V_1^{n_3} V_2^{m_3-1} h_1 h_2 \\ I_1 & I_2 & I_3 & I_4 \end{vmatrix}$$

where

$$\begin{aligned} I_1 &= V_1^{-n_4} V_2^{-m_4} T_1^{n_4-1} T_2^{m_4} (1 + T_1 + T_2)^{n_4+m_4-1} \\ &\quad \times \{[n_4(1 + T_1 + T_2) + (n_4 + m_4)T_1] l_1 + T_1(1 + T_1 + T_2) l_1'\} \\ I_2 &= -n_4 V_1^{-n_4-1} V_2^{-m_4} T_1^{n_4} T_2^{m_4} (1 + T_1 + T_2)^{n_4+m_4} l_1 l_2 \\ I_3 &= V_1^{-n_4} V_2^{-m_4} T_1^{n_4} T_2^{m_4-1} (1 + T_1 + T_2)^{n_4+m_4-1} \\ &\quad \times \{[m_4(1 + T_1 + T_2) + (n_4 + m_4)T_2] l_2 + T_2(1 + T_1 + T_2) l_2'\} \\ I_4 &= -m_4 V_1^{-n_4} V_2^{-m_4-1} T_1^{n_4} T_2^{m_4} (1 + T_1 + T_2)^{n_4+m_4} l_1 l_2. \end{aligned}$$

We note that

$$(1 + T_1 + T_2)^{n_4+m_4-1} \mid I_1, I_3, \qquad (1 + T_1 + T_2)^{n_4+m_4} \mid I_2, I_4.$$

We will expand $\partial(P_1, P_2, P_3, P_4)/\partial(T_1, V_1, T_2, V_2)$ using the last row (I_1, I_2, I_3, I_4) and will, in fact, show that $(1 + T_1 + T_2)$ divides each one of the 4 summands in the expansion. The first summand is

$$-I_1 \begin{vmatrix} n_1 \cdots & * & m_1 \cdots \\ n_2 \cdots & * & m_2 \cdots \\ n_3 \cdots & * & m_3 \cdots \end{vmatrix}$$

If $n_4 = 0$ then $n_1 = n_2 = n_3 = 0$ and this summand is 0.

If $m_4 = 0$ then $m_1 = m_2 = m_3 = 0$ and this summand is 0.

If $n_4, m_4 \geq 1$ then $n_4 + m_4 - 1 \geq 1$ and since $(1 + T_1 + T_2)^{n_4+m_4-1} \mid I_1$ it follows that $(1 + T_1 + T_2)$ divides the first summand.

The third summand is treated the same.

The second summand vanishes if $m_4 = 0$ and if $m_4 \geq 1$ then $n_4 + m_4 \geq 1$ and since $(1+T_1+T_2)^{n_4+m_4} \mid I_2$ it follows that $(1+T_1+T_2)$ divides the second summand.

The fourth summand is handled the same.

Identical arguments hold in the other cases, namely, the factor $(1 + T_1 + T_2)$ gets raised to a power that is a sum of a contribution that comes from the pair (U_1, V_1) and a contribution that comes from the pair (U_2, V_2) (sometimes, minus 1). If any of these contributions is zero then the whole determinant vanishes and so we may assume that both contributions are at least 1 in which case $(1 + T_1 + T_2)$ factors the determinant.

Remark 6. It is very plausible that the obvious extension of the above technique—of assigning weights that are independent over Z in order to separate the intermediates—works for all the varieties of Winkelmann's type in any even dimension (see Theorem 1). If so, it provides us with examples of etale exotic varieties of any even dimension. However, we do not know of an example of an etale exotic variety of odd dimension.

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