ILLINOIS JOURNAL OF MATHEMATICS
Volume 40, Number 3, Fall 1996

# ON COMPLEX INTERPOLATION OF COMPACT OPERATORS

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This paper is concerned with various aspects of the following question:

QUESTION 1. *Given Banach couples* $\mathbf{A} = (A_0, A_1)$ *and* $\mathbf{B} = (B_0, B_1)$ *and a bounded linear operator* $T\colon \mathbf{A} \to \mathbf{B}$ *such that* $T\colon A_0 \to B_0$ *is compact, is* $T\colon [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ *compact for* $\theta \in (0, 1)$?

Question 1 was first considered by Calderón thirty years ago in the course of his development of the complex interpolation method [Ca]. He was able to answer it in the affirmative under suitable additional hypotheses on the couple **B**. (See [Ca], Sections 9.6 and 10.4.)

Since then, Question 1 has been answered affirmatively in quite a number of other special cases. For example, see [CwK], [CKS], [M] and [P]. The analogous question for the real interpolation method has also been answered ([Cw2], p. 334; cf. also [CKS], p. 286). However Question 1 is still open in general.

We shall show here that the solution of Question 1 is equivalent to the solution of any of a number of its special cases, where $T$ has a simple form, or where the spaces $A_j$ and $B_j$ have additional properties, or they are certain special sequence spaces. In this last case we also show that it suffices to establish an apparently rather weaker condition than the compactness of $T\colon [A_0, A_1]_\theta \to [B_0, B_1]_\theta$. This last result leads us to a connection between Question 1 and some natural questions from a more recently developed generalization of interpolation theory, where couples of Banach spaces are replaced by families of infinitely many spaces.

We refer to the above-mentioned papers [Ca], [CKS], [Cw2], [CwK], [M] and [P] for further background and for definitions of the relevant notions from interpolation theory which we shall use here.

## 1. Reduction to the case of an inclusion operator

Let $X_0$ be the subspace of $B_0$ consisting of all those elements of the form $x = Ta$ where $a \in A_0$ with norm $\|x\|_{X_0} = \inf\{\|a\|_{A_0}\colon x = Ta\}$. The boundedness of

Received August 11, 1994.
1991 Mathematics Subject Classification. Primary 46B70; Secondary 47B07.
[1]Research supported by the Technion V.P.R. Fund.
[2]This research was done during a visit to the Technion as a Lady Davis Fellow.

$T\colon A_0 \to B_1$ ensures that $X_0$ is a Banach space, and the compactness of the same operator implies that the inclusion map $J\colon X_0 \to B_0$ is compact. Now consider $J$ as a map from the couple $\mathbf{X} = (X_0, B_1)$ into the couple $\mathbf{B} = (B_0, B_1)$. If we can show that $J\colon [X_0, B_1]_\theta \to [B_0, B_1]_\theta$ is compact, then it will follow that $T\colon [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ is compact, since $T\colon \mathbf{A} \to \mathbf{B}$ is the composition of the operators $S\colon \mathbf{A} \to \mathbf{X}$ and $J\colon \mathbf{X} \to \mathbf{B}$ where $S$ is defined by $Sa = Ta$ for all $a \in A_0 + A_1$ and thus defines a bounded operator from $[A_0, A_1]_\theta$ into $[X_0, B_1]_\theta$. In other words we have shown:

PROPOSITION 1. *In order to answer Question* 1 *in general, it suffices to resolve the special case where the couple* $\mathbf{A} = (A_0, A_1)$ *is such that* $A_1 = B_1$ *and* $A_0$ *is contained in* $B_0$ *and the operator* $T$ *is simply the inclusion map,* (*which of course is assumed to be compact from* $A_0$ *to* $B_0$).

*Remark.* In fact it would suffice to answer Question 1 in the context of Proposition 1 while further specializing to the case where $\mathbf{B}$ is the special couple $\mathbf{F}$ defined in [Cw2] or in Section 3 below, or the special couple $\mathbf{G}$ also defined in Section 3.

## 2. Reduction to the case of reflexive spaces

This is a strengthening of Proposition 1.

PROPOSITION 2. *Suppose we can prove that for every Banach couple* $\mathbf{Y} = (Y_0, Y_1)$ *of* **reflexive** *Banach spaces and every* **reflexive** *Banach space* $X_0$ *which is compactly embedded in* $Y_0$*, we always have* $[X_0, Y_1]_\theta$ *compactly embedded in* $[Y_0, Y_1]_\theta$*. Then this answers Question* 1.

*Proof.* Let $\mathbf{B} = (B_0, B_1)$ be an arbitrary Banach couple and let $A_0$ be compactly embedded in $B_0$. Then, by Proposition 1 of [Bz], p. 32, we deduce that $(A_0, B_0)_{\alpha,p}$ is reflexive for each $\alpha$ in (0,1) and each $p$ in $(1, \infty)$. Let us choose $X_0$ and $Y_0$ to be the reflexive spaces $X_0 = (A_0, B_0)_{1/3,2}$ and $Y_0 = (A_0, B_0)_{2/3,2}$. Then the inclusions $A_0 \subset X_0 \subset Y_0 \subset B_0$ are all compact embeddings. This follows by interpolation of compact operators for the real method [Cw2] (but in fact for this case even the classical lemma of Lions-Peetre [LP] would suffice) and also the reiteration theorem for the real method. Given any fixed $\theta$ in (0,1) we choose some $\beta$ in $(0, \theta)$ and let $Y_1 = [Y_0, B_1]_\beta$. It follows from [Ca] sections 12.2 and 32.2 that $Y_1$ is reflexive. Thus by our supposition above we can deduce that $[X_0, Y_1]_\gamma$ is compactly embedded in $[Y_0, Y_1]_\gamma$ for every $\gamma$ in (0,1). Now we observe, using the reiteration formula for the complex method in the form presented in [CwK], that $[A_0, B_1]_\theta = [A_0, [A_0, B_1]_\beta]_{\theta/\beta}$ and this space is of course continuously embedded in $[X_0, [Y_0, B_1]_\beta]_{\theta/\beta}$ which in turn is, as we have just seen, compactly embedded in $[Y_0, [Y_0, B_1]_\beta]_{\theta/\beta}$. Another application of the reiteration formula shows that this space is $[Y_0, B_1]_\theta$ which is of course continuously embedded in $[B_0, B_1]_\theta$. All this shows that $[A_0, B_1]_\theta$ is compactly embedded in $[B_0, B_1]_\theta$ which by Proposition 1 is all that we need to establish our claim. □

*Remarks.* (i) The proof of reflexivity of $(A_0, B_0)_{\alpha,p}$ in [Bz] only requires the embedding of $A_0$ in $B_0$ to be *weakly* compact. Can we use the fact that we have a stronger condition on the embedding to obtain a stronger and more useful property for $(A_0, B_0)_{\alpha,p}$? By [CwK], Theorem 9, it would suffice to show that $(A_0, B_0)_{\alpha,p}$ is a UMD space for some value of $\alpha$ and some value of $p$ to resolve Question 1, but easy examples where $A_0$ and $B_0$ are weighted $\ell_\infty$ spaces indicate that this is not possible in general. Thus this indicates that the problem can be attacked on two fronts, on the one hand to try to replace the UMD condition required in [CwK] by something weaker, on the other hand, as just suggested, to try to replace the reflexivity property of $(A_0, B_0)_{\alpha,p}$ by something stronger.

(ii) As a further simplification, we mention that it suffices to deal with the case of spaces which are separable, (as well as reflexive). We need only show separately that $\{Ta_m\}$ has a convergent subsequence in $[B_0, B_1]_\theta$ for each bounded sequence $\{a_m\}$ in $[A_0, A_1]_\theta$. By obvious density considerations it suffices to consider the case where each element $a_m$ is in $A_0 \cap A_1$. Now we construct a countable subset $\mathcal{A}$ of $A_0 \cap A_1$ such that for each $m$ and $n$ there exists a function $f_{mn}$ which is a finite sum of elements of the form $\phi(z)a$ where $a \in \mathcal{A}$ and $\phi$ is a scalar valued bounded analytic function on the unit strip which is continuous on the closed strip, and $f_{mn}(\theta) = a_m$ and $\|f_{mn}\|_{\mathcal{F}(\mathbf{A})} \le \|a\|_{[A_0,A_1]_\theta} + 1/n$ (cf. [S] and [Ca]). It suffices to know that the general result holds when $A_j$ is replaced by the closure of $\mathcal{A}$ in $A_j$ and $B_j$ by the closure of $T(\mathcal{A})$ in $B_j$ for $j = 0, 1$.

## 3. Reduction to the cases of special "Fourier" couples and point evaluation operators

Let us recall that, as explained in [Cw2, pp. 339–340] (cf. also [CP]), it suffices to answer Question 1 in the case where $\mathbf{A} = \mathbf{E} = (\ell^1(FL^1), \ell^1(FL^1(e^\nu))$ and $\mathbf{B} = \mathbf{F} = (\ell^\infty(FL^\infty), \ell^\infty(FL^\infty(e^\nu))$.

Let us now refine this a little further, and show that it suffices to consider variants of the couples $\mathbf{E}$ and $\mathbf{F}$ which are slightly simpler than those introduced in [Cw2]. Rather than explaining how to modify the reasoning in [Cw2] it seems preferable to present a new version of the whole argument. This approach will also give additional insights and enable us to make yet another reduction of the problem.

As in [Ja] and [Cw2], for each $\alpha \in \mathbb{R}$ and each $p \in [1, \infty]$ we let $FL^p(e^{\alpha\nu})$ denote the space of complex valued sequences $\{x_\nu\}_{\nu\in\mathbb{Z}}$ such that $\{e^{\alpha\nu}x_\nu\}$ is the sequence of Fourier coefficients of a function $f \in L^p(\mathbb{T})$. We set $\|\{x_\nu\}_{\nu\in\mathbb{Z}}\|_{FL^p(e^{\alpha\nu})} = \|f\|_{L^p}$. Analogously we also define the space $FC(e^{\alpha\nu})$ by replacing the $L^p$ space by the space $C = C(\mathbb{T})$ of continuous functions on $\mathbb{T}$. For any $p \in [1, \infty]$ and any Banach space $B$ let $\ell^p(B)$ be the space of $B$ valued sequences $\{b_n\}_{n\in\mathbb{N}}$ such that $\{\|b_n\|_B\} \in \ell^p$ with the obvious norm. Let $E_\alpha = \ell^1(FL^1(e^{\alpha\nu}))$, $F_\alpha = \ell^\infty(FL^\infty(e^{\alpha\nu}))$ and $G_\alpha = \ell^\infty(FC(e^{\alpha\nu}))$ and consider the Banach couples $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ and $\mathbf{G} = (G_0, G_1)$.

PROPOSITION 3. *In order to answer Question* 1 *it suffices to resolve it in the particular case when* $\mathbf{A} = \mathbf{E}$ *and* $\mathbf{B}$ *is either* $\mathbf{G}$ *or* $\mathbf{F}$.

*Proof.* Let us first use the fact that $[FL^1(e^{\alpha_0 \nu}), FL^1(e^{\alpha_1 \nu})]_\theta = FL^1(e^{\beta\nu})$ where $\beta = (1-\theta)\alpha_0 + \theta\alpha_1$ (cf. [Ja, p. 68]). So by [Ca], Section 13.6, p. 125, we have $[E_{\alpha_0}, E_{\alpha_1}]_\theta = E_\beta$ (and in particular $[E_0, E_1]_\theta = E_\theta$). Since for each $\alpha$ the dual of $E_\alpha$ is $F_{-\alpha}$ (where the bilinear functional for the duality is defined in the obvious way and is independent of $\alpha$), it follows by Calderón's duality theorem that $[F_0, F_1]^\theta = F_\theta$.

Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary Banach couples and let $T\colon \mathbf{A} \to \mathbf{B}$ be as in the statement of Question 1. Fix $\theta \in (0, 1)$ and let $\{a_m\}_{m\in\mathbb{N}}$ be an arbitrary sequence in $[A_0, A_1]_\theta$ with $\|a_m\|_{[A_0,A_1]_\theta} \le 1$. Our goal will of course be to show that $\{Ta_m\}$ has a subsequence which is Cauchy in $[B_0, B_1]_\theta$. In fact it will be convenient, and will not cause any loss of generality (as already remarked above) if we make the additional assumption that each $a_m$ is in $A_0 \cap A_1$. It is also very easy to see (cf. e.g. the proof of Theorem 9 in [CwK]) that we may also suppose without loss of generality that the couples $\mathbf{A}$ and $\mathbf{B}$ are both *regular*; i.e., $A_0 \cap A_1$ is dense in $A_0$ and also in $A_1$, and similarly for the $B_j$'s.

By Janson's description of the complex method [Ja], for each integer $m$ there exists a (possibly infinite) subset $K_m$ of $\mathbb{N}$ and for each $k \in K_m$ there exist an element $x_k \in FL^1(e^{\theta\nu})$ and a norm one operator $U_k\colon (FL^1(1), FL^1(e^\nu)) \to \mathbf{A}$ such that $\sum_{\cdot\in K_m} \|x_k\|_{FL^1(e^{\theta\nu})} \le C$ and $a_m = \sum_{k\in K_m} U_k x_k$ for each integer $m$. Here the constant $C$ depends only on $\theta$. Clearly we may choose the sets $K_m$ to be pairwise disjoint for different values of $m$. Thus the norm one operators $U_k$ and the elements $x_k$ will be defined for all $k \in \mathbb{N}$. (They can be taken to be zero for all $k \notin \bigcup_{m\in\mathbb{N}} K_m$.) For each $z = \{z_n\}_{n\in\mathbb{N}}$ in $E_0 + E_1$ let $\mathcal{U}z = \sum_{n\in\mathbb{N}} U_n z_n$. Obviously $\mathcal{U}\colon \mathbf{E} \to \mathbf{A}$ with norm 1 and clearly $\mathcal{U}y_m = a_m$ and $\|y_m\|_{[E_0,E_1]_\theta} \le C$ for each $m$, where $y_m = \{y_{mk}\}_{k\in\mathbb{N}}$ is the element of $[E_0, E_1]_\theta$ defined by $y_{mk} = x_k$ for all $k \in K_m$ and $y_{mk} = 0$ for all $k \notin K_m$.

For $j = 0, 1$ we shall let $B_j'$ denote the dual space of $B_j$. For each $x \in B_0 \cap B_1$ and each $y$ in the dual space $(B_0 \cap B_1)'$ of $B_0 \cap B_1$ we denote the value of $y$ at $x$ by $\langle x, y\rangle$. Since we are assuming that $\mathbf{B}$ is regular, $B_0'$ and $B_1'$ can of course be identified as subspaces of $(B_0 \cap B_1)'$.

For each element $b \in B_0 \cap B_1$ we can use the argument of [Cw1], p. 1006 to obtain a norm one element $h \in \mathcal{F}(B_0', B_1')$ such that $\langle b, h(\theta)\rangle$ is arbitrarily close to $\|b\|_{[B_0,B_1]_\theta}$. Clearly we can suppose furthermore that the element $h$ is in the dense subclass $\mathcal{G}(B_0', B_1')$ of $\mathcal{F}(B_0', B_1')$. (Cf. [Ca].)

Let $\Omega$ be the annulus $\{z \in \mathbb{C}\colon 1 \le |z| \le e\}$. Via the argument of [Cw1], p. 1008, we may use $h$ to contruct a continuous function $H\colon \Omega \to B_0' \cap B_1'$ which is analytic in the interior of $\Omega$ and satisfies $H(e^\theta) = h(\theta)$ and also $\sup_{t\in[0,2\pi], j=0,1} \|H(e^{j+it})\|_{B_j'} \le C_\theta$ for some absolute constant $C_\theta$ depending only on $\theta$.

For each element $x \in B_0 \cap B_1$ we shall introduce a scalar valued function $\phi_{x,H}$ on $\Omega$. We define it by setting $\phi_{x,H}(z) = \langle x, H(z)\rangle$. Clearly $\phi_{x,H}$ is continuous on $\Omega$ and analytic in its interior. We denote its Laurent series by $\sum_{k\in\mathbb{Z}} \widehat{\phi}_{x,H}(k) z^k$. Let us now define the linear operator $V\colon B_0 \cap B_1 \to FC(1) + FC(e^\nu)$ by $Vx = \{\widehat{\phi_{x,H}}(k)\}_{k\in\mathbb{Z}}$. Then clearly $\|Vx\|_{FC(e^{j\nu})} \le C_\theta \|x\|_{B_j}$ for all $x \in B_0 \cap B_1$ and so $V$ can be uniquely extended (since $\mathbf{B}$ is regular) to an operator $V\colon B_j \to FC(e^{j\nu})$ which is bounded with norm not exceeding $C_\theta$ for $j = 0, 1$. Furthermore, if $\zeta_\theta\colon FC(e^{\theta\nu}) \to \mathbb{C}$ is the continuous linear functional of point evaluation at $e^\theta$, i.e. $\zeta_\theta(\{\alpha_\nu\}_{\nu\in\mathbb{Z}}) = \sum_{\nu\in\mathbb{Z}} \alpha_\nu e^{\theta\nu}$, then, for the element $b$ above, $\zeta_\theta(Vb)$ is arbitrarily close to $\|b\|_{[B_0,B_1]_\theta}$. (Of course in general the above series $\sum_{\nu\in\mathbb{Z}} \alpha_\nu e^{\theta\nu}$ has to be summed using its (C,1) means.)

Let us construct such an operator $V = V_n$ for each element $b_n$ in the countable set $\mathcal{B}$ of all elements of the form $Ta_m - Ta_{m'} = T\mathcal{U}y_m - T\mathcal{U}y_{m'}$ for all $m, m' \in \mathbb{N}$. Thus we obtain a sequence of operators $V_n\colon \mathbf{B} \to (FC(1), FC(e^\nu))$ each with norm bounded by $C_\theta$ such that

$$(*) \qquad \|b\|_{[B_0,B_1]_\theta} \le 2 \sup_{n\in\mathbb{N}} |\zeta_\theta(V_n b)| \le 2 \sup_{n\in\mathbb{N}} \|V_n b\|_{FC(e^{\theta\nu})}$$

for each $b \in \mathcal{B}$. For each $b \in B_0 + B_1$ let $\mathcal{V}b = \{V_n b\}_{n\in\mathbb{N}}$. Then $\mathcal{V}\colon \mathbf{B} \to \mathbf{G}$ with norm $C_\theta$. (The term $2\sup_{n\in\mathbb{N}} |\zeta_\theta(V_n b)|$ in $(*)$ is irrelevant for the proof of the current proposition, but it will be useful later.)

Let $C(\ell^\infty)$ denote the space of all *continuous* functions $f\colon \mathbb{T} \to \ell^\infty$ with norm $\|f\|_{C(\ell^\infty)} = \sup_{t\in[0,2\pi]} \|f(e^{it})\|_{\ell^\infty}$. The Fourier coefficients

$$\widehat{f}(\nu) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu t} f(t)\, dt$$

of each $f \in C(\ell^\infty)$ are of course well defined elements of $\ell^\infty$. Let $\Gamma$ denote the space of $\ell^\infty$ valued sequences $\gamma = \{\gamma(\nu)\}_{\nu\in\mathbb{Z}} = \{\{\gamma_n(\nu)\}_{n\in\mathbb{N}}\}_{\nu\in\mathbb{Z}}$ which arise as Fourier transforms of functions $f$ in $C(\ell^\infty)$, i.e. $\gamma(\nu) = \widehat{f}(\nu)$ with norm $\|\gamma\|_\Gamma = \|f\|_{C(\ell^\infty)}$. For each real $\alpha$ let $\Gamma_\alpha$ be the space of doubly indexed sequences $\gamma = \{\gamma(\nu)\}_{\nu\in\mathbb{Z}} = \{\{\gamma_n(\nu)\}_{n\in\mathbb{N}}\}_{\nu\in\mathbb{Z}}$ such that $\{e^{\alpha\nu}\gamma(\nu)\}_{\nu\in\mathbb{Z}} \in \Gamma$, with the obvious norm. The elements $f$ of $C(\ell^\infty)$ can of course also be considered as bounded sequences of scalar continuous functions and clearly $C(\ell^\infty)$ is a closed subspace of $\ell^\infty(C)$ and of $\ell^\infty(L^\infty)$. Then obviously also $\Gamma_\alpha$ is a closed subspace of $G_\alpha$ and of $F_\alpha$.

We present the next part of the proof of Proposition 3 as a separate lemma.

LEMMA 1. *For each $\theta \in (0,1)$ the spaces $[F_0, F_1]_\theta, [G_0, G_1]_\theta, [\Gamma_0, \Gamma_1]_\theta$ and $\Gamma_\theta$ all coincide with equivalent norms.*

*Proof of the Lemma.* Obviously $[\Gamma_0, \Gamma_1]_\theta \subset [G_0, G_1]_\theta \subset [F_0, F_1]_\theta$ continuously. So we have only to establish the continuous inclusions (i) $\Gamma_\theta \subset [\Gamma_0, \Gamma_1]_\theta$ and (ii) $[F_0, F_1]_\theta \subset \Gamma_\theta$. To show (i) let us first suppose that $\gamma = \{\gamma(\nu)\}_{\nu\in\mathbb{Z}}$ is an arbitrary finitely supported $\ell^\infty$ valued sequence. More precisely, for some fixed $N$ we have $\gamma(\nu) = 0$ for all $\nu$ such that $|\nu| > N$ (but for each other value of $\nu$ the elements of the sequence $\gamma_n(\nu)$ may possibly be non-zero for infinitely many values of $n$.) It is clear that such elements $\gamma$ constitute a dense subset of $\Gamma_\theta$. The function $g\colon \mathbb{C} \to \Gamma_0 \cap \Gamma_1$ defined by $g(z) = \{e^{\nu\theta} z^{-\nu}\gamma(\nu)\}_{\nu\in\mathbb{Z}}$ is clearly analytic and continuous for all $z$ and satisfies $g(e^\theta) = \gamma$. Furthermore $\|g(e^{j+it})\|_{\Gamma_j} = \|\gamma\|_{\Gamma_\theta}$ for $j = 0, 1$ and $t \in [0, 2\pi]$. Consequently, using the function $G(z) = e^{\delta z^2} g(e^z) \in \mathcal{F}(\Gamma_0, \Gamma_1)$ with $\delta > 0$ arbitrarily small, we have $\gamma \in [\Gamma_0, \Gamma_1]_\theta$ and $\|\gamma\|_{[\Gamma_0,\Gamma_1]_\theta} \le \|\gamma\|_{\Gamma_\theta}$. By density this extends to all elements $\gamma \in \Gamma_\theta$ and yields (i).

We now turn our attention to (ii). First, suppose that $\gamma = \{\gamma(\nu)\}_{\nu\in\mathbb{Z}}$ is in $F_0 \cap F_1$ and, for each $n$, let $f_n(z)$ be the scalar valued function whose Laurent expansion is $\sum_{\nu\in\mathbb{Z}} \gamma_n(\nu) z^\nu$. Then via Fejér's theorem and the maximum modulus principle it is clear that $f_n$ is analytic in the interior of $\Omega$ and that $|f_n(z)| \le \|\gamma\|_{F_0\cap F_1}$ for almost all $z$ on the boundaries of $\Omega$ and also for all $z$ in its interior. Thus for all $z$ on the circle $|z| = e^\theta$ we have $|f_n'(z)| \le C\|\gamma\|_{F_0\cap F_1}$ where the constant $C$ depends only on $\theta$. Consequently the functions $f_n$ are equicontinuous on this circle and the map $\phi\colon \mathbb{T} \to \ell^\infty$ defined by $\phi(e^{it}) = \{f_n(e^{\theta+it})\}_{n\in\mathbb{N}}$ is an element of $C(\ell^\infty)$. In other words we have shown that $F_0 \cap F_1$ is contained in $\Gamma_\theta$. It is also a dense subset of $\Gamma_\theta$ since it contains the elements of finite support used in the proof of (i) above.

As observed earlier, $[F_0, F_1]^\theta = F_\theta$ and so by Bergh's theorem [Bg] it follows that $[F_0, F_1]_\theta$ is a closed subspace of $F_\theta$. Clearly $\Gamma_\theta$ is also a closed subspace of $F_\theta$. Since $F_0 \cap F_1$ is dense in both $\Gamma_\theta$ and $[F_0, F_1]_\theta$ we deduce that $\Gamma_\theta = [F_0, F_1]_\theta$. This establishes (ii) and completes the proof of Lemma 1.

We can now easily finish off the proof of Proposition 3. Clearly the operator $\mathcal{VTU}\colon \mathbf{E} \to \mathbf{G}$ is bounded and, since $T\colon A_0 \to B_0$ is compact, so is $\mathcal{VTU}\colon E_0 \to G_0$. We are supposing that Question 1 has been resolved in the affirmative for the case of operators mapping from $\mathbf{E}$ to either $\mathbf{G}$ or to $\mathbf{F}$. In both of these cases $\mathcal{VTU}\colon [E_0, E_1]_\theta \to \Gamma_\theta$ must be compact (by Lemma 1) and so some subsequence $\{\mathcal{VTU}y_{m(j)}\}$ must be a Cauchy sequence in $\Gamma_\theta$ and thus also in $G_\theta$. By (*) this implies that the corresponding sequence $\{Ta_{m(j)}\}$ must be Cauchy in $[B_0, B_1]_\theta$. This shows that $T\colon [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ is compact. □

*Remarks.* (i) If we reverse the rôles of $\mathbf{E}$ and $\mathbf{B}$, i.e., take $\mathbf{A} = \mathbf{F}$ and $\mathbf{B} = \mathbf{E}$, then Question 1 can be resolved in the affirmative for this case. We can show this by an argument which uses a result of W. B. Johnson (Theorem 2 of [Jo]) and the classical Grothendieck theorem.

(ii) It is tantalizing to note that in Theorem 10 of [CwK], Question 1 is answered in the affirmative for all couples which satisfy the condition that $A_0$ is reflexive, and also the condition that $A_0 = [X, A_1]_\alpha$ for some Banach space $X$. Propositions 2 and 3 show that it would suffice if we could remove either one of these two conditions from that theorem.

One of the pleasant features of working with the space $C(\ell^\infty)$ (and thus also its isometric images $\Gamma_\alpha$) is that the compact subsets of this space have the following simple "Arzelà–Ascoli" style characterization:

LEMMA 2. *The set $K$ is relatively compact in $C(\ell^\infty)$ if and only if*

(i) *there exists a compact subset $M$ of $\ell^\infty$ such that $f(e^{it}) \in M$ for all $f \in K$ and $t \in [0, 2\pi]$ and also*
(ii) $\lim_{\delta\to 0} \sup\{\|f(e^{i(t+h)}) - f(e^{it})\|_{\ell^\infty}\colon f \in K,\ |h| < \delta\} = 0$.

*Proof.* This is an easy exercise. □

Using Lemma 2 it is not hard to show that if $K_0$ is a relatively compact subset of $\Gamma_0$ and if $K_\theta$ is defined to be the set of all elements $\gamma = f(\theta)$ where $f$ ranges over all functions in $\mathcal{F}(\Gamma_0, \Gamma_1)$ such that $f(it) \in K_0$ and $f(1+it) \in B_{\Gamma_1}$ for all real $t$, then $K_\theta$ is a relatively compact subset of $\Gamma_\theta$. In other words the answer to Question 1 is affirmative in the case where $\mathbf{B} = (\Gamma_0, \Gamma_1)$ and also, via the reiteration theorem, in the cases $\mathbf{B} = (\Gamma_0, F_1)$ and $\mathbf{B} = (\Gamma_0, G_1)$. This observation also leads readily to an alternative proof of Theorem 11 of [CwK] (and so also of Theorem 2.1 of [Cw2], p. 339).

We can now present our additional reduction of Question 1. It is a strengthening of Proposition 3. Let us begin with the trivial remark that if $K$ is a compact subset of $C(\ell^\infty)$ then it necessarily satisfies the following condition:

(i′) The set $\{f(1)\colon f \in K\}$ is relatively compact in $\ell^\infty$.

This is of course rather weaker than the conditions (i) and (ii) of Lemma 2.

Now suppose that we try to use Proposition 3 to answer Question 1. We consider an operator $T\colon \mathbf{E} \to \mathbf{G}$ such that $T\colon E_0 \to G_0$ is compact. We have to show that the set $K = T(B_{E_\theta})$ is relatively compact in $\Gamma_\theta$. So it would seem that we would have to check the analogues of Conditions (i) and (ii) of Lemma 2. But it turns out to be sufficient to merely check the analogue of condition (i′).

Let us define an (obviously bounded and linear) "evaluation map" $Z_\theta\colon G_\theta \to \ell^\infty$ by setting $Z_\theta(\gamma) = \{\zeta_\theta(\gamma_n(\cdot))\}_{n\in\mathbb{N}}$ where $\zeta_\theta\colon FC(e^{\theta\nu}) \to \mathbb{C}$ is the functional of point evaluation at $e^\theta$ defined above.

PROPOSITION 4. *In order to obtain an affirmative answer to Question* 1 *it suffices to show that for every bounded linear operator $T\colon \mathbf{E} \to \mathbf{G}$ such that $T\colon E_0 \to G_0$ is compact, the map $Z_\theta T\colon E_\theta \to \ell^\infty$ is compact.*

*Remark.* In fact, in view of Lemma 1 and the reiteration theorem, it suffices in Proposition 4 to consider the subclass of operators $T$ which satisfy the additional condition that $T\colon E_1 \to \Gamma_1$.

*Proof.* We proceed almost exactly as in the proof of Proposition 3. The only place where we need to make a small change is in the last paragraph of that proof. Although now we cannot obtain that $\mathcal{V}T\mathcal{U}\colon [E_0, E_1]_\theta \to \Gamma_\theta$ is compact, we can suppose that

$Z_\theta\{\mathcal{V}T\mathcal{U}y_{m(j)}\}$ is a Cauchy sequence in $\ell^\infty$. This is still enough to imply, using the term in $(*)$ which we ignored earlier, that the corresponding sequence $\{Ta_{m(j)}\}$ is Cauchy in $[B_0, B_1]_\theta$. As before this implies that $T\colon [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ is compact. □

*Some further natural questions.* 1. Look for a counterexample. The simplest case we can think of now where the theorem might conceivably fail is the case where T is the inclusion/identity operator, and $A_0 = \ell^\infty(w)$, $B_0 = \ell^\infty$, and $A_1 = B_1$ is a sequence space which is not a lattice with respect to the natural componentwise ordering. Here $w = \{w_n\}$ has to be a weight sequence which tends to infinity as $n \to \pm\infty$. $w$ should have arbitrarily large sets on which it assumes constant values, to prevent the compact embedding from $A_0$ into $B_0$ from being too nice. The first choice for a "non-lattice" for $A_1 = B_1$ might seem to be $FL^p$ for some $p$ but in fact in this case the answer to Question 1 will be positive, because the sequence $\{P_n\}$ of multiplication/convolution operators defined by the de la Vallée Poussin kernels will be norm bounded on this space and furthermore $\lim_{n\to\infty} \|P_n - T\|_{A_0 \to B_0} = 0$. Possible choices for $A_1$ might be the space of Fourier coefficients of functions in a non-translation invariant function space on $[0, 2\pi]$ such as weighted $L^p$, or a space defined by some sort of conditions on differences of the elements of the sequence.

2. Formulate a quantitative version of the result for the case of finite dimensional spaces which is equivalent to the infinite dimensional case.

3. Suppose we know that the answer to Question 1 is affirmative when $T\colon A_1 \to B_1$ is also compact. Is this sufficient to give the answer in general?

## 4. Some related questions in the theory of interpolation of infinite families of spaces

Proposition 4 opens up some intriguing new possibilities. It appears that one can attempt to answer Question 1 via some other questions related to the theory of interpolation of infinite families of Banach spaces (cf. [CCRSW], [CwJ], [HRW]).

Before we explain this in detail let us introduce some further notation. For any Banach couple $\mathbf{X} = (X_0, X_1)$ let $\mathcal{F}_\Omega(\mathbf{X})$ be the space of functions $f\colon \Omega \to X_0 + X_1$ which are analytic on the interior of $\Omega$ and such that $f(e^{j+it})$ is an $X_j$ -valued and $X_j$ -continuous function of the real variable $t$ for $j = 0, 1$. This space is normed by

$$\|f\|_{\mathcal{F}_\Omega(\mathbf{X})} = \sup_{t\in[0,2\pi],\, j=0,1} \|f(e^{j+it})\|_{X_j}.$$

Let $\mathcal{G}_\Omega(\mathbf{X})$ be the subspace of $\mathcal{F}_\Omega(\mathbf{X})$ consisting of those functions $f$ which have the form $f(\omega) = \sum_{k=-N}^{N} x_k\omega^k$ for $\omega \in \Omega$, where each $x_k$ is in $X_0 \cap X_1$. It is clear (cf. [Ca]) that $\mathcal{G}_\Omega(\mathbf{X})$ is dense in $\mathcal{F}_\Omega(\mathbf{X})$. As shown in [Cw1], the space $[X_0, X_1]_\theta$ coincides with the space of all elements of the form $f(e^\theta)$ as $f$ ranges over $\mathcal{F}_\Omega(\mathbf{X})$

and the norm $\|x\|_{[X_0,X_1]_\theta}$ is equivalent to

$$\inf_{x=f(e^\theta),\, f\in\mathcal{F}_\Omega(\mathbf{X})} \|f\|_{\mathcal{F}_\Omega(\mathbf{X})}.$$

In the case $x \in X_0 \cap X_1$ the preceding infimum is the same when $f$ ranges over $\mathcal{G}_\Omega(\mathbf{X})$. (This is shown by using an obvious variant of an argument in [S].)

It is more convenient at this stage to reformulate Proposition 4 in terms of spaces of vector valued (analytic) functions instead of vector valued sequences (of Laurent and/or Fourier coefficients of those same functions). This is essentially a rather trivial modification, but we shall take a few moments to work through it carefully:

Let $A(\Omega)$ denote the space of all continuous functions $f\colon \Omega \to \mathbb{C}$ which are analytic on the interior of $\Omega$ with norm $\|f\|_{A(\Omega)} = \sup_{z\in\Omega}|f(z)|$. Suppose that $\gamma = \{\gamma(\nu)\}_{\nu\in\mathbb{Z}}$ is in $G_0 \cap G_1$. Then Fejér's theorem and the maximum modulus principle show that for each $n$, the Fejér means of the Laurent series $\sum_{\nu\in\mathbb{Z}}\gamma_n(\nu)z^\nu$ converge uniformly on $\Omega$ to a function $f_n \in A(\Omega)$. Clearly $\gamma \in G_0 \cap G_1$ if and only if $\{f_n\}_{n\in\mathbb{N}} \in \ell^\infty(A(\Omega))$. Furthermore $G_0 \cap G_1$ and $\ell^\infty(A(\Omega))$ are isometrically isomorphic. For our purposes it will be convenient to work with the Banach couple $\widehat{\mathbf{G}} = (\widehat{G}_0, \widehat{G}_1)$ defined by taking $\widehat{G}_j$ to be the Cauchy completion of $\ell^\infty(A(\Omega))$ with respect to the norm $\|\{f_n\}_{n\in\mathbb{N}}\|_{\widehat{G}_j} = \sup_{n\in\mathbb{N},0\le t\le 2\pi}|f_n(e^{j+it})|$. Of course we have $\widehat{G}_0 \cap \widehat{G}_1 = \ell^\infty(A(\Omega))$, and the map $\gamma \mapsto \{\sum_{\nu\in\mathbb{Z}}\gamma_n(\nu)z^\nu\}_{n\in\mathbb{N}}$ extends to an isometry of $G_j$ onto $\widehat{G}_j$ for $j = 0, 1$. This means that we can replace the couple $\mathbf{G}$ by $\widehat{\mathbf{G}}$ in Propostion 3. We can also do so in Proposition 4 provided we also replace the map $Z_\theta$ by the map $\widehat{Z}_\theta\colon \ell^\infty(A(\Omega)) \to \ell^\infty$ defined by $\widehat{Z}_\theta(\{f_n\}_{n\in\mathbb{N}}) = \{f_n(e^\theta)\}_{n\in\mathbb{N}}$.

In fact in this modified Propostion 4 we also need $\widehat{Z}_\theta$ to have a unique extension to a continuous map of $[\widehat{G}_0, \widehat{G}_1]_\theta$ into $\ell^\infty$. This is clearly true (cf. Lemma 1) since $[\widehat{G}_0, \widehat{G}_1]_\theta$ is the closure of $\ell^\infty(A(\Omega))$ with respect to the norm $\|\{f_n\}_{n\in\mathbb{N}}\|_\theta = \sup_{n\in\mathbb{N},0\le t\le 2\pi}|f_n(e^{\theta+it})|$.

Now suppose that we are given a bounded linear operator $T\colon \mathbf{E} \to \widehat{\mathbf{G}}$ and a compact subset $K_0 \subset \widehat{G}_0$ such that $T(B_{E_0}) \subset K_0$. In order to answer Question 1 we are going to try to show that $\widehat{Z}_\theta T\colon E_\theta \to \ell^\infty$ is compact. Clearly we may suppose without loss of generality that $K_0$ is absolutely convex and that $\|T\|_{E_j\to\widehat{G}_j} \le 1$.

For each element $g(\omega) = \sum_{k=-N}^N a_k\omega^k$ in the unit ball of $\mathcal{G}_\Omega(\mathbf{E})$ consider $f(\omega) = Tg(\omega) = \sum_{k=-N}^N f_k(z)\omega^k$, where each $f_k = Ta_k \in \ell^\infty(A(\Omega))$. Clearly the $\ell^\infty$ valued function $\phi(z) = \sum_{k=-N}^N f_k(z)z^k$ is also an element of $\ell^\infty(A(\Omega))$ and $\widehat{Z}_\theta(T(g(e^\theta)) = \phi(e^\theta)$.

For each $\sigma \in \Omega$ we define the evaluation map $\Delta_\sigma\colon \ell^\infty(A(\Omega)) \to \ell^\infty$ by $\Delta_\sigma(\{f_n\}_{n\in\mathbb{N}}) = \{f(\sigma)\}_{n\in\mathbb{N}}$. Clearly if $|\sigma| = e^j$ then $\Delta_\sigma$ extends to a norm one map of $\widehat{G}_j$ into $\ell^\infty$ for $j = 0, 1$. Thus, in particular, if $\sigma = e^{it}$, $M_\sigma = \Delta_\sigma(K_0)$ must be an absolutely convex compact subset of $\ell^\infty$. Also, for $\sigma = e^{1+it}$, $M_\sigma = \Delta_\sigma(B_{\widehat{G}_1})$ must be an absolutely convex subset of $B_{\ell^\infty}$.

We can now state a new question. In view of the preceding remarks and Proposition 4, an affirmative answer to this question would imply an affirmative answer to Question 1.

*Question* 2. *Suppose that for each point $\sigma$ of the boundary of the annulus $\Omega$ we are given an absolutely convex subset $M_\sigma$ of the unit ball of $\ell^\infty$ and that $M_\sigma$ is relatively compact for each $\sigma$ on the unit circle.*

*Let $\mathcal{M}$ be the space of all functions $\phi \in \ell^\infty(A(\Omega))$ such that $\phi(\sigma) \in M_\sigma$ for each $\sigma \in \partial\Omega$.*

*For $0 < \theta < 1$, let $M_{[e^\theta]}$ be the subset of $\ell^\infty$ consisting of all elements of the form $\phi(e^\theta)$ where $\phi \in \mathcal{M}$. Is $M_{[e^\theta]}$ relatively compact in $\ell^\infty$?*

*Remark.* In fact it would suffice to resolve a special case of Question 2 where the compactness of the sets $M_\sigma$ is "uniform" for all $\sigma$, $|\sigma| = 1$, in the sense that for each $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that for each $\sigma$, $|\sigma| = 1$, $M_\sigma$ is contained in the union of $N(\epsilon)$ balls of radius $\epsilon$ in $\ell^\infty$. It would also suffice (via reiteration) to consider the particular case where the elements of $\mathcal{M}$ have the additional property that their restrictions to the circle $|\sigma| = e$ are continuous $\ell^\infty$ valued functions on that circle.

Question 2 implicitly contains the definition of a variant of the interpolation spaces introduced and studied in a number of papers, including [CCRSW], [CwJ] and [HRW]: For each $\sigma \in \Omega$ let $X_\sigma$ be a Banach space which is continuously embedded in $\ell^\infty$. (For the particular application which we have in mind $X_\sigma$ will of course be the Cauchy completion of the normed space whose unit ball is $M_\sigma$.) Let $\mathcal{H} = \mathcal{H}(\{X_\sigma\}_{\sigma\in\partial\Omega})$ be the subspace of all functions in $\ell^\infty(A(\Omega))$ which satisfy $\{f_n(\sigma)\}_{n\in\mathbb{N}} \in X_\sigma$ for each $\sigma \in \partial\Omega$ and for which $\|\{f_n\}\|_{\mathcal{H}} := \sup_{\sigma\in\partial\Omega} \|\{f_n(\sigma)\}\|_{X_\sigma} < \infty$. Then for each $z$ in the interior of $\Omega$ we define the space $X_{[z]}$ to consist of all elements of $\ell^\infty$ of the form $\{f_n(z)\}_{n\in\mathbb{N}}$ as $\{f_n\}$ ranges over $\mathcal{H}$. $X_{[z]}$ is normed by the obvious quotient norm with respect to $\mathcal{H}$.

The spaces $X_{[z]}$ have interpolation properties: Here is an obvious special case of an interpolation theorem which is relevant to our purposes:

PROPOSITION 5. *If $U$: $\ell^\infty \to \ell^\infty$ is a bounded operator which also maps $X_\sigma$ into $\ell^\infty$ with norm not exceeding* 1 *for each $\sigma \in \partial\Omega$, then it follows that $U$: $X_{[z]} \to \ell^\infty$ with norm not exceeding* 1.

For our purposes $U$ is of course the identity operator and to answer Question 2 corresponds to obtaining a version of the Lions-Peetre lemma (cf. [LP], [CP], [M] etc.) in the context of Proposition 5, i.e. if $U$: $X_\sigma \to \ell^\infty$ is compact for each $\sigma$ on the unit circle, then we must show that $U$: $X_{[z]} \to \ell^\infty$ is compact. To do this it may perhaps be more convenient to work with a real method analogue of $X_{[z]}$ which contains $X_{[z]}$ continously (cf. [CwJ]).

Finally let us give a partial positive result.

PROPOSITION 6. *Question 2 does have a positive answer in the case where each $M_\sigma$ is contained in $c_0$ for each $\sigma \in E$, where $E$ is a subset of the unit circle having positive arc length measure.*

*Proof.* In this case there exist numbers $0 \le w(\sigma, n) \le 1$ for each $n \in \mathbb{N}$ and each $\sigma \in \partial\Omega$ such that $\lim_{n\to\infty} w(\sigma, n) = 0$ for each fixed $\sigma \in E$ and $|x_n| \le w(\sigma, n)$ for each $x = \{x_n\}$ in $M_\sigma$. Note that we do not require the functions $w(\sigma, n)$ to be measurable functions of $\sigma$.

Given any sequence $\{x_k\} = \{\{x_{kn}\}\}$ in $M_{[e^\theta]}$ choose functions $\phi_k = \{\phi_{kn}\}_{n\in\mathbb{N}} \in \mathcal{M}$ such that $\phi_k(e^\theta) = x_k$. The functions $v(n, \sigma) = \sup_k |\phi_{kn}(\sigma)|$ are measurable functions of $\sigma$ and of course satisfy $0 \le v(n, \sigma) \le w(\sigma, n)$. For each $k$ and $n$ we have

$$(**) \qquad |x_{kn}| = |\phi_{kn}(e^\theta)| \le \exp\left(\int_{\partial\Omega} \log|\phi_{kn}(\sigma)| P_\theta(\sigma)\, d\sigma\right)$$

where $P_\theta(\sigma)$ is a strictly positive integrable function which is an appropriate kernel for $\Omega$. (To be more explicit about $P_\theta(\sigma)$ observe that we can, for instance, apply formula (i) of [Ca], Section 9.4, p. 177 to the function $f(z) = \phi_{kn}(e^z)$ with $B_0 = B_1 = \mathbb{C}$ (even though $f$ does not vanish at infinity). It follows immediately that we can take $P_\theta(\sigma)d\sigma = \sum_{k\in\mathbb{Z}} \mu_j(\theta, t + 2\pi k)dt$ where $\sigma = e^{j+it}$, $j = 0, 1$, $t \in [0, 2\pi]$ and $\mu_j(\theta, t)$ are the Poisson kernels defined in [Ca]. )

From $(**)$ we deduce that

$$|x_{kn}| \le v_n := \exp\left(\int_{\partial\Omega} \log v(n,\sigma) P_\theta(\sigma)\, d\sigma\right) \le \exp\left(\int_E \log v(n,\sigma) P_\theta(\sigma)\, d\sigma\right).$$

We shall show that $\lim_{n\to\infty} v_n = 0$ and this will imply that $\{x_k\}$ has a convergent subsequence in $\ell^\infty$ as required. For each positive integer $N$, let $h(N, n, \sigma) = \max\{-N, \log v(n, \sigma)\}$. Then $-N \le h(N, n, \sigma) \le 0$ and $\lim_{n\to\infty} h(N, n, \sigma) = -N$ for each $\sigma \in$ E. Thus,

$$\limsup_{n\to\infty} v_n \le \limsup_{n\to\infty} \exp\left(\int_E h(N, n, \sigma) P_\theta(\sigma)\, d\sigma\right)$$

and, by dominated convergence, this last limit equals $\exp(-N \int_E P_\theta(\sigma)\, d\sigma)$. Since $N$ is arbitrary and $E$ has positive measure we have completed the proof. □

*Remark.* We are grateful to Yuri Brudnyi for the very interesting observation that via the Banach-Mazur theorem one should be able to extend Proposition 6 to the case where, for all $\sigma$ on the unit circle, all the sets $M_\sigma$ are contained in some fixed separable subset of $\ell^\infty$.

*Acknowledgements.* We thank several of our colleagues, in particular, Svante Janson, Nigel Kalton and Richard Rochberg, for a number of stimulating discussions and e-mail exchanges about this problem over the past few years.

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