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ON COUNTEREXAMPLES TO KELLER'S PROBLEM

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1. Introduction

Let F: $K^n \to K^n$ be a polynomial map where $K = R$ or C. Let us denote by $J(F)$ the determinant of the Jacobian of F. The Jacobian conjecture [2,13] is the following statement:

If $J(F)$ never vanishes then the map F is injective.

Originally, the conjecture was stated for $K = C$ with polynomials over Z by O. Keller [5]. When $K = C$ the assumption can be rewritten as $J(F) \in C^*$ (by the Fundamental Theorem of Algebra) and the conclusion can be rewritten as: F is invertible in the ring $C[X_1, \dots, X_n]$ [3].

For $K = R$, injectivity of F implies its surjectivity [6,12] (a result that was generalized by A. Borel to real algebraic varieties).

The conjecture for $K = C$ is still open for $n \geq 2$.

The conjecture for $K = R$, the so called real Jacobian conjecture, was recently shown to be false by S. Pinchuk [11].

The main purpose of this paper is to give a proof of the fact that there is no counterexample to the complex Jacobian conjecture of the type constructed by S. Pinchuk for the real case (Theorem 5).

This proof depends on properties of the asymptotic values of polynomial maps. In Section 2 we will mention other results that are consequences of these properties and motivate further research of the asymptotic values of polynomial maps.

A detailed study of the asymptotic values of polynomial maps is given in [9,10].

2. Global diffeomorphisms and asymptotic identities

Let f: $R^n \to R^n$ be a map. A point $X_0 \in R^n$ is called an asymptotic value of f if there exists a curve $\sigma(t) = (X_1(t), \dots, X_n(t))$, $0 \leq t < \infty$, that extends to ∞ such that

$$\lim_{t\to\infty} f(\sigma(t)) = X_0.$$

$\sigma(t)$ is called an asymptotic curve of f.

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If $f = (f_1, \dots, f_n)$ and $X_0 = (X_{01}, \dots, X_{0n})$ then we say that X_{0j} is an asymptotic value of the function f_j, $j = 1, \dots, n$.

A well known theorem of J. Hadamard [4] gives a necessary and sufficient condition for a local diffeomorphism $f\colon R^n \to R^n$ to be a global diffeomorphism. One consequence of Hadamard's result is that in order to show that f is a global diffeomorphism it suffices to exclude the existence of asymptotic values for f.

Pinchuk's counterexample [11] is of a polynomial map (P, Q) which consists of two polynomials in the ring $R[t, h, f]$ where

$$t = XY - 1 \ , \ \ h = t(Xt+1) \ , \ \ f = (Xt+1)^2(h+1)/X.$$

It is arranged so that $J(P, Q)$ is a sum of three squares of real polynomials that have no common real zero and hence $J(P, Q)$ is positive on all of R^2. Let us define

$$u_1 = Y \ , \ \ u_2 = XY \ , \ \ u_3 = X^2Y - X;$$

then it is easy to check that Pinchuk's polynomials belong to the ring $R[u_1, u_2, u_3]$. To see the significance of this ring let us consider the birational map

$$R(X, Y) = (X^{-1}, X + X^2Y).$$

The ring $R[u_1, u_2, u_3]$ consists of all the polynomials $P(X, Y) \in R[X, Y]$ with the property

$$(P \circ R)(X, Y) = P(X^{-1}, X + X^2Y) = A(X, Y) \in R[X, Y]$$

(Theorem 1). We denote this ring by $I(R(X, Y))$. We say that the polynomials in $I(R)$ satisfy an asymptotic identity with respect to the rational map $R(X, Y)$.

If (P, Q) is any pair of polynomials in $I(R)$ then the map they induce satisfies the following double asymptotic identity

$$P(X^{-1}, X + X^2Y) = A(X, Y) \in R[X, Y],$$

$$Q(X^{-1}, X + X^2Y) = B(X, Y) \in R[X, Y].$$

The consequence of this is that

$$\lim_{X\to 0}(P, Q)(X^{-1}, X + X^2Y) = (A(0, Y), B(0, Y))$$

so that the points of the affine algebraic curve $(A(0, Y), B(0, Y))$ are all asymptotic values of the map (P, Q). Thus we conclude from this and from the Theorem of Hadamard that in order to construct a counterexample to the real Jacobian conjecture it suffices to find a Jacobian pair within $I(R(X, Y))$. This was demonstrated nicely in Pinchuk's work.

It is not difficult to construct pairs of polynomials in $I(X^{-1}, X + X^2Y)$ whose Jacobian hardly vanishes. For example

$$J(u_3 - u_1, (u_3 + u_1)(u_2 - 1/2)) = (u_3 + u_1)^2 + 4(u_2 - 1/2)^2$$

or equivalently

$$J(X^2Y - X - Y, (X^2Y - X + Y)(XY - 1/2)) = (X^2Y - X + Y)^2 + 4(XY - 1/2)^2.$$

This Jacobian vanishes at only two points, $(\pm 1, \pm 1/2)$, and the map is not etale at those points, so this does not give an example of an etale map which is not a global diffeomorphism (as in Pinchuk's construction).

However the above example as well as Pinchuk's construction can be based on the following fundamental relations among the generators of $I(X^{-1}, X + X^2Y)$. These relations are of two types. One type is algebraic and the second is differential.

ALGEBRAIC RELATIONS.

$$u_1u_3 = u_2(u_2 - 1).$$

DIFFERENTIAL RELATIONS.

$$\begin{aligned} J(u_1, u_2) &= -u_1, \\ J(u_1, u_3) &= 1 - 2u_2, \\ J(u_2, u_3) &= -u_3. \end{aligned}$$

To motivate further study of the properties of the asymptotic values of polynomial maps and of the related rings (of the type of the ring $I(R(X, Y))$ that was mentioned above) we will now mention results that emerge from that theory.

A well known theorem of S. S. Abhyankar [1] asserts the following:

> *Every Jacobian pair is an automorphic pair iff the Newton polygons of every Jacobian pair are triangles with vertices on the coordinate axes.*

With the aid of the properties of the asymptotic values of polynomial maps we are able to prove the following stronger statement (see [9]):

> *Every Jacobian pair is an automorphic pair iff the Jacobian condition implies that the Newton polygons have no edges of a positive slope.*

This theorem was proved by J. Lang [7] using other techniques. Another application is the following (see [9]):

> *The complex Jacobian conjecture in dimension 2 is true iff every Jacobian pair $P(U, V)$, $Q(U, V) \in C[U, V]$ induces a finite map $f = (P, Q)$.*

There are also geometric aspects of the theory. These are related to exotic surfaces. We are led to define a new type of exoticity [9]. We say that an affine surface S is *etale exotic* if and only if

(a) There is a diffeomorphism ϕ: $C^2 \to S$ which is realized by a birational map ϕ.
(b) There is no regular etale map $S \to C^2$ (into C^2).

An example of such a surface S_3 in C^3 is the surface which is parametrized by

$$X = V\ ,\ Y = VU\ ,\ Z = VU^2 + U.$$

Its affine closure is given by $XZ = Y(Y+1)$ which is isomorphic to $X^2+Y^2+Z^2 = 1$. In fact there is a whole family of such surfaces all of which are doubly ruled surfaces. A question which we can not answer is the following: Is any doubly ruled surface which is not isomorphic to C^2 an etale exotic surface? Clearly those surfaces are related to the rings of the type $I(R(X, Y))$ that was used in Pinchuk's construction.

Finally we mention that the structure of the variety of the asymptotic values of a real polynomial map is completely understood and is given in [10].

3. The ring $K[u_1, u_2, u_3]$ and its relations to the map $R(X, Y) = (X^{-1}, X + X^2Y)$

We now give the algebraic structure of $I(X^{-1}, X + X^2Y)$ as explained in the previous section.

THEOREM 1.

$$I(X^{-1}, X + X^2Y) = K[Y, XY, X^2Y - X] = K[u_1, u_2, u_3].$$

Proof. Let $P(X, Y) \in I(X^{-1}, X + X^2Y)$. Then by definition we have

$$P(X^{-1}, X + X^2Y) \in C[X, Y].$$

Let

$$P(U, V) = \sum a_{ij} U^i V^j,$$

$$P_+(U, V) = \sum_{0 \le i \le j} a_{ij} U^i V^j,$$

$$P_-(U, V) = \sum_{0 \le j < i} a_{ij} U^i V^j.$$

Then since we clearly have

$$\begin{aligned} P_+(X^{-1}, X+X^2Y) &= \sum_{0\le i\le j} a_{ij}X^{-i}(X+X^2Y)^j \\ &= \sum_{0\le i\le j} a_{ij}X^{j-i}(1+XY)^j \in C[X,Y] \end{aligned}$$

we deduce that $P_+(X,Y) \in I(X^{-1}, X+X^2Y)$ and since $P = P_+ + P_-$ we deduce that $P_-(X,Y) \in I(X^{-1}, X+X^2Y)$. We have

$$\begin{aligned} P_-(X^{-1}, X+X^2Y) &= \sum_{0\le j<i} a_{ij}X^{j-i}(1+XY)^j \\ &= \sum_{0\le j<i} a_{ij} \sum_{k=0}^{j} \binom{j}{k} X^{j+k-i}Y^k \end{aligned}$$

and this last sum should be a polynomial in (X,Y). So in the inner summation we can take only those values of k for which $0 \le j+k-i$, i.e., $i-j \le k$. Also since $k \le j$, the effective values of j must satisfy $i-j \le j$ or $i \le 2j < 2i$. Summarizing up all these detailes, we obtain

$$P_-(X^{-1}, X+X^2Y) = \sum_{0<i/2\le j<i} a_{ij} \sum_{k=i-j}^{j} \binom{j}{k} X^{j+k-i}Y^k.$$

We make the change of variables

$$U = X^{-1}\ ,\ V = X+X^2Y;$$

thus

$$X = U^{-1}\ ,\ Y = U^2V - U$$

and, substituting, we obtain

$$\begin{aligned} P_-(U,V) &= \sum_{0<i/2\le j<i} a_{ij} \sum_{k=i-j}^{j} \binom{j}{k} U^{-(j+k-i)}(U^2V-U)^k \\ &= \sum_{0<i/2\le j<i} a_{ij} \sum_{k=i-j}^{j} \binom{j}{k} U^{i-j}(UV-1)^k. \end{aligned}$$

All the above shows that $P_+(U,V)$ is generated by U^iV^j where $0 \le i \le j$, and $P_-(U,V)$ is generated by $U^l(UV-1)^k$ where $0 \le l \le k$. Hence both $P_+(U,V)$ and $P_-(U,V)$ are generated by (V, UV, U^2V-U) and since these generators belong to $I(U^{-1}, U^2V+U)$ we have proved our assertion. □

This theorem is a special case of the following:

THEOREM 2.

$$I(X^{-1}, A_1X + A_2X^2 + \cdots + A_{N-1}X^{N-1} + YX^N)$$
$$= K[Y, XY, X^2Y - A_1X, \ldots, X^NY - A_1X^{N-1} - \cdots - A_{N-1}X]$$

This is explained in [9] (we will not need that result here).

4. There is no counterexample to the case $K = C$ of the type constructed by Pinchuk

One might be tempted, following Pinchuk, to construct a counterexample to the complex Jacobian conjecture whose components are polynomials in $I(X^{-1}, X+X^2Y)$ (over C). Indeed if it were possible to find two polynomials $P(X, Y), Q(X, Y) \in I(X^{-1}, X+X^2Y)$ such that $J(P, Q) \equiv 1$ then this would have been a counterexample to Keller's problem. According to Moh [8] any such example must be of high degree (≥ 100). However, such an attempt will fail as the following theorems of this section will show.

As always, $K = R$ or C. According to Theorem 2, if $N \geq 2$, $A_1, \ldots, A_{N-1} \in K$ where at least one is different from zero then the set of all the polynomials $P(X, Y) \in K[X, Y]$ that satisfy the asymptotic identity

$$P(X^{-1}, A_1X + \cdots + A_{N-1}X^{N-1} + YX^N) = A(X, Y) \in K[X, Y]$$

denoted by $I(X^{-1}, A_1X + \cdots + A_{N-1}X^{N-1} + YX^N)$, equals the ring

$$K[Y, XY, X^2Y - A_1X, \ldots, X^NY - A_1X^{N-1} - \cdots - A_{N-1}X]$$

Indeed, it is clear that this last ring is contained in $I(X^{-1}, A_1X + \cdots + A_{N-1}X^{N-1} + YX^N)$, for all of its $N + 1$ generators satisfy the asymptotic identity. The opposite inclusion is more involved (and we will not need it in the sequel).

We do want to emphasize the following fact which was explained in Section 2 for the special case $N = 2$.

THEOREM 3. *If $P(X, Y), Q(X, Y) \in K[Y, XY, X^2Y - A_1X, \ldots, X^NY - A_1X^{N-1} - \cdots - A_{N-1}X]$ then the map (P, Q) is not a global diffeomorphism.*

Proof. Let us consider the birational map

$$R(X, Y) = (X^{-1}, A_1X + \cdots + A_{N-1}X^{N-1} + YX^N).$$

Let us assume that $F(X, Y) = (P(X, Y), Q(X, Y))$ is a global diffeomorphism. By the Theorem of Hadamard, F can not have asymptotic values. There are $A(X, Y)$, $B(X, Y) \in K[X, Y]$ such that

$$F(R(X, Y)) = (A(X, Y), B(X, Y)).$$

Hence

$$\lim_{X\to 0} F(X^{-1}, A_1X + \cdots + A_{N-1}X^{N-1} + YX^N) = (A(0, Y), B(0, Y)) \in K^2.$$

So the curve $(A(0, Y), B(0, Y))$ consists of asymptotic values of F which is a contradiction. □

REMARK 1. *There are polynomials*

$$P(X, Y), Q(X, Y) \in R[Y, XY, X^2Y - A_1X, \dots, X^NY - A_1X^{N-1} - \cdots - A_{N-1}X]$$

such that the map (P, Q) has a nonvanishing Jacobian. For after a regular linear change of variables we can assume that $A_1 = 1$, and then use the map of Pinchuk [11].

As opposed to the remark, the situation is very different over $K = C$. The proof of the next theorem is based on an idea of L. Makar-Limanov to whom I am very grateful.

THEOREM 4. *If $P(U, V), Q(U, V) \in C[V, VU, VU^2 + \alpha U, \dots, VU^N + \alpha U^{N-1}]$, where $\alpha \in C^*$, then the Jacobian $J(P, Q)$ must have a zero.*

Proof. We can assume after a regular linear change of variables that $\alpha = 1$. We shall use weighted grading on $C[U, V]$. The weights are choosed as follows

$$\deg U = -1\ ,\ \deg V = 1$$

Thus $\deg U^iV^j = j - i$ and for the generators of our ring we have the following facts:

$V, VU, VU^2 + U, \dots, VU^N + U^{N-1}$ are homogeneous (with respect to the weights).

$$\begin{aligned}\deg V &= 1, \deg(VU)\\ &= 0, \deg(VU^2 + U) = -1, \dots, \deg(VU^N + U^{N-1}) = -(N-1).\end{aligned}$$

Let $T = VU$. We will need to know the structure of homogeneous polynomials with respect to the chosen weights.

If $P_k(U, V) \in C[U, V]$ is homogeneous of degree $k \geq 0$ then there exists a polynomial $q(T) \in C[T]$ such that

$$P_k(U, V) = V^kq(T) \tag{1}$$

To see this we write

$$P_k(U,V) = \sum a_{ij}U^iV^j = \sum_{\deg(U^iV^j)=k} a_{ij}U^iV^j = \sum_{j-i=k} a_{ij}U^iV^j$$
$$= \sum a_{i(i+k)}U^iV^{i+k} = V^k \sum a_{i(i+k)}T^i$$

The structure of homogeneous polynomials of negative degree is a little more complicated. Here we shall confine ourselves to $C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$.

If $P_{-k}(U,V) \in C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$ is homogeneous of degree $-k<0$ then

$$P_{-k}(U,V) = \sum_{j_2+2j_3+\cdots+(N-1)j_N=k} (VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N}P_{j_2\cdots j_N}(T) \quad (2)$$

where $P_{j_2\cdots j_N}(T) \in C[T]$. To see this we write

$$\begin{aligned} P_{-k}(U,V) &= \sum_{i_0-i_2-2i_3-\cdots-(N-1)i_N=-k} a_{i_0\cdots i_N}V^{i_0}(VU)^{i_1}\cdots(VU^N+U^{N-1})^{i_N} \\ &= \sum_{i_0+k=i_2+\cdots+(N-1)i_N} a_{i_0\cdots i_N}V^{i_0}\cdots(VU^N+U^{N-1})^{i_N} \\ &= \sum [(VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N}] \\ &\quad \times[a_{i_0\cdots i_N}V^{i_0}(VU)^{i_1}(VU^2+U)^{i_2-j_2}\cdots(VU^N+U^{N-1})^{i_N-j_N}] \end{aligned}$$

where in the last sum the indices vary as follows:

$$i_0 = (i_2-j_2)+2(i_3-j_3)+\cdots+(N-1)(i_N-j_N),$$

$$i_2-j_2, \ldots, i_N-j_N \geq 0,$$

$$j_2+2j_3+\cdots+(N-1)j_N = k.$$

Since $V^{i_0}(VU)^{i_1}(VU^2+U)^{i_2-j_2}\cdots(VU^N+U^{N-1})^{i_N-j_N}$ belongs to $C[T]$, by $i_0 = (i_2-j_2)+\cdots+(N-1)(i_N-j_N)$, this proves (2).

We now consider the Jacobian of a pair of polynomials in $C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$ which are homogeneous of degrees k and $-k$ respectively, for some $k \geq 0$. In fact, we shall confine ourselves to those polynomials of negative degree that appear in the sum in equation (2).

If $k \geq 0$ and $P_k(U,V) = V^kf(T)$, $Q_{-k}(U,V) = (VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N}g(T)$ where $f(T), g(T) \in C[T]$ and $j_2+\cdots(N-1)j_N = k$ then

$$\partial(P_k, Q_{-k})/\partial(U,V) = -kd/dT\{T^k(T+1)^{j_2+\cdots+j_N}f(T)g(T)\}. \quad (3)$$

The verification of this equation is done by a straightforward computation. Namely, we have

$$\begin{aligned}
\partial P_k/\partial U &= V^{k+1}f'(T)\ ,\ \partial P_k/\partial V = kV^{k-1}f(T) + V^k U f'(T),\\
\partial Q_{-k}/\partial U &= (VU^2+U)^{j_2-1}\cdots(VU^N+U^{N-1})^{j_N-1}\\
&\quad \times\{j_2(VU^3+U^2)\cdots(VU^N+U^{N-1})(2UV+1)\\
&\quad +\cdots+ j_N(VU^2+U)\cdots(VU^{N-1}+U^{N-2})\\
&\quad \times(NVU^{N-1}+(N-1)U^{N-2})\}g(T)\\
&\quad +(VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N}Vg'(T),\\
\partial Q_{-k}/\partial V &= (VU^2+U)^{j_2-1}\cdots(VU^N+U^{N-1})^{j_N-1}\\
&\quad \times\{j_2(VU^3+u^2)\cdots(VU^N+U^{N-1})U^2\\
&\quad +\cdots+ j_N(VU^2+U)\cdots(VU^{N-1}+U^{N-2})U^N\}g(T)\\
&\quad +(VU^2+U)^{j_2}\cdots(VU^N+U^{N-1})^{j_N}Ug'(T).
\end{aligned}$$

Now on multiplying and subtracting in

$$\partial(P_k, Q_{-k})/\partial(U,V) = (\partial P_k/\partial U)(\partial Q_{-k}/\partial V) - (\partial P_k/\partial V)(\partial Q_{-k}/\partial U)$$

we find that the coefficient of $f'(T)g'(T)$ is zero while $f'(T)g(T)$ and $f(T)g'(T)$ have the common coefficient

$$-kT^k(T+1)^{j_2+\cdots+j_N}$$

and $f(T)g(T)$ has as its coefficient

$$-kd/dT\{T^k(T+1)^{j_2+\cdots+j_N}\}$$

where, as above, $k = j_2 + \cdots + (N-1)j_N$.

In order to prove the theorem we argue by contradiction. Suppose that $P(U,V)$, $Q(U,V) \in C[V, VU, VU^2+U, \ldots, VU^N+U^{N-1}]$ and $\partial(P,Q)/\partial(U,V) \equiv 1$. Let us represent P and Q according to our weighted grading:

$$P = \sum P_n\ ,\ Q = \sum Q_m,$$

where P_n and Q_m are homogeneous of degrees n and m respectively. Then

$$\partial(P,Q)/\partial(U,V) = \sum_{n,m} \partial(P_n, Q_m)/\partial(U,V).$$

However

$$\partial(P_n, Q_m)/\partial(U,V) = (\partial P_n/\partial U)(\partial Q_m/\partial V) - (\partial P_n/\partial V)(\partial Q_m/\partial U)$$

and degree calculations give

$$\deg\{(\partial P_n/\partial U)(\partial Q_m/\partial V)\} = (n+1)+(m-1) = n+m,$$

$$\deg\{(\partial P_n/\partial V)(\partial Q_m/\partial U)\} = (n-1)+(m+1) = n+m.$$

It follows that the homogeneous polynomial $\partial(P_n, Q_m)/\partial(U, V)$ is either identically zero or of degree $n+m$. Since we assumed that $\sum_{n,m}\partial(P_n, Q_m)/\partial(U, V) \equiv 1$ it follows that

$$\sum_k \partial(P_k, Q_{-k})/\partial(U, V) \equiv 1.$$

Hence by (1), (2) and (3) we obtain an equation of the form

$$\sum_k \left\{-k \sum_{j_2+\cdots+(N-1)j_N=k} d/dT[T^k(T+1)^{j_2+\cdots+j_N} f(T)g(T)]\right\} \equiv 1$$

where the polynomials $f(T)$ and $g(T)$ depend on the indices of the summation. Hence we deduce that

$$\sum_k \left\{-k \sum_{j_2+\cdots+(N-1)j_N=k} T^k(T+1)^{j_2+\cdots+j_N} f(T)g(T)\right\} = T+\lambda$$

for some $\lambda \in C$. We note that $T(T+1)$ divides the left hand side of the last equation and so $T(T+1) \mid (T+\lambda)$. This contradiction proves the theorem. □

An immediate consequence of the theorem is the following:

THEOREM 5. *There is no counter example to the complex Jacobian conjecture with coordinate polynomials in* $C[V, VU, VU^2+\alpha U, \ldots, VU^N+\alpha U^{N-1}]$, *where* $N \geq 2$, $\alpha \in C^*$.

Thus Pinchuk's construction is a "real" construction and can not be modified (in the above sense) to Keller's problem.

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