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EXTREMAL FUNCTIONS IN INVARIANT SUBSPACES OF BERGMAN SPACES

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1. Introduction

For $0 < p < \infty$, the Bergman space A^p consists of all functions f analytic in the unit disk $\mathbb{D}$ for which

$$\|f\|_p^p = \int_{\mathbb{D}} |f(z)|^p \, d\sigma < \infty,$$

where $d\sigma = \frac{1}{\pi} dA$ denotes the normalized element of area. A recent development in the theory of Bergman spaces was the construction of *contractive zero-divisors*, weak analogues of Blaschke products produced by an extremal problem analogous to one that leads to Blaschke products in the Hardy spaces H^p. Specifically if $\{\zeta_j\}$ is a given A^p zero-set with $\zeta_j \neq 0$ for all j, and N^p is the subspace of all functions f in A^p which vanish at least on $\{\zeta_j\}$ with the prescribed multiplicity or higher, then the *canonical divisor* G is the unique normalized solution to the extremal problem of maximizing $|f(0)|$ among all $f \in N^p$ with $\|f\|_p = 1$. It is known [6], [3], [4] that the canonical divisor has no extraneous zeros, that $\|Gf\|_p \geq \|f\|_p$ for all $f \in A^p$, and that $\|f/G\|_p \leq \|f\|_p$ for all $f \in N^p$. In the case of *finite* zero-sets, it was shown further [3] that G has the structure

$$G(z) = J(0,0)^{-1/p} B(z) J(z,0)^{2/p}, \tag{1}$$

where B is the finite Blaschke product with zeros ζ_j and $J(z, \zeta)$ is the reproducing kernel in the Bergman space A_w^2 with weight $w = |B|^p$. In particular, $J(z, 0) \neq 0$ in $\mathbb{D}$. Regularity properties of the kernel function J then implied that G has an analytic continuation to a larger disk.

Some of these results have been extended [4] to arbitrary invariant subspaces of A^p, not necessarily defined by zero-sets. Here an invariant subspace means a proper closed subspace of A^p that is invariant under multiplication by polynomials. Thus I is an invariant subspace if $Pf \in I$ for each $f \in I$ and every polynomial P. The invariant subspaces of A^p are much more complicated than those of H^p and have never been described completely. They need not be singly generated and they may have weird properties (see the recent paper by Hedenmalm [7]). The present paper

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will focus on singly generated subspaces $[f]$, defined as the closure of the set of polynomial multiples of a given function $f \in A^p$.

For an arbitrary invariant subspace $I \subset A^p$, the extremal problem that defined the canonical divisors is generalized to

$$\max_{f \in I, \|f\|_p = 1} |f(0)|. \tag{2}$$

Here it is assumed that $0 \notin Z(I)$, the common zero-set of functions in I. For $1 \le p < \infty$ it follows from the strict convexity of A^p that an extremal function exists and is unique under the normalization $f(0) > 0$. (See [4] for further details.) For general invariant subspaces I, neither the existence nor the uniqueness of the extremal function is clear for $0 < p < 1$; we consider this an open problem. For subspaces generated by inner functions, however, we are able to establish the existence and uniqueness of extremal functions in the case $0 < p < 1$.

Here is a summary of results in the paper. For any invariant subspace of A^p $(0 < p < \infty)$ generated by an inner function it is shown that the extremal function has a regularity property, a generalization of the result in [3] for subspaces defined by finite zero-sets. Along the way the structural formula in terms of kernel functions is generalized to this setting. The nonvanishing of the kernel function is found to be a phenomenon far more general than needed for this application. The regularity of the extremal function shows in particular that the canonical divisor of a Blaschke sequence extends analytically over each boundary arc containing no cluster points of the zeros. In this case it is shown further that $N^p = [B] = [G]$; thus the invariant subspace is generated both by the Blaschke product B and by the canonical divisor G.

The first version of this paper was written in January 1993 and was narrowly circulated. Since that time, three preprints [5], [8], [9] have appeared, treating similar questions from a rather different point of view. However, the overlap is small and the papers actually complement each other.

2. Regularity of extremal functions

Let $h = BS$ be an arbitrary inner function with $h(0) > 0$, where B is a Blaschke product and S is a singular inner function. (See [2] for instance, for terminology.) The *singular set* of h is the closed subset of the unit circle defined as the union of the cluster points of the zeros of B and the support of the singular measure of S. Let $I = [h]$ be the invariant subspace of A^p generated by h. Assume first that $1 \le p < \infty$, deferring the case $0 < p < 1$ to Section 3. Let F be the extremal function of problem (2) with $F(0) > 0$. We are going to show that F has an analytic continuation over each arc of the unit circle complementary to the singular set of h. Our main result (Theorem 2 below) is closely related to Theorem D in [8], but there is no inclusion.

As usual, the case $p = 2$ is simplest. Here we can give a direct argument that leads to a stronger form of the result. Thus we will first suppose that $p = 2$. The underlying idea is to show that $f = \int F$ is not a cyclic element for the backward shift operator $T^\star$ in H^2. From this it is known to follow [1] that f has a meromorphic pseudocontinuation to the exterior of the unit circle, denoted by $\mathbb{D}_e$. But to say that f is noncyclic for $T^\star$ is equivalent to saying that f is orthogonal to some invariant subspace of T. In fact, we will show that $f \perp z^2 h\varphi$ for every $\varphi \in H^2$. Having established that, we will be able to conclude directly, without reference to the backward shift, that f, and therefore F, has an analytic continuation to $\mathbb{D}_e$ except possibly for poles at the reflections of the zeros of h.

By a simple variational argument (cf. [3]), it can be shown in general that the extremal function F for the problem (2) has the orthogonality property

$$\int_{\mathbb{D}} |F|^{p-2}\overline{F}k\,d\sigma = 0, \qquad k \in I, \quad k(0) = 0. \tag{3}$$

For $p = 2$ and $I = [h]$, this reduces to

$$\int_{\mathbb{D}} \overline{F}h\psi\,d\sigma = 0, \qquad \psi \in A^2, \quad \psi(0) = 0. \tag{4}$$

Invoking the Cauchy-Green formula, we conclude that

$$\begin{aligned}\int_0^{2\pi} \overline{f(e^{i\theta})}e^{2i\theta}h(e^{i\theta})\varphi(e^{i\theta})\,d\theta &= -i\int_{\mathbb{T}} \overline{f(z)}zh(z)\varphi(z)\,dz \\ &= 2\pi\int_{\mathbb{D}} \frac{\partial}{\partial\bar{z}}\left\{\overline{f(z)}zh(z)\varphi(z)\right\}d\sigma \\ &= 2\pi\int_{\mathbb{D}} \overline{F(z)}zh(z)\varphi(z)\,d\sigma = 0\end{aligned}$$

for arbitrary $\varphi \in H^2$. In other words, $f \perp z^2h\varphi$ for $\varphi \in H^2$. But in view of the theorem of F. and M. Riesz (see [2], p. 41), this implies that $zh(z)\overline{f(z)} = g(z)$ almost everywhere on $\mathbb{T}$ for some $g \in H^2$. Since $|h(z)| = 1$ a.e., it is equivalent to write

$$f(z) = zh(z)\overline{g(z)} \qquad \text{a.e. on } \mathbb{T}. \tag{5}$$

Now observe that $zh(z)\overline{g(z)} = z\tilde{h}(z)\tilde{g}(z)$ a.e. on $\mathbb{T}$, where $\tilde{h}(z) = 1/\overline{h(1/\bar{z})}$ is meromorphic and $\tilde{g}(z) = \overline{g(1/\bar{z})}$ is analytic in $\mathbb{D}_e$. Thus $f(z) = \tilde{f}(z)$ a.e. on $\mathbb{T}$, where $\tilde{f}(z) = z\tilde{h}(z)\tilde{g}(z)$ is analytic in $\mathbb{D}_e$ except at the reflections of the zeros of h, and $\tilde{f}$ is locally in H^2 near each arc of the circle outside the singular set of h. By a general version of the analytic continuation principle, it follows that f has a single-valued extension across each such arc to a function analytic in $\mathbb{D}_e$ except perhaps for poles at the reflections of the zeros of h.

The general case $1 \le p < \infty$ is more difficult, and our final conclusion is weaker. However, our proof is based on a result of independent interest, a generalization of the structural formula (1) derived in [3] for canonical divisors of finite zero-sets. Given a continuous function $w(z) \ge 0$ in $\mathbb{D}$, let A^2_w denote the Hilbert space of analytic functions f such that

$$\|f\|_{2,w}^2 = \int_{\mathbb{D}} |f|^2 w \, d\sigma < \infty.$$

Let $\mathbb{P}_w^2$ be the closure of the polynomials in the norm of A_w^2. If w is bounded, it is the same to take the closure of the functions analytic in $\overline{\mathbb{D}}$. At present we are interested only in the weight function $w = |h|^p$, where h is the given inner function. Then, because the zeros of w are isolated, standard techniques show that each point-evaluation is a bounded linear functional, and that norm convergence implies uniform convergence on compact subsets of the disk. Thus $\mathbb{P}_w^2 \subset A_w^2$ and $\mathbb{P}_w^2$ has a reproducing kernel $J_\zeta(z) = J(z,\zeta)$, characterized by the properties $J_\zeta \in \mathbb{P}_w^2$ and

$$f(\zeta) = \int_{\mathbb{D}} f(z)\overline{J(z,\zeta)} w(z) \, d\sigma, \qquad f \in \mathbb{P}_w^2.$$

The kernel function has the symmetry property $J(\zeta, z) = \overline{J(z,\zeta)}$ and a scalar multiple of it uniquely maximizes $|f(\zeta)|$ up to rotation among all functions $f \in \mathbb{P}_w^2$ of unit norm.

THEOREM 1. *Let h be an arbitrary inner function with $h(0) > 0$. Let F be the normalized extremal function for problem* (2) *over the invariant subspace $I = [h]$ generated in A^p by the polynomial multiples of h, where $1 \le p < \infty$. Let $J(z,\zeta)$ be the reproducing kernel for the Hilbert space $\mathbb{P}_w^2$ with weight $w = |h|^p$. Then*

$$F(z) = J(0,0)^{-1/p} h(z) J(z,0)^{2/p}. \tag{6}$$

Moreover, $J(z,0) \neq 0$ in $\mathbb{D}$ and F has no extraneous zeros.

This result was established in [3] for the special case where h is a finite Blaschke product. The proof in [3] can be extended to the more general case, but we prefer to give a shorter, more direct proof that makes the relation (6) transparent.

Proof of theorem. The first step is to show that F has no extraneous zeros; its zero-set is precisely the same as that of h, multiplicities counted. Suppose on the contrary that $F(\alpha) = 0$ at some point $\alpha \in \mathbb{D}$ where $h(\alpha) \neq 0$, or that F has a zero at α of higher multiplicity than that of h. The canonical divisor G_α of the zero-set $\{\alpha\}$ was given explicitly in [3]. It is analytic in $\overline{\mathbb{D}}$, has a simple zero at α and no other zeros in $\overline{\mathbb{D}}$, and satisfies $G_\alpha(0) < 1$. Thus F/G_α is analytic in $\mathbb{D}$, and $\|F/G_\alpha\|_p \le \|F\|_p = 1$ by the contractive property of canonical divisors. Furthermore, $F/G_\alpha \in I$. To see this, observe first that for each $\varepsilon > 0$ there is a polynomial Q with $Q(\alpha) = 0$ such that $\|hQ - F\|_p < \varepsilon$. But $hQ/G_\alpha \in I$ and the contractive property gives

$$\|hQ/G_\alpha - F/G_\alpha\|_p \le \|hQ - F\|_p < \varepsilon.$$

Thus $F/G_\alpha \in I$, and the inequality $F(0)/G_\alpha(0) > F(0)$ contradicts the extremal property of F.

Having shown that F has no extraneous zeros, we can write $F = hk^{2/p}$, where $k \in A^2_w$ and $k(z) \neq 0$ in $\mathbb{D}$, with $k(0) > 0$. Recall now that by definition F maximizes $f(0)$ among all functions $f \in [h]$ with $\|f\|_p \leq 1$ and $f(0) > 0$. The definition of $[h]$ therefore shows that $k(0)^{2/p}$ is the supremum of $g(0)$ among all functions g analytic in $\overline{\mathbb{D}}$ with $g(0) > 0$ and $\|hg\|_p \leq 1$. In view of the contractive property of the canonical divisor and the fact that the canonical divisor of a finite zero-set is analytic in $\overline{\mathbb{D}}$ and does not vanish on the boundary (see [3]), it is easily seen that the supremum need be taken only over such functions g with $g(z) \neq 0$ in $\mathbb{D}$. Thus $k(0)$ is the supremum of $f(0)$ over all *nonvanishing* functions $f \in \mathbb{P}^2_w$ with $\|f\|_{2,w} \leq 1$ and $f(0) > 0$. The argument shows in particular that $k \in \mathbb{P}^2_w$.

On the other hand, the function $K(z) = J(0,0)^{-1/2} J(z,0)$ uniquely maximizes $f(0)$ among *all* functions $f \in \mathbb{P}^2_w$ of norm $\|f\|_{2,w} \leq 1$ with $f(0) > 0$. We claim that the same maximum value persists under the additional constraint that $f(z) \neq 0$ in $\mathbb{D}$, allowing us to conclude that $k = K$. Indeed, we will show in Section 4 (in greater generality) that $J(z,0) \neq 0$ in $\mathbb{D}$. Since the extremal function is nonvanishing, it is clear that nothing changes if this additional constraint is imposed. Thus $k = K$, and the derivation of formula (6) is complete except for the proof that $J(z,0) \neq 0$. This will be given in Theorem 3.

Having established the structural formula (6), we are now ready to show that F has an analytic continuation.

THEOREM 2. *For $1 \leq p < \infty$, let F be the normalized extremal function in the invariant subspace $I = [h]$ generated by an inner function h with $h(0) > 0$. Then F has an analytic continuation across each arc of the unit circle complementary to the singular set of h. For $p = 2$ the extremal function extends meromorphically to the full exterior of the unit disk and is single-valued and analytic except perhaps for poles at the points $1/\overline{\zeta_j}$ symmetric to the zeros ζ_j of h.*

Proof. The special case $p = 2$ was treated above. For $1 \leq p < \infty$ the proof exploits the representation (6) of F in terms of the kernel function J. Since h has an analytic continuation across each arc complementary to its singular set, and $|h(z)| = 1$ on the boundary, it is known that $J(z,0)$ also extends analytically across each such "regular" arc (see [3], Lemma 9). From formula (6) it follows that F has a continuous extension to each regular arc. But it was shown in [4] that F is an expansive multiplier: $\|FQ\|_p \geq \|Q\|_p$ for every polynomial Q. Thus Hedenmalm's "peaking function" argument (see [3], Lemma 5) can be applied to show that $|F(z)| \geq 1$ on each regular arc. In particular, the extended function $J(z,0)$ has no zeros near the arc, so $J(z,0)^{2/p}$ remains analytic. This shows that F also has an analytic continuation across each regular arc of the circle.

For the special case where h is a singular inner function whose singular measure μ is supported on finitely many points of the circle, it is shown by Hedenmalm, Korenblum, and Zhu [8] that $J(z,0)$ is a rational function with simple poles on the support of μ and no other poles in the extended complex plane.

3. Further applications of the structural formula

The representation (6) of the extremal function in terms of the kernel function can be extended to the range $0 < p < 1$. Here the existence and uniqueness of the extremal function can not be asserted *a priori*, but it is shown that the kernel function $J(z,0)$ is uniquely determined by the associated extremal problem in the space $\mathbb{P}^2_w$. Thus the proof of Theorem 1 can be adapted to establish the existence and uniqueness of the extremal function F even for $0 < p < 1$, and to extend the structural formula (6) to that case. The analytic continuation property (Theorem 2) then generalizes at once to $0 < p < 1$.

It may also be noted that formula (6) gives an effective way to calculate the kernel function whenever the extremal function is known. Here are two examples.

First let

$$h(z) = S(z) = \exp\left\{-\frac{1+z}{1-z}\right\},$$

an atomic singular inner function. For $0 < p < \infty$, the normalized extremal function for $[S]$ in A^p was found in [4] to be

$$F(z) = (1+p)^{-1/p} S(z) \left(1 + \frac{p}{1-z}\right)^{2/p}.$$

Combining this with (6), we conclude that

$$J(z,0) = 1 + \frac{p}{1-z}.$$

Hansbo [5] has given a more general formula for two mass-points.

As a second example, let h be a singular inner function S whose associated singular measure μ puts no mass on any Carleson thin set. It is known (see [11]) that these are precisely the cyclic inner functions in A^p; their polynomial multiples generate the whole space. Thus $[S] = A^p$ and the extremal function is trivial: $F(z) \equiv 1$. It now follows from (6) that

$$J(z,0) = e^{p/2} S(z)^{-p/2} \qquad \text{if } \|\mu\| = 1.$$

Finally, formula (6) displays a simple relation between certain extremal functions. If F is the extremal function for $[h]$ in A^p and G is the extremal function for $[h^m]$ in A^q, where $p = mq$ and m is a positive integer, then $G = F^m$. This result also follows directly from the definition of an extremal function (*cf.* [3], Lemma 8).

4. Nonvanishing of kernel functions

In proving Theorems 2 and 3 we needed to know that the kernel function $J(z, 0)$ of the space $\mathbb{P}^2_w$ does not vanish anywhere in the unit disk. This is actually true for a wide class of weight functions, as the following theorem asserts.

THEOREM 3. *Let $w(z) \geq 0$ be a bounded continuous function on $\mathbb{D}$. Suppose that the zeros (if any) of w are isolated, and that $\log w$ is subharmonic in $\mathbb{D}$. Let $J(z, \zeta)$ be the kernel function of the Hilbert space $\mathbb{P}^2_w$ defined as the closure of the polynomials in the norm $\|\cdot\|_{2,w}$. Then $J(z, 0) \neq 0$ for all z in $\mathbb{D}$.*

Proof. First note that $J(0,0) = \|J(\cdot, 0)\|^2_{2,w} > 0$. Define $K(z) = J(0,0)^{-1/2} J(z, 0)$, so that $\|K\|_{2,w} = 1$. Suppose on the contrary that $K(\alpha) = 0$ for some $\alpha \in \mathbb{D}$. Then $\tilde{K} = K/G_\alpha \in \mathbb{P}^2_w$, where G_α is the canonical divisor of $\{\alpha\}$ in A^2. According to an integral formula established in [4] (see remark at the end of Section 3 in [4]), the inequality

$$\int_{\mathbb{D}} |G_\alpha|^2 \varphi \, d\sigma \geq \int_{\mathbb{D}} \varphi \, d\sigma \tag{7}$$

holds for all bounded subharmonic functions φ. Since G_α is analytic in $\overline{\mathbb{D}}$, a simple limiting argument allows us to extend (7) to all integrable subharmonic functions. Take $\varphi = w|\tilde{K}|^2$ and observe that $\log \varphi$ is subharmonic because of the hypothesis that $\log w$ is subharmonic. Thus φ is subharmonic, and (7) gives $\|\tilde{K}\|_{2,w} \leq \|K\|_{2,w} = 1$. On the other hand, $G_\alpha(0) < 1$ and so $\tilde{K}(0) > K(0)$, which violates the extremal property of the kernel function. This contradiction shows that $J(z, 0)$ cannot vanish anywhere in $\mathbb{D}$.

The nonvanishing of the kernel function was first found in a special case by Håkan Hedenmalm (private communication), and the main idea of the above proof is his. Our integral formula allows a stronger result to be obtained by his method.

5. Subspaces generated by Blaschke products

For $0 < p < \infty$, let $\{\zeta_j\}$ be an A^p zero-set with $\zeta_j \neq 0$ for $j = 1, 2, \ldots$. Again let $N^p \subset A^p$ be the subspace of functions vanishing on $\{\zeta_j\}$, as defined in the introduction. Let G be the normalized extremal function of the invariant subspace N^p, also known as the canonical divisor of $\{\zeta_j\}$. Is it true that $[G] = N^p$? In other words, are the polynomial multiples of G dense in N^p? For *Blaschke sequences* $\{\zeta_j\}$, defined by the requirement that $\sum(1 - |\zeta_j|) < \infty$, we can give an affirmative answer.

THEOREM 4. *Let $\{\zeta_j\}$ be a Blaschke sequence with $\zeta_j \neq 0$, and let $B(z)$ be the corresponding Blaschke product. For $0 < p < \infty$, let N^p be the invariant subspace of functions in A^p which vanish on $\{\zeta_j\}$, and let G be the canonical divisor of $\{\zeta_j\}$ in A^p. Then $[B] = [G] = N^p$; in other words, both B and G are generators of N^p.*

Proof. The fact that $[B] = N^p$ was proved in [10], but because that publication was not widely circulated the proof will be repeated here. We have to show that each $f \in N^p$ can be approximated by polynomial multiples of B. Write $B = B_n R_n$, where B_n is the partial product consisting of the first n factors of B. Then $R_n(z) \to 1$ for each z in $\mathbb{D}$, while $\|R_n\|_\infty = 1$. This shows that $\|R_n f - f\|_p \to 0$ for each $f \in A^p$, by the Lebesgue dominated convergence theorem. Given $\varepsilon > 0$, choose n so large that $\|R_n f - f\|_p < \frac{\varepsilon}{2}$. Because $f/B_n \in A^p$, there is a polynomial Q such that $\|Q - f/B_n\|_p < \frac{\varepsilon}{2}$. But obviously $\|Bg\|_p \le \|g\|_p$ for each $g \in A^p$, since $|B(z)| \le 1$ in $\mathbb{D}$. Thus $\|BQ - R_n f\|_p < \frac{\varepsilon}{2}$, and the triangle inequality gives $\|BQ - f\|_p < \varepsilon$ if $1 \le p < \infty$. For $0 < p < 1$ the triangle inequality is not strictly valid and must be replaced by the inequality $\|f + g\|_p^p \le \|f\|_p^p + \|g\|_p^p$.

To prove $[G] = N^p$ it is enough to show $N^p \subset [G]$, because the reverse inclusion follows trivially from the fact that $G \in N^p$. But we know that $N^p = [B]$, so it is enough to show $B \in [G]$. Again let B_n denote a partial product of B and let G_n be the extremal function of $[B_n]$, or the canonical divisor of $\{\zeta_1, \dots, \zeta_n\}$. It was shown in [3] (as a special case of Theorems 1 and 2 above) that $G_n = B_n H_n$, where H_n is analytic and nonvanishing in $\overline{\mathbb{D}}$. Furthermore, $|H_n(z)| \ge 1$ on $\mathbb{T}$, since $|G_n(z)| \ge 1$ and $|B_n(z)| \equiv 1$ there. Thus by the maximum modulus principle, $|1/H_n(z)| \le 1$ in $\mathbb{D}$. But it was also shown in [3] that $G_n \to G$ in A^p norm as $n \to \infty$. It follows that $G = BH$ for some nonvanishing function H with $|1/H(z)| \le 1$ in $\mathbb{D}$. This implies that $B \in [G]$, and the proof is complete.

COROLLARY. *For $0 < p < \infty$ the canonical divisor in A^p of a Blaschke sequence has an analytic continuation across each arc of the unit circle that contains no cluster points of the zeros.*

Proof. Theorem 4 says that $[B] = N^p$, so the canonical divisor is the extremal function of $[B]$. But by Theorem 2, the extremal function of $[B]$ has an analytic continuation across each regular arc of the boundary, free of cluster points of the zeros.

The corollary gives a partial answer to a question raised in [3]. For $p \neq 2$ we still do not know whether the canonical divisor of a general A^p zero-set has an analytic continuation over each regular arc.

In the course of the proof of Theorem 4, it was observed that the canonical divisor of a Blaschke sequence is the quotient of a Blaschke product and a bounded analytic function. In particular, G belongs to the Nevanlinna class N (see [2]). What can be said more geneally about the extremal function F of an invariant subspace $[h]$ generated by an inner function? Is it always true that $F \in N$? (In their recent preprint [8], Hedenmalm, Korenblum, and Zhu have given an affirmative answer. In fact, their Theorem 3.3 implies our Theorem 4. Our proof is more direct, but their result is more general.) For which inner functions h does F belong to the Smirnov class N^+? For which h does the kernel function $J(z, 0)$ lie in N^+? Is it true that

$J \in N^+$ or at least that $F \in N^+$ when h is a Blaschke product? What can be said for *interpolating* Blaschke products?

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Addendum. After this paper was accepted for publication, a preprint by A. Aleman, S. Richter, and C. Sundberg appeared with the title "Beurling's theorem for the Bergman space." Among other results, these authors generalize Theorem 4 above by showing that the canonical divisor of *any* A^p zero-set generates the corresponding invariant subspace N^p.

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