SOME PROPERTIES OF TOTALLY COREGULAR MATRICES

BY
B. E. RHOADES

1. This paper is an extension of some of the results obtained by Hurwitz [4] to totally coregular matrices, according to Definition 1 listed below. In order to make this paper somewhat self-contained, the following definitions have been included.

Let $A = (a_{nk})$ denote an infinite matrix. The norm of A, written $\| A \|$, is defined as $\sup_n \sum_k | a_{nk} |$. If A has finite norm, $a_k = \lim_{n\to\infty} a_{nk}$ exists for each k, and $t = \lim_{n\to\infty} \sum_k a_{nk}$ exists, then A is called *conservative*. Associated with each conservative matrix A is a number $\chi(A)$ defined by

$$\chi(A) = t - \sum_k a_k .$$

When only one matrix is being considered, I shall simply write χ. A conservative matrix A is said to be *coregular* if $\chi \neq 0$. For a matrix A and a sequence x, I shall write $A_n(x) = \sum_k a_{nk} x_k$, and $Ax = \{A_n(x)\}$, considered as a column matrix. A matrix A is said to be *triangular* if $a_{nk} = 0$ for $k > n$, and a *triangle* if A is triangular and $a_{nn} \neq 0$ for each n. A is said to be *row-finite* if each row of A contains only a finite number of nonzero elements, and A is called *multiplicative* if A is conservative and $a_k = 0$ for each k.

DEFINITION 1. *Let A be a coregular matrix. Then A is said to be totally coregular if, for any sequence x with $x_n \to +\infty$, $A_n(x) \to +\infty$.*

Throughout this paper the matrices and sequences discussed are real.

2. The theorems in this section are concerned with row-finite coregular matrices.

THEOREM 1. *Let A be a coregular matrix. Then a sufficient condition for A to be totally coregular is that*

($*$) *there exists an integer q such that a_{nk} is nonnegative for all $k \geqq q$.*

Condition ($*$) states that, except possibly for those elements in the first $q - 1$ columns, all of the elements of the matrix are nonnegative.

Before proving the theorem, I give a proof of the following lemma.

LEMMA 1. *If A is a coregular matrix satisfying ($*$), then $\chi > 0$.*

Proof. From the definition of χ,

$$\chi = \lim_n \sum_{k=q}^{\infty} a_{nk} - \sum_{k=q}^{\infty} a_k \geqq \sum_{k=q}^{\infty} \lim_n a_{nk} - \sum_{k=q}^{\infty} a_k = 0,$$

Received March 7, 1959; received in revised form May 23, 1960.

where q is any integer satisfying $(*)$. But A is coregular. Therefore χ is positive.

To prove the theorem, let x be a sequence with $x_k \to +\infty$. Then there exists a positive integer N such that $x_k \geqq 0$ for all $k > N$. By hypothesis, there exists an integer q such that $a_{nk} \geqq 0$ for all $k > q$. Let $p = \max(N, q)$.

For any $M > 0$ there exists an integer $r > p$ such that $k > r$ implies $x_k > M$.

The remainder of the proof follows from the inequality

$$\begin{aligned} A_n(x) &\geqq \textstyle\sum_{k=1}^{r} a_{nk}\, x_k + M \sum_{k=r+1}^{\infty} a_{nk} \\ &= \textstyle\sum_{k=1}^{r} a_{nk}\, x_k + M(\sum_{k=r+1}^{\infty} a_{nk} - \sum_{k=r+1}^{\infty} a_k) + M \sum_{k=r+1}^{\infty} a_k \,. \end{aligned}$$

The remainder of the proof is basically the same as that of the sufficiency of [4, Theorem I].

To show that (i) A row-finite, (ii) A coregular, and (iii) $\chi > 0$ are not sufficient to imply $(*)$, consider the Hausdorff matrix method $H \sim \mu$ with $\mu_n = 2(n+1)/(n+2)$. $H \sim \mu$ is regular and hence satisfies (i) to (iii) with $\chi = 1$. But μ_n can be written as $2 - 2/(n+2)$. Therefore

$$\Delta\mu_n = -\Delta\lambda_n \,,$$

where $\lambda_n = 2/(n+2)$ and

$$\Delta^n \mu_k = -\frac{n!}{2^n} \prod_{i=0}^{n} \lambda_{k+i} \,, \qquad n = 1, 2, 3, \cdots; \quad k = 0, 1, 2, \cdots .$$

Hence all of the terms of the matrix H_μ are negative except those on the main diagonal. Therefore $(*)$ is not satisfied.

THEOREM 2. *If A is row-finite and has infinitely many columns with negative entries, then there is a sequence x such that $x_n \to \infty$ and $\liminf_{n\to\infty} A_n(x) < \infty$.*

Proof. Because of the hypotheses, there are integers n_r and k_r $(r = 1, 2, 3, \cdots)$ such that $n_1 < n_2 < \cdots$, $a_{n_r k_r} < 0$, and $k_r \leqq N_r < k_{r+1}$, where $N_0 = 0$ and otherwise N_r is the largest k for which $a_{n_r k} \neq 0$. If $x_k = r$ when $N_{r-1} < k \leqq N_r$ and $k \neq k_r$, and

$$x_{k_r} = \max\left\{ r,\ -\frac{1}{a_{n_r k_r}} \sum_{j=1, j\neq k_r}^{N_r} a_{n_r j}\, x_j \right\}$$

when $k = k_r$, then $x_k \geqq r$ for $N_{r-1} < k \leqq N_r$. Therefore, $x_k \to \infty$ as $k \to \infty$. On the other hand, $A_{n_r}(x) \leqq 0$, and hence $\liminf A_n(x) \leqq 0$.

THEOREM 3. *If A is a row-finite and coregular matrix, then a necessary and sufficient condition for A to be totally coregular is that A satisfy condition $(*)$.*

Proof. The sufficiency is Theorem 1, and the necessity is included in Theorem 2.

For the remainder of this paper, let $l = \liminf x_n$, $L = \limsup x_n$.

We now consider restrictions on A, other than coregularity, which will give us inequalities for $\liminf A_n(x)$ and $\limsup A_n(x)$ analogous to the familiar one $l \leqq \liminf A_n(x) \leqq \limsup A_n(x) \leqq L$ for a totally regular matrix A.

If x is a convergent sequence, it turns out that no additional restrictions on A are required; in fact,

$$\lim A_n(x) = \sum_{k=1}^{\infty} a_k x_k + \chi \lim x_n .$$

In the next few theorems, we shall consider the more general case when l is not necessarily equal to L.

THEOREM 4. *Let A be a matrix for which χ is defined. Then* $(*)$ *is a sufficient condition for*

$$\text{(A)} \qquad \liminf A_n(x) \geqq \sum_{k=1}^{\infty} a_k x_k + \chi l$$

and

$$\text{(B)} \qquad \limsup A_n(x) \leqq \sum_{k=1}^{\infty} a_k x_k + \chi L$$

whenever the series $\sum a_k x_k$ is convergent.

(If $l = -\infty$, then (A) *is true without* $(*)$, *provided that $\chi > 0$, and similarly for* (B) *when $L = \infty$.)*

Proof of (A). Assume $l > -\infty$. To prove $(*)$ sufficient, fix $\varepsilon > 0$. Then there exists an integer N such that $x_k \geqq l - \varepsilon$ and $a_{nk} \geqq 0$ for all $k \geqq N$. If $r \geqq N$, then

$$A_n(x) = (l - \varepsilon) \sum_{k=1}^{\infty} a_{nk} + \sum_{k=1}^{r} a_{nk}(x_k - l + \varepsilon) + \sum_{k=r+1}^{\infty} a_{nk}(x_k - l + \varepsilon).$$

The third series is nonnegative, and

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = t,$$

$$\lim_{n\to\infty} \sum_{k=1}^{r} a_{nk}(x_k - l + \varepsilon) = \sum_{k=1}^{r} a_k(x_k - l + \varepsilon).$$

Therefore,

$$\liminf A_n(x) \geqq (l - \varepsilon)\left(t - \sum_{k=1}^{r} a_k\right) + \sum_{k=1}^{r} a_k x_k .$$

Since $r \geqq N$ is otherwise arbitrary,

$$\liminf A_n(x) \geqq (l - \varepsilon)\chi + \sum_{k=1}^{\infty} a_k x_k ,$$

and (A) follows.

(B) is obtained from (A) by considering $-x$ instead of x.

We shall say that a matrix A has *property q* if there exists an integer q such that $a_k = 0$ for $k \geqq q$.

THEOREM 5. *If A is a coregular triangular matrix with property q, then* $(*)$ *is a necessary and sufficient condition for* (A) *and* (B).

Proof. The sufficiency is included in Theorem 4. To prove the necessity of $(*)$, use the necessity proof of [4, Theorem II′], selecting $k_1 > q$.

There is a class of matrices for which (A) and (B) are true for bounded sequences, but which do not necessarily satisfy (*). The following two theorems consider this class.

THEOREM 6. *Let A be a matrix for which χ is defined. Then*

$$(**) \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} | a_{nk} | = t$$

is a sufficient condition for (A) *and* (B) *to hold for all bounded sequences x for which $\sum a_k x_k$ is convergent* (*in particular, for all bounded sequences x if A is conservative*).

Proof. If $b_{nk} = (| a_{nk} | + a_{nk})/2$ and $c_{nk} = (| a_{nk} | - a_{nk})/2$, then

$$a_{nk} = b_{nk} - c_{nk} .$$

By hypothesis,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{nk} = t \quad \text{and} \quad \lim_{n\to\infty} \sum_{k=1}^{\infty} c_{nk} = 0.$$

Since x is bounded, there exists a number $X > 0$ such that $| x_k | < X$ for all k. Fix $\varepsilon > 0$. There exist integers M, $N > q$ such that

$$l - \varepsilon \leqq x_k \leqq L + \varepsilon$$

for all $k > N$ and, for $n > M$,

$$\sum_{k=1}^{\infty} c_{nk} < \varepsilon/(X + m + \varepsilon),$$

where $m = \max (| l | , | L |)$. Let $r > \max (M, N)$.

To prove (A), for all $n > r$,

$$A_n(x) = (l - \varepsilon)\sum_{k=1}^{\infty} a_{nk} + \sum_{k=1}^{r} a_{nk}(x_k - l + \varepsilon) + \sum_{k=r+1}^{\infty} b_{nk}(x_k - l + \varepsilon) - \sum_{k=r+1}^{\infty} c_{nk}(x_k - l + \varepsilon).$$

The third quantity on the right is nonnegative, and

$$\sum_{k=r+1}^{\infty} c_{nk}(x_k - l + \varepsilon) < (X + | l | + \varepsilon)\sum_{k=r+1}^{\infty} c_{nk} < \varepsilon,$$

$$\lim_{n\to\infty} \sum_{k=1}^{r} a_{nk}(x_k - l + \varepsilon) = \sum_{k=1}^{r} a_k x_k - (l - \varepsilon)\sum_{k=1}^{r} a_k .$$

Therefore,

$$\liminf A_n(x) \geqq (l - \varepsilon)\left(t - \sum_{k=1}^{r} a_k\right) + \sum_{k=1}^{r} a_k x_k - \varepsilon$$

for each $r > \max (M, N)$. Letting $r \to \infty$ and then $\varepsilon \to 0$, we obtain (A).

The proof of (B) is similar to that of (A) and has thus been omitted.

THEOREM 7. *Let A be a coregular triangular multiplicative matrix. Then* (**) *is a necessary and sufficient condition for*

$$\limsup A_n(x) \leqq \chi L \quad \textit{and} \quad \liminf A_n(x) \geqq \chi l$$

for each bounded sequence x.

Proof. To show that (**) is necessary, suppose that there exists a number $\lambda > 0$ such that, repeatedly, $\sum_{k=1}^{n} |a_{nk}| > |t|(1 + 3\lambda)$. (For any conservative matrix A,

$$\liminf_{n\to\infty} \sum_{k=1}^{n} |a_{nk}| \geqq |\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}| = |t|.)$$

Choose n_1 so that $\sum_{k=1}^{n} |a_{n_1k}| > |t|(1 + 3\lambda)$. Since A is multiplicative, choose an integer n_2 so that

$$\sum_{k=1}^{n_1} |a_{n_2k}| < |t|\lambda, \qquad \sum_{k=1}^{n_2} |a_{n_2k}| > |t|(1 + 3\lambda),$$

and, generally, n_p so that

$$\sum_{k=1}^{n_{p-1}} |a_{n_pk}| < \lambda|t|, \qquad \sum_{k=1}^{n_p} |a_{n_pk}| > |t|(1 + 3\lambda).$$

Define $x_k = (-1)^p \operatorname{sgn} a_{n_pk}$ $(n_{p-1} < k \leqq n_p)$, where $n_0 = 0$. Then

$$A_{n_p}(x) = \sum_{k=1}^{n_p} a_{n_pk}\, x_k = \sum_{k=1}^{n_{p-1}} a_{n_pk}\, x_k + (-1)^p \sum_{k=n_{p-1}+1}^{n_p} |a_{n_pk}|.$$

$$(-1)^p A_{n_p}(x) = (-1)^p \sum_{k=1}^{n_{p-1}} a_{n_pk}\, x_k - \sum_{k=1}^{n_{p-1}} |a_{n_pk}| + \sum_{k=1}^{n_p} |a_{n_pk}|$$
$$> -2\lambda|t| + |t|(1 + 3\lambda) = |t|(1 + \lambda).$$

For p even, $A_{n_p}(x) > |t|(1 + \lambda) \geqq t(1 + \lambda)$, and for p odd

$$A_{n_p}(x) = -|t|(1 + \lambda) \leqq -t(1 + \lambda).$$

Therefore $\limsup A_n(x) > t$ and $\liminf A_n(x) < -t$.

Since the sequence x is defined in terms of the sign of a_{n_pk} there are two possibilities: (i) x is a sequence containing an infinite number of 1's and -1's, or (ii) all but a finite number of the x_k are of the same sign. If (i) is true, then $l = -1$, $L = 1$, and (A) and (B) do not hold. If (ii) is true, then x is a convergent sequence, and the above discussion shows that A is not conservative and hence not coregular, since Ax is not convergent.

The sufficiency follows from Theorem 6.

That the condition "multiplicative" in the necessary part of Theorem 7 cannot be replaced by "A has property q" is shown by the following example.

Let $A = (a_{nk})$ be defined as follows: $a_{n1} = -1$, $a_{nn} = 1$ for $n > 1$, and $a_{nk} = 0$ otherwise. Then $t = 0$, $\| A \| = 2$, $a_1 = -1$, $a_k = 0$ for $k > 1$, and $\chi = 1$. Therefore A is coregular. For any bounded sequence x,

$$A_n(x) = -x_1 + x_n .$$

Therefore $\limsup A_n(x) = -x_1 + L$ and $\liminf A_n(x) = -x_1 + l$, (see Theorem 4 for the form of the A-limits), but $\sum_{k=1}^{n} |a_{nk}| = 2$ for all $n > 1$.

If $S = \{A \mid A \text{ satisfies } (*)\}$ and $T = \{A \mid A \text{ satisfies } (**)\}$, then $S \cap T$ contains all matrices A for which $a_{nk} \geqq 0$ for all n and k. However, S and

T are incompatible; in other words, $S \not\subset T$ and $T \not\subset S$. The example following Theorem 7 shows that $T \not\subset S$, and we now exhibit a matrix $B \in T - S$.

Let $B = (b_{nk})$ be defined as follows: $b_{nn} = 1$, $b_{n,n-1} = -1/n$, and $b_{nk} = 0$ otherwise. Then

$$\sum_{k=1}^{n} b_{nk} = -1/n + 1 \to 1 \quad \text{and} \quad \sum_{k=1}^{n} | b_{nk} | = 1/n + 1 \to 1.$$

Therefore $B \in T$, but $B \notin S$, since $b_{n,n-1} = -1/n$.

3. Most of the results in Section 2 have obvious analogues for sequence-to-function transformations.

4. We now apply the theorems of Section 2 to Hausdorff matrices. A Hausdorff matrix $H_\mu = (h_{nk})$ is generated by a sequence $\mu = \{\mu_n\}$ in the following manner: $h_{nk} = \Delta^{n-k}\mu_k$, $k \leqq n$, $h_{nk} = 0$ for $k > n$, where

$$\Delta\mu_k = \mu_k - \mu_{k+1}, \qquad \Delta^n \mu_k = \Delta(\Delta^{n-1}\mu_k) = \sum_{j=0}^{n} (-1)^j C_{n,j}\, \mu_{k+j}.$$

It is well known that H_μ having finite norm is a necessary and sufficient condition for H_μ to be conservative. (For a discussion of these and other properties of Hausdorff matrices see [2, Chapter XI] or [3].) Let **H** denote the set of Hausdorff matrices with finite norm, i.e., the set of conservative Hausdorff matrices. Then $t = \mu_0$, and if $h_k = \lim_{n\to\infty} h_{nk}$, $h_k = 0$ for all $k > 0$.

LEMMA 2. *Let $H \in \mathbf{H}$. If there exists an integer $r > 0$ such that $h_{nr} \geqq 0$ for all n, then $h_{nk} \geqq 0$ for all n and k for $0 < k \leqq r$, and $h_{n,0} \downarrow$.*

Proof. Assume $r > 1$. By hypothesis $\Delta^{n-r}\mu_r \geqq 0$ for all n.

$$\Delta^{n-r+1}\mu_{r-1} = \Delta^{n-r}\mu_{r-1} - \Delta^{n-r}\mu_r \leqq \Delta^{n-r}\mu_{r-1}.$$

Therefore $\Delta^{n-r+1}\mu_{r-1} \downarrow$ in n. But $\Delta^{n-r+1}\mu_{r-1} \to 0$ for each $r > 1$. Therefore $h_{n-1,r-1} \geqq 0$ for all $n > 0$. Similarly, we can show that $h_{nk} \geqq 0$ for $0 < k < r - 1$ and all n.

To show $h_{n,0} \downarrow$, observe that $\Delta^n\mu_k \geqq 0$ for $n \geqq 0$, $k > 0$. Therefore $\Delta^n\mu_0 = \Delta^{n-1}\mu_0 - \Delta^{n-1}\mu_1 \leqq \Delta^{n-1}\mu_0$.

COROLLARY 1. *Let H_μ be a coregular Hausdorff matrix. Then H_μ is totally coregular if and only if $h_{nk} \geqq 0$ for $k = 1, 2, 3, \cdots$.*

Proof. Since $H_\mu \in \mathbf{H}$, $h_k = 0$ for $k > 0$. From Theorem 3, H_μ is totally coregular if and only if $h_{nk} \geqq 0$ for all $k > q$ for some integer q. However, from Lemma 2, $h_{nk} \geqq 0$ for all $k > q$ implies $h_{nk} \geqq 0$ for $k = 1, 2, 3 \cdots$.

THEOREM 8. *Let $H \sim \mu$ be a multiplicative Hausdorff method. Then H_μ satisfies $(*)$ or $(**)$ if and only if μ is totally monotone.*

Proof of sufficiency. $| h_{nk} | = h_{nk}$. Therefore $\sum_{k=0}^{n} | h_{nk} | = \mu_0 = t$.

Proof of necessity. For the proof using (*), note that $h_0 = 0$, and then refer to Corollary 1 and Lemma 2.

The proof using (**) is basically the same as that of [4, Theorem VI] and will be omitted.

Theorem 8 is an extension of [4, Theorem VI] to multiplicative Hausdorff matrices.

We now generalize a result of [1, p. 452].

THEOREM 9. *Two coregular Hausdorff methods with nonvanishing moment sequences cannot be totally equivalent unless they are identical.*

Two triangles A and B are said to be totally equivalent if and only if AB^{-1} and BA^{-1} are totally regular.

Let μ, μ' be two totally equivalent nonvanishing coregular moment sequences. Then μ/μ' and μ'/μ are totally regular. Therefore $\Delta^r(\mu_n/\mu'_n) \geqq 0$ and $\Delta^r(\mu'_n/\mu_n) \geqq 0$ for $n, r = 0, 1, 2, \cdots$. In particular

$$\Delta(\mu_n/\mu'_n) = \mu_n/\mu'_n - \mu_{n+1}/\mu'_{n+1} \geqq 0,$$

and

$$\Delta(\mu'_n/\mu_n) = \mu'_n/\mu_n - \mu'_{n+1}/\mu_{n+1} \geqq 0,$$

which is a contradiction unless $\mu'_n/\mu_n = \mu'_{n+1}/\mu_{n+1}$; i.e., $\mu_n/\mu'_n = \mu_{n+1}/\mu'_{n+1}$, for $n = 0, 1, 2, \cdots$. Because μ_n/μ'_n is regular, $\mu_n/\mu'_n = \mu_0/\mu'_0 = 1$. Therefore $\mu_n = \mu'_n$ for $n = 0, 1, 2, \cdots$.

Since $\mathbf{H}$ is the set of Hausdorff matrices with finite norm, one might conjecture that the only totally coregular Hausdorff matrices are those which are multiples of totally regular Hausdorff matrices. The following theorem demonstrates that such is not the case.

THEOREM 10. *There exists a totally coregular Hausdorff method that is not multiplicative.*

Proof. Let $\mu_0 = 1$, $\mu_n = (n+2)^{-1}$, $n > 0$. Let $\lambda_n = (n+1)^{-1}$. Then $\mu_n = \lambda_{n+1}$, $n > 0$. Therefore $\Delta^n \mu_k > 0$ for $k = 1, 2, 3, \cdots$, $n = 0, 1, 2, \cdots$.

$$\begin{aligned}\Delta^n \mu_0 &= \textstyle\sum_{j=0}^n (C_{n,j})(-1)^j \mu_j = \mu_0 + \sum_{j=1}^n (C_{n,j})(-1)^j \lambda_{j+1} \\ &= \mu_0 + \Delta^n \lambda_1 - \lambda_1 = \tfrac{1}{2} + \Delta^n \lambda_1 .\end{aligned}$$

Since $\Delta^n \lambda_1 > 0$, $\Delta^n \mu_0 > 0$. Therefore μ is totally monotone.

$$\chi = 1 - \tfrac{1}{2} \neq 0,$$

since $h_0 = \frac{1}{2}$. Therefore H_μ is totally coregular and is not multiplicative.

I wish to express my thanks to the referee for his careful critique and helpful suggestions in the preparation of this paper.

References

1. S. K. Basu, *On the total relative strength of the Hölder and Cesàro methods*, Proc. London Math. Soc. (2), vol. 50 (1949), pp. 447–462.
2. G. H. Hardy, *Divergent series*, Oxford, 1949.
3. F. Hausdorff, *Summationsmethoden and Momentfolgen. I*, Math. Zeitschrift, vol. 9 (1921), pp. 74–109.
4. W. A. Hurwitz, *Some properties of methods of evaluation of divergent sequences*, Proc. London Math. Soc. (2), vol. 26 (1927), pp. 231–248.

Lafayette College
Easton, Pennsylvania