# HOMOGENEOUS SYMPLECTIC MULTIPLIERS

BY

R. C. GUNNING[1]

## 1. Introduction

Some properties of modular forms in one and several complex variables have recently been extended by R. Godement to more general classes of vector-valued modular forms associated to various matrix factors of automorphy for the symplectic modular group [6]; the factors of automorphy considered extend trivially to factors of automorphy, or multipliers, for the full symplectic group. This suggested the exercise of generalizing the classification of scalar multipliers given in [8] to a classification of matrix multipliers, which proved quite simple. The general technique is discussed in Section 2, and its application to the symplectic group in Section 3.

## 2. The general formulation of the problem

Let $D$ be a complex analytic manifold, $G$ a transitive real Lie group of complex analytic automorphisms of $D$, and $K$ the isotropy subgroup of $G$ at a point $z_0 \in D$; the image of a point $z \in D$ under a mapping $g \in G$ will be denoted by $gz$. Assume further that

(1) $D$ is a Stein manifold, [2];

(2) the mapping $G \to D$ defined by $g \to gz_0$ admits $C^\infty$ local cross-sections, (in the sense that there is a covering of $D$ by open neighborhoods $U_j$ to each of which corresponds a $C^\infty$ mapping $z \to g_j(z)$ of $U_j$ into $G$ such that $g_j(z)z_0 = z$ for all $z \in U_j$).

A *multiplier* (of rank $m$) for $G$ on $D$ is a $C^\infty$ mapping $(g, z) \to M(g, z)$ of $G \times D$ into the group $\mathrm{Gl}(m, C)$ of nonsingular $m \times m$ complex matrices, which is complex analytic in $z$ for each fixed $g \in G$ and which satisfies the functional equation

$$(3) \qquad M(g\, g^1, z) = M(g, g^1 z)M(g^1, z).$$

Two such multipliers $M(g, z)$ and $M_0(g, z)$ will be called *equivalent*, written $M \sim M_0$, if there is a complex analytic mapping $z \to F(z)$ of $D$ into $\mathrm{Gl}(m, C)$ such that

$$(4) \qquad F(gz) = M(g, z)F(z)M_0(g, z)^{-1}.$$

From (3) it follows that the mapping $k \to M(k, z_0)$ is a complex representation of the group $K$, and from (4) that two equivalent multipliers induce

Received October 5, 1959.

[1] The author holds an Alfred P. Sloan Fellowship.

equivalent representations. It thus suffices for a classification to consider the inequivalent complex representations of $K$, and for each such representation $\rho$ to determine the set $\mathfrak{M}(\rho)$ of multipliers associated thereto and the set $\mathfrak{M}^*(\rho)$ of equivalence classes in $\mathfrak{M}(\rho)$.

From each nonempty set $\mathfrak{M}(\rho)$ select in some manner a canonical element $M_0(g, z)$, henceforth to be assumed fixed. For any $M(g, z) \in \mathfrak{M}(\rho)$ there is a unique $C^\infty$ mapping $z \to F(z)$ of $D$ into $\mathrm{Gl}(m, C)$ which satisfies (4) and in addition $F(z_0) = I$. (The uniqueness is a clear consequence of the transitivity of $G$; as for the existence, the function $\tilde{F}(g) = M(g, z_0)\, M_0(g, z)^{-1}$ on $G$ is right-invariant under $K$, and hence induces a function $F(z)$ on $G/K = D$ which is easily seen to be the desired function.) Then associate to $M(g, z)$ the $m \times m$ matrix $\phi(z)$ of complex differential forms on $D$ defined by

$$\phi(z) = F(z)^{-1}\bar{\partial}F(z); \tag{5}$$

here the differential operator $\bar{\partial}$ (in the notation of [5] )is applied componentwise, while multiplication and inverse are taken in the usual matrix sense. This determines a mapping $\mathfrak{D}$ from $\mathfrak{M}(\rho)$ onto the particular set $\Phi(\rho)$ of differential forms described in the following:

LEMMA 1. *$\Phi(\rho)$ is the set of $m \times m$ matrices $\phi(z)$ of complex differential forms of type $(0, 1)$ on $D$ such that*

$$\delta g \cdot \phi(z) = M_0(g, z)\phi(z)M_0(g, z)^{-1} \tag{6}$$

*(where $\delta g$ is the left translation of differential forms on $D$ associated to the automorphism $z \to gz$ as in [4]), and such that*

$$\bar{\partial}\phi(z) + \phi(z) \wedge \phi(z) = 0. \tag{7}$$

*Proof.* If $\phi(z) = \mathfrak{D}M(g, z)$, then (6) and (7) follow upon applying the operator $\bar{\partial}$ to (4) and (5) respectively. Conversely if $\phi(z)$ satisfies (7) there is a $C^\infty$ mapping $z \to F(z)$ of $D$ into $\mathrm{Gl}(m, C)$ which satisfies (5). To see this recall that the existence of such mappings locally was proved for instance in [9] (especially Section 19); that is, there is an open covering $\{U_i\}$ of $D$ together with a family of $C^\infty$ mappings $z \to F_i(z)$ of the various sets $U_i$ into $\mathrm{Gl}(m, C)$ such that $F_i(z)^{-1}\bar{\partial}F_i(z) = \phi(z)$. The mappings $z \to F_{ij}(z) = F_i(z)F_j(z)^{-1}$ of $U_i \cap U_j$ into $\mathrm{Gl}(m, C)$ are consequently holomorphic and define a complex vector bundle on $D$ which is moreover obviously topologically trivial. Since $D$ is a Stein manifold the bundle must be analytically trivial as well, [3], [7]; that is, there must exist holomorphic mappings $z \to G_i(z)$ of $U_i$ into $\mathrm{Gl}(m, C)$ such that $G_i(z) = F_{ij}(z)G_j(z)$ in $U_i \cap U_j$. The function $F(z) = G_i(z)^{-1}\, F_i(z)$ is the desired mapping. Then $M(g, z) = F(gz)M_0(g, z)F(z)^{-1}$ clearly satisfies the functional equation (3); and since

$$\bar{\partial}M(g, z) = F(gz)\cdot[\delta g\cdot\phi(z) - M_0(g, z)\phi(z)M_0(g, z)^{-1}]M_0(gz)F(z)^{-1},$$

the function $M(g, z)$ will be holomorphic in $z$ under the further assumption of (6). Thus $M(g, z)$ is a multiplier for which $\phi(z) = \mathfrak{D}M(g, z)$, and the proof is thereby concluded.

Two differential forms $\phi(z)$, $\phi^1(z) \in \Phi(\rho)$ will be called equivalent, also written $\phi \sim \phi^1$, if there is a $C^\infty$ mapping $z \to G(z)$ of $D$ into $\mathrm{Gl}(m, C)$ such that

$$(8) \qquad G(gz) = M_0(g, z)G(z)M_0(g, z)^{-1},$$

and

$$(9) \qquad \bar{\partial}G(z) = G(z)\phi(z) - \phi^1(z)G(z);$$

and the set of equivalence classes of elements in $\Phi(\rho)$ will be denoted by $\Phi^*(\rho)$.

LEMMA 2. *The mapping* $\mathfrak{D}:\mathfrak{M}(\rho) \to \Phi(\rho)$ *induces a one-to-one mapping* $\mathfrak{D}^*:\mathfrak{M}^*(\rho) \to \Phi^*(\rho)$.

*Proof.* Let $\phi(z) = \mathfrak{D}M(g, z) = F(z)^{-1}\bar{\partial}F(z)$ and $\phi^1(z) = \mathfrak{D}M^1(g, z) = F^1(z)^{-1}\bar{\partial}F^1(z)$, with the obvious notation; the assertion of the lemma is that $\phi \sim \phi^1$ if and only if $M \sim M^1$. Now $M \sim M^1$ is just the condition that there exist a complex analytic mapping $z \to H(z)$ of $D$ into $\mathrm{Gl}(m, C)$ such that

$$(10) \qquad H(gz) = M^1(g, z)H(z)M(g, z)^{-1}.$$

Setting

$$(11) \qquad G(z) = F^1(z)^{-1}H(z)F(z),$$

equation (8) is clearly equivalent to equation (10); and since the operation $\bar{\partial}$ applied to equation (11) yields the result that

$$\bar{\partial}H(z) = F^1(z)[\bar{\partial}G(z) - G(z)\phi(z) + \phi^1(z)G(z)]F(z)^{-1},$$

it is also clear that equation (9) is equivalent to the condition that $H(z)$ be complex analytic. Thus the desired result is demonstrated.

*Remark.* These simple observations also hold practically without change for a properly discontinuous group $\Gamma$ of complex analytic automorphisms of $D$. It is only necessary to replace the sets $\mathfrak{M}(\rho)$ by the sets $\mathfrak{M}(M_0)$ of multipliers $M(g, z)$ for which there is a $C^\infty$ mapping $z \to F(z)$ of $D$ into $\mathrm{Gl}(m, C)$ satisfying (4). The problem under consideration can then be rephrased as that of classifying the complex analytic vector bundles on the complex space $D/\Gamma$; and $\mathfrak{M}(M_0)$ is the set of those complex vector bundles which are topologically equivalent to $M_0(g, z)$. In particular if $M_0(g, z) = I$ (the trivial bundle), the set of topologically trivial complex analytic vector bundles on $D/\Gamma$ can be identified with the space of matrices $\phi(z)$ of differential forms of type (0, 1) on $D/\Gamma$ satisfying (7) modulo the equivalence relation (9); for complex line bundles this reduces to a well-known result, [11].

## 3. Multipliers for the symplectic group

The case to be considered here is that in which $D$ is the set of $p \times p$ complex symmetric matrices $Z = (z_{ij})$ such that $I - Z\bar{Z}$ is positive definite Hermitean: $G$ is the $2p \times 2p$ complex matrix group

$$G = \left\{ g = \begin{pmatrix} \bar{D} & \bar{C} \\ C & D \end{pmatrix} \middle| \, {}^t g \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\},$$

where $C$, $D$ are $p \times p$ matrix blocks and ${}^t g$ denotes the transpose of the matrix $g$; and the action of $G$ on $D$ is defined by $gZ = (\bar{D}Z + \bar{C})(CZ + D)^{-1}$. Thus the isotropy subgroup of $G$ at the zero matrix $0 \in D$ is

$$K = \left\{ k = \begin{pmatrix} \bar{U} & 0 \\ 0 & U \end{pmatrix} \middle| \, U \text{ unitary} \right\},$$

which is naturally isomorphic to the group $\mathfrak{U}(p)$ of $p \times p$ unitary matrices, [1], [10]. Any continuous complex representation $\rho$ of $\mathfrak{U}(p)$ can be extended to a representation of $\mathrm{Gl}(p, C)$ and determines a canonical multiplier

$$M_0(g, Z) = \rho(CZ + D)$$

as in [6].

The technique of classification used requires the determination of the set $\Phi(\rho)$; however (6) and the transitivity of $G$ show that a differential form $\phi(Z) \in \Phi(\rho)$ is determined uniquely by its value $\phi(0)$ at the origin, so that it is enough to describe merely the set $\Phi_0(\rho) = \{\phi(0) \mid \phi(Z) \in \Phi(\rho)\}$. An element $\phi \in \Phi_0(\rho)$ of course has the form $\phi = \sum_{ij} \phi_{ij}\, d\bar{z}_{ij}$, where $\phi_{ij}$ are constant matrices and $\phi_{ij} = \phi_{ji}$; to be explicit, note that the action of a transformation $U = (u_{ij})$ of the isotropy subgroup $\mathfrak{U}(p)$ on such an element is given by

$$\delta U \cdot \phi = \sum_{ijkl} \phi_{kl}\, u_{ki}\, u_{lj}\, d\bar{z}_{ij}\,.$$

LEMMA 3. *$\Phi_0(\rho)$ is the set of matrix differential forms $\phi$ of type $(0, 1)$ at the origin such that*

$$\delta U \cdot \phi = \rho(U)\, \phi\, \rho(U)^{-1} \qquad \textit{for all } U \in \mathfrak{U}(p), \tag{12}$$

*and*

$$\phi \wedge \phi = 0. \tag{13}$$

*Proof.* Clearly (12) is just the condition that $\phi = \phi(0)$ for some $\phi(Z)$ satisfying (6). The main portion of the proof, contained in the subsequent paragraphs, consists of the demonstration that $\bar{\partial}\phi(Z) = 0$ whenever $\phi(Z)$ satisfies (6); as a consequence of that (13) is equivalent to (7) and the lemma follows as stated.

To carry out the proof suppose that $\rho(U)$ is the direct sum of irreducible representations $\rho(U) = \oplus_r \rho^r(U)$; the associated canonical multiplier is then the direct sum $M(g, Z) = \oplus_r M^r(g, Z)$, where $M^r(g, Z) = \rho^r(CZ + D)$. Now if $\phi(Z)$ satisfies (6) it can be decomposed into matrix blocks

$\phi(Z) = (\phi^{rs}(Z))$ such that (6) becomes

$$(14) \qquad \delta g \cdot \phi^{rs}(Z) = M^r(g, Z)\phi^{rs}(Z)M^s(g, Z)^{-1}.$$

Write the components of the matrix $\phi^{rs}(Z)$ in the form

$$\phi^{rs}(Z) = (\textstyle\sum_{ij} \phi^{rs}_{abij}(Z)d\bar{z}_{ij})$$

and the components of the matrix $M^r(g, Z)$ in the form

$$M^r(g, Z) = (M^r_{ab}(g, Z)) = (\rho^r_{ab}(CZ + D));$$

throughout this proof the range of the indices $i, j, k, l$ will be from 1 to $p$, and the ranges of the indices $a, b, c, e$ will be the sizes of the matrices under consideration. For each transformation $W = gZ$ introduce a complex Jacobian matrix $J(g, Z) = (J_{klij}(g, Z))$ such that $dw_{kl} = \sum_{ij} J_{klij}(g, Z)dz_{ij}$ and $J_{klij}(g, Z) = J_{lkji}(g, Z)$. Then (14) can be written componentwise

$$(15) \quad \textstyle\sum_{ijkl} \phi^{rs}_{abkl}(gZ)\bar{J}_{klij}(g, Z)d\bar{z}_{ij} = \sum_{ijce} M^r_{ac}(g, Z)M^{*s}_{be}(g, Z)\phi^{rs}_{ceij}(Z)d\bar{z}_{ij},$$

where $M^{*s}(g, Z) = (M^{*s}_{be}(g, Z)) = {}^t(M^s(g, Z))^{-1}$; of course (15) implies the equality of the coefficients of the $d\bar{z}_{ij}$ for each $i, j$ since the coefficients are symmetric in $i, j$. From the explicit form $W = gZ = (\bar{D}Z + \bar{C})\,(CZ + D)^{-1}$ an easy calculation gives $dW = {}^t(CZ + D)^{-1}dZ(CZ + D)^{-1}$; at $Z = 0$ in particular, $dW = D^* dZ\, {}^tD^*$ so that $J_{klij}(g, 0) = d^*_{ki}\, d^*_{lj}$, where $D^* = (d^*_{ij}) = {}^tD^{-1}$. Then since $g0 = \bar{C}D^{-1}$ and $M(g, 0) = \rho(D)$, equation (15) implies that

$$(16) \qquad \textstyle\sum_{kl} \phi^{rs}_{abkl}(\bar{C}D^{-1})\bar{d}^*_{ki}\, \bar{d}^*_{lj} = \sum_{ce} \rho^r_{ac}(D)\rho^{*s}_{be}(D)\phi^{rs}_{ceij},$$

where $\phi = \phi(0)$. Now the components $\phi^{rs}_{abij}(Z)$ can also be considered as forming an $m^2 \times p^2$ matrix

$$\tilde{\phi}^{rs}(Z) = (\phi^{rs}_{(ab)(ij)}(Z)),$$

and in this matrix form (16) is

$$(17) \qquad \tilde{\phi}^{rs}(\bar{C}D^{-1})\cdot(\bar{D}^* \otimes \bar{D}^*) = (\rho^r(D) \otimes \rho^{*s}(D))\tilde{\phi}^{rs}.$$

Select a suitable matrix $T$ to exhibit the direct sum decomposition

$$(18) \qquad \rho^r(U) \otimes \rho^{*s}(U) = T(\oplus_q \sigma^q(U))T^{-1},$$

where $\sigma^q(U)$ are irreducible representations, and set

$$(19) \qquad \tilde{\psi}^{rs}(Z) = T^{-1}\tilde{\phi}^{rs}(Z).$$

Then upon decomposing $\tilde{\psi}^{rs}(Z)$ into matrix blocks

$$(20) \qquad \tilde{\psi}^{rs}(Z) = \begin{pmatrix} \tilde{\psi}^{rs1}(Z) \\ \cdots \\ \tilde{\psi}^{rsq}(Z) \\ \cdots \end{pmatrix}$$

compatibly with the decomposition (18), equation (17) becomes

$$\tilde{\psi}^{rsq}(\bar{C}D^{-1})(\bar{D}^* \otimes \bar{D}^*) = \sigma^q(D)\tilde{\psi}^{rsq}. \tag{21}$$

For a transformation $g$ in the isotropy subgroup in particular, equation (21) reduces to

$$\tilde{\psi}^{rsq}(U \otimes U) = \sigma^q(U)\tilde{\psi}^{rsq}. \tag{22}$$

Now the rows of the matrix $\tilde{\psi}^{rsq}$ are symmetric tensors of rank 2, and the representation $U \otimes U$ is irreducible on such tensors; so by Schur's lemma either $\tilde{\psi}^{rsq} = 0$ or $\sigma^q(U)$ is also the symmetric representation of degree 2 and $\tilde{\psi}^{rsq}$ is a scalar matrix [12]. That is, if $\tilde{\psi}^{rsq} \neq 0$ then $\sigma^q(U) = U \otimes U$ and $\tilde{\psi}^{rsq}$ has components

$$\psi^{rsq}_{abij} = \tfrac{1}{2}h(\delta_{ai}\,\delta_{bj} + \delta_{aj}\,\delta_{bi}) \tag{23}$$

for some complex constant $h$. Therefore (21) becomes

$$\begin{aligned}\psi^{rsq}_{abij}(\bar{C}D^{-1}) &= \tfrac{1}{2}h\textstyle\sum_{cekl}\, d_{ac}\, d_{be}\, \bar{d}_{ik}\, \bar{d}_{jl}(\delta_{ck}\,\delta_{el} + \delta_{cl}\,\delta_{ek})\\ &= \tfrac{1}{2}h\textstyle\sum_{ce}\, d_{ac}\, d_{be}(\bar{d}_{ic}\,\bar{d}_{je} + \bar{d}_{ie}\,\bar{d}_{jc}).\end{aligned}$$

These components can also be considered as being the coefficients of a new matrix differential form

$$\psi^{rsq}(\bar{C}D^{-1}) = \left(\textstyle\sum_{ij}\psi^{rsq}_{abij}(\bar{C}D^{-1})d\bar{z}_{ij}\right) = \left(h\textstyle\sum_{ceij}\, d_{ac}\, d_{be}\,\bar{d}_{ic}\,\bar{d}_{je}\, d\bar{z}_{ij}\right); \tag{24}$$

and on setting $Z = \bar{C}D^{-1}$ and recalling from the definition of the symplectic group that $I = {}^t\bar{D}D - {}^tC\bar{C}$, it follows that $D\,{}^t\bar{D} = (I - \bar{Z}Z)^{-1}$ and hence that (24) can be written

$$\psi^{rsq}(Z) = h(I - \bar{Z}Z)^{-1}d\bar{Z}(I - Z\bar{Z})^{-1}. \tag{25}$$

Note that hence

$$\begin{aligned}&\bar{\partial}\psi^{rsq}(Z)\\ &\quad= h(I - \bar{Z}Z)^{-1}d\bar{Z}[Z(I - \bar{Z}Z)^{-1} - (I - Z\bar{Z})^{-1}Z] \wedge d\bar{Z}(I - Z\bar{Z})^{-1}\\ &\quad= 0\end{aligned}$$

since $(I - Z\bar{Z})Z = Z(I - \bar{Z}Z)$, so that the components $\sum_{ij}\psi^{rsq}_{abij}(Z)d\bar{z}_{ij}$ of the matrices $\psi^{rsq}(Z)$ are $\bar{\partial}$-closed differential forms. If the matrix $T$ of (18) is decomposed into blocks $T = (T^1 \cdots T^q \cdots)$ compatibly with the decomposition (20), then (19) can be written

$$\textstyle\sum_{ij}\phi^{rs}_{abij}(Z)d\bar{z}_{ij} = \sum_{qcdij} T^q_{abcd}\,\psi^{rsq}_{cdij}(Z)d\bar{z}_{ij},$$

and consequently the components of the matrices $\phi^{rs}(Z)$ are also $\bar{\partial}$-closed differential forms; since this was all that remained to be demonstrated, the proof is thereby completed.

The mapping $\mathfrak{D}:\mathfrak{M}(\rho) \to \Phi(\rho)$ followed by the restriction isomorphism

$\Phi(\rho) \rightarrow \Phi_0(\rho)$ gives a mapping $\mathfrak{D}_0 : \mathfrak{M}(\rho) \rightarrow \Phi_0(\rho)$. Two differential forms $\phi, \phi^1 \in \Phi_0(\rho)$ will be called equivalent, written $\phi \sim \phi^1$, if there is a nonsingular matrix $G$ such that

$$G = \rho(U)\, G\, \rho(U)^{-1} \qquad \text{for all } U \in \mathfrak{U}(p) \tag{26}$$

and

$$\phi^1 = G\phi G^{-1}; \tag{27}$$

and the set of equivalence classes of elements in $\Phi_0(\rho)$ will be denoted by $\Phi_0^*(\rho)$.

LEMMA 4. *The mapping* $\mathfrak{D}_0 : \mathfrak{M}(\rho) \rightarrow \Phi_0(\rho)$ *induces a one-to-one mapping* $\mathfrak{D}_0^* : \mathfrak{M}^*(\rho) \rightarrow \Phi_0^*(\rho)$.

*Proof.* In view of Lemma 2 it is only necessary to show that $\phi(Z), \phi^1(Z) \in \Phi(\rho)$ are equivalent in the sense of (8), (9) if and only if their restrictions $\phi(0), \phi^1(0) \in \Phi_0(\rho)$ are equivalent in the sense of (26), (27). If $G(Z)$ satisfies (8) then $G(0)$ satisfies (26); consequently, as in the unitarian trick, $G(0) = \rho(D)G(0)\rho(D)^{-1}$ for all $D \in \mathrm{Gl}(m, C)$, so that (8) implies that $G(g0) = G(0)$ and hence that $G(Z)$ is constant. Then (8) and (26) are equivalent conditions, and therefore so are (9) and (27), which suffices to prove the lemma.

The net result of this chain of lemmas can be summarized in the following:

THEOREM. *The set* $M^*(\rho)$ *is in a natural one-to-one correspondence with the set of matrix differential forms of type* (0, 1) *at the origin which satisfy* (12) *and* (13) *modulo the equivalence relation defined by* (26) *and* (27).

COROLLARY. *If* $\rho$ *is an irreducible representation, all the multipliers in* $\mathfrak{M}(\rho)$ *are equivalent to the canonical multiplier* $M_0(g, Z) = \rho(CZ + D)$.

*Proof.* If $\rho$ is irreducible, then for the particular unitary matrices $U = \varepsilon I$, where $\varepsilon$ is a complex number of absolute value one, $\rho(U) = \varepsilon^r I$ for some integer $r$ and equation (12) becomes $\varepsilon^{2r}\phi = \phi$; hence only the zero differential form satisfies (12) and (13), so the desired result follows from the above theorem.

The corollary shows that for irreducible representations the multipliers considered by Godement are indeed the most general type. Perhaps the simplest case in which there are multipliers other than the canonical multiplier is that of the representation

$$\rho(U) = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}, \tag{28}$$

for which the canonical multiplier is of course

$$M_0(g, Z) = \begin{pmatrix} CZ + D & 0 \\ 0 & {}^t(CZ + D)^{-1} \end{pmatrix}.$$

To analyze the remaining multipliers decompose any matrix differential form $\phi \in \Phi_0(\rho)$ into blocks

$$\phi = \begin{pmatrix} \phi^{11} & \phi^{12} \\ \phi^{21} & \phi^{22} \end{pmatrix}$$

compatibly with (28), so that condition (12) becomes

$$\delta U \cdot \phi^{11} = U \phi^{11} U^{-1}, \qquad \delta U \cdot \phi^{12} = U \phi^{12}\, {}^t U,$$
$$\delta U \cdot \phi^{21} = U^* \phi^{21} U^{-1}, \qquad \delta U \cdot \phi^{22} = U^* \phi^{22}\, {}^t U.$$

Then as in the above corollary it follows that $\phi^{11} = \phi^{22} = \phi^{21} = 0$, while $\phi^{12}$ satisfies the same condition as the differential forms $\psi^{rsq}$ of Lemma 3 and hence can be written as in (23) or (25); that is,

$$\phi = h \begin{pmatrix} 0 & d\bar{Z} \\ 0 & 0 \end{pmatrix} \tag{29}$$

for some complex constant $h$, from which it follows that (13) is automatically fulfilled. The only nonsingular matrices satisfying (26) in this case are of the form

$$G = \begin{pmatrix} aI & 0 \\ 0 & bI \end{pmatrix};$$

hence two differential forms $\phi$, $\phi^1$ of type (29) corresponding to constants $h$, $h^1$ respectively are equivalent precisely when $h^1 = (a/b)h$ for some nonzero constants $a$ and $b$, that is, if and only if either $h = h^1 = 0$ or $hh^1 \neq 0$. Consequently there is but a single multiplier not equivalent to the canonical multiplier, an explicit representation for which can be calculated quite easily. As in (25) the differential form $\phi(Z) \in \Phi(\rho)$ corresponding to (29) is

$$\phi(Z) \begin{pmatrix} 0 & (I - \bar{Z}Z)^{-1} d\bar{Z} (I - Z\bar{Z})^{-1} \\ 0 & 0 \end{pmatrix}$$

for $h = 1$; and since

$$F(Z) = \begin{pmatrix} I & \bar{Z}(I - Z\bar{Z})^{-1} \\ 0 & I \end{pmatrix}$$

is a nonsingular matrix function for which $F(0) = I$ and $F(Z)^{-1} \bar{\partial} F(Z) = \phi(Z)$, the desired multiplier is given by

$$\begin{aligned} M(g, Z) &= F(gZ) M_0(g, Z) F(Z)^{-1} \\ &= \begin{pmatrix} CZ + D & C \\ 0 & {}^t(CZ + D)^{-1} \end{pmatrix}. \end{aligned}$$

The situation in one complex variable is of course particularly simple, the only multipliers other than the canonical multiplier being derived as in the example considered above. Indeed, it will probably suffice merely to list the possible $3 \times 3$ multipliers, this case being sufficiently general to illuminate the entire pattern. The isotropy subgroup at the origin is the circle group (the multiplicative group of complex numbers $\varepsilon$ of absolute value one), so the relevant representations are of the form

$$\rho(\varepsilon) = \begin{pmatrix} \varepsilon^{q_1} & 0 & 0 \\ 0 & \varepsilon^{q_2} & 0 \\ 0 & 0 & \varepsilon^{q_3} \end{pmatrix}$$

for some integers $q_1$, $q_2$, $q_3$, and the corresponding canonical multipliers are

$$M_0(g, z) = \begin{pmatrix} (cz+d)^{q_1} & 0 & 0 \\ 0 & (cz+d)^{q_2} & 0 \\ 0 & 0 & (cz+d)^{q_3} \end{pmatrix}. \tag{30}$$

There are no other multipliers unless at least two of the integers $q_1$, $q_2$, $q_3$ differ by 2; if, say, $q_1 - q_2 = 2$, then in addition to (30) there is the multiplier

$$M_1(g, z) = \begin{pmatrix} (cz+d)^{q_1} & c(cz+d)^{q_1-1} & 0 \\ 0 & (cz+d)^{q_2} & 0 \\ 0 & 0 & (cz+d)^{q_3} \end{pmatrix}; \tag{31}$$

and if both $q_1 - q_2 = 2$ and $q_2 - q_3 = 2$, then there are the three multipliers (30), (31), and

$$M_2(g, z) = \begin{pmatrix} (cz+d)^{q_1} & c(cz+d)^{q_1-1} & \frac{1}{2}c^2(cz+d)^{q_1-2} \\ 0 & (cz+d)^{q_2} & c(cz+d)^{q_2-1} \\ 0 & 0 & (cz+d)^{q_3} \end{pmatrix}. \tag{32}$$

## BIBLIOGRAPHY


1. E. CARTAN, *Sur les domaines bornés homogènes de l'espace de n variables complexes*, Abh. Math. Sem. Univ. Hamburg, vol. 11 (1936), pp. 111–162.
2. H. CARTAN, *Séminaire H. Cartan 1951/1952*, (mimeographed lecture notes), Paris, École Normale Supérieure, 1952.
3. ———, *Espaces fibrés analytiques*, Report of the Mexico Symposium in Topology 1956, Mexico, 1959.
4. C. CHEVALLEY, *Theory of Lie Groups I*, Princeton, Princeton University Press, 1946.
5. P. R. GARABEDIAN AND D. C. SPENCER, *A complex tensor calculus for Kähler manifolds*, Acta Math., vol. 89 (1953), pp. 279–331.
6. R. GODEMENT, lectures in *Séminaire H. Cartan 1957/1958*, (mimeographed lecture notes), Paris, École Normale Supérieure, 1958.
7. H. GRAUERT, *Analytische Faserungen über holomorph-vollständigen Räumen*, Math. Ann., vol. 135 (1958), pp. 263–273.
8. R. C. GUNNING, *Factors of automorphy and other formal cohomology groups for Lie groups*, Ann. of Math. (2), vol. 69 (1959), pp. 314–326.
9. K. KODAIRA AND D. C. SPENCER, *On deformations of complex analytic structures, II*, Ann. of Math (2), vol. 67 (1958), pp. 403–466.
10. C. L. SIEGEL, *Analytic functions of several complex variables*, (mimeographed lecture notes), Princeton, Institute for Advanced Study, 1949.
11. A. WEIL, *Introduction à l'étude des variétés kählériennes*, Actualités Scientifiques et Industrielles, no. 1267, Paris, 1958.
12. H. WEYL, *The classical groups*, Princeton, Princeton University Press, 1946.


PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY