QUOTIENT FIELDS OF RESIDUE CLASS RINGS OF FUNCTION RINGS

BY

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The ring of all continuous functions from a topological space X into the reals, $\mathbf{R}$, is denoted by C or $C(X)$. In [1, 14E], an example is presented of a residue class field of one ring of continuous functions that is isomorphic in a natural way with the quotient field of a residue class ring of another ring of functions. The first ring is $C(\mathbf{N})$, $\mathbf{N}$ denoting the discrete space of positive integers. The second is $C(\Sigma)$, where $\Sigma = \mathbf{N} \cup \{\sigma\}$ is the subspace of the Stone-Čech compactification of $\mathbf{N}$ obtained by adjoining a single point σ to $\mathbf{N}$. The set M of all functions in $C(\mathbf{N})$ that vanish on a set having σ in its closure is a maximal ideal in $C(\mathbf{N})$; the set Q of all functions in $C(\Sigma)$ that vanish on a neighborhood of σ is a nonmaximal prime ideal in $C(\Sigma)$. In the manner to be described in §2, the mapping that sends each function in $C(\Sigma)$ into its restriction to $\mathbf{N}$ induces an isomorphism of the integral domain $C(\Sigma)/Q$ onto a subring of $C(\mathbf{N})/M$, and $C(\mathbf{N})/M$ is the quotient field of that subring.

In the present paper, we investigate the possibility of obtaining, in a similar way, the quotient field of $C(Y)/Q$, where Q is a prime ideal in an arbitrary function ring $C(Y)$. We shall find that a necessary condition is that Q be a *z-ideal*, i.e., if $h \in C(Y)$, and if there exists $g \in Q$ such that $h(y) = 0$ wherever $g(y) = 0$, then $h \in Q$. A sufficient condition is that Q have an immediate successor in the family of all z-ideals in $C(Y)$. On the other hand, if Q is the intersection of a countable family of z-ideals different from itself, then the quotient field of $C(Y)/Q$ is not isomorphic with a residue class field of *any* function ring. The question is left open as to what may happen in case Q neither has an immediate successor nor is a countable intersection; whether such a prime z-ideal Q exists at all is also left unsettled.

1. Preliminaries

The terminology and notation of [1] will be used throughout the paper. In this section, we summarize the material from [1] that will be used. Most of the information about prime ideals can also be found in [2] and [3].

When dealing with algebraic properties of a ring $C(X)$, one loses no generality by supposing X to be completely regular. We adopt this standing assumption.

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For $f \in C(X)$, $Z(f)$—or when more precision is needed, $Z_X(f)$—denotes the *zero-set* of f, that is, the set $\{x \in X: f(x) = 0\}$. The family of all zero-sets of functions in $C(X)$ is denoted by $Z(X)$. A family of nonempty zero-sets that contains the intersection of any two of its members and contains every zero-set containing any of its members is called a *z-filter*. For any (proper) ideal I, the family $Z[I] = \{Z(f): f \in I\}$ is a z-filter. Thus, I is a z-ideal provided that $Z(f) \in Z[I]$ implies $f \in I$. Every maximal ideal is a z-ideal, and its corresponding z-filter is a maximal z-filter or *z-ultrafilter*. If $\mathfrak{F}$ is a z-filter, then the family of functions $\{f \in C: Z(f) \in \mathfrak{F}\}$ is a z-ideal; and this ideal is maximal if and only if $\mathfrak{F}$ is a z-ultrafilter.

If I is an ideal such that $|f| \leqq |g|$ and $g \in I$ implies $f \in I$, then the residue class ring C/I is a lattice-ordered ring,[2] and the canonical homomorphism of C onto C/I is a lattice homomorphism. A z-ideal obviously possesses the stated property. Every prime ideal also has this property, and, in addition, its residue class ring is totally ordered. This implies that the prime ideals containing a given prime ideal form a chain (under set inclusion), and hence that every prime ideal is contained in a unique maximal ideal. A prime ideal may or may not be a z-ideal, and a z-ideal may or may not be prime. However, a z-ideal containing a prime ideal is necessarily prime.

A z-filter $\mathfrak{F}$ is said to be *prime* if whenever the union of two zero-sets belongs to $\mathfrak{F}$, at least one of them does. If P is a prime z-ideal, then $Z[P]$ is a prime z-filter. Conversely, if $\mathfrak{F}$ is a prime z-filter, then the z-ideal $\{f \in C: Z(f) \in \mathfrak{F}\}$ is prime.

A (completely regular) space X is a dense subspace of its Stone-Čech compactification βX. The points of βX are in one-one correspondence with the z-ultrafilters on X and hence with the maximal ideals in $C(X)$. Corresponding to the point $p \in \beta X$ is the maximal ideal

$$M^p = \{f \in C(X): p \in \mathrm{cl}_{\beta X}\, Z(f)\} \tag{1a}$$

in $C(X)$. When $p \in X$ and we wish to emphasize this fact, we denote M^p by M_p; clearly, M_p is the set of all functions in $C(X)$ that vanish at p.

The terms *homomorphism* and *isomorphism*, unmodified, refer exclusively to mappings that preserve the ring operations. Any homomorphism t of a ring $C(Y)$ into $C(X)$ is automatically a lattice homomorphism. A homomorphism t has a particularly nice representation when Y is *realcompact*,[3] to wit: There exist an open-and-closed set E in X and a continuous mapping

[2] The order in the residue class ring is defined by $I(f) \geqq 0$ if $f \equiv g \pmod{I}$ for some $g \geqq 0$. Here, $I(f)$ denotes the residue class to which f belongs, and the residue class $I(r)$ containing the constant function r is identified with the real number r. Ideals with the property described in the text are called *absolutely convex* in [1].

[3] The maximal ideal M^p in $C(S)$ is hyper-real (defined in §4) if and only if p is contained in a zero-set in βS disjoint from S. The space S is *realcompact* (i.e., a "Q-space") if M^p is hyper-real for every $p \in \beta S - S$. For a detailed discussion of realcompact spaces, see [1, Chapter 8].

$\tau: E \to Y$ such that for all $g \in C(Y)$, $\mathrm{t}g(x) = g(\tau x)$ if $x \in E$, and $\mathrm{t}g(x) = 0$ if $x \in X - E$. Since the subspace $X - E$ has no effect upon any of the algebraic questions under consideration here, we may assume that it is empty. Then τ maps X into Y, and t is the homomorphism τ':

$$\tau' g = g \circ \tau. \tag{1b}$$

The assumption that Y be realcompact is also devoid of algebraic consequences, because every ring of functions is isomorphic with $C(Y)$ for a uniquely determined realcompact Y.

2. Embedding of C/Q in a residue class field

Let Q be a prime ideal in a ring $C(Y)$. We are searching for conditions on Q under which there will exist a space X, a maximal ideal M in $C(X)$, and a homomorphism $\mathrm{t}: C(Y) \to C(X)$ such that the formula

$$\mathrm{t}^*(Q(g)) = M(\mathrm{t}g) \tag{2a}$$

defines an isomorphism

$$\mathrm{t}^*: C(Y)/Q \to C(X)/M,$$

and $C(X)/M$ is the quotient field of $\mathrm{t}^*[C(Y)/Q]$. When these exist, we shall say that the quotient field of $C(Y)/Q$ is *realized* as $C(X)/M$, and that the latter is a *realization* of the former. As we pointed out earlier, no generality is lost if it is required in addition that $\mathrm{t} = \tau'$ for some continuous mapping $\tau: X \to Y$.

In this section, we shall characterize the prime ideals Q in $C(Y)$ for which X, M, and t exist so that t^* is an isomorphism—without considering as yet the question of the quotient field.

LEMMA 2.1. *Given X, M, and* t, *the formula* (2a) *defines* t^* *as a mapping if and only if* $\mathrm{t}[Q] \subset M$. *If* t^* *is defined, then it is a homomorphism and a lattice homomorphism. It is an isomorphism if and only if* $Q = \mathrm{t}^{\leftarrow}[M]$.[4]

The verification of this lemma is a routine matter. Actually, the result is valid in a far more general setting.

LEMMA 2.2. *Let τ be a continuous mapping from a space X into Y. If P is a prime z-ideal in $C(X)$, then $Q = \tau'^{\leftarrow}[P]$ is a prime z-ideal in $C(Y)$.*

Proof. The mapping τ', as given by (1b), is a homomorphism from $C(Y)$ into $C(X)$. Since the residue class ring $C(Y)/Q$ is isomorphic with a subring of the integral domain $C(X)/P$, Q is a prime ideal. Consider any $h \in C(Y)$, and suppose that $Z_Y(h) = Z_Y(g)$ for some $g \in Q$. For any $x \in X$, $\tau' h(x) = h(\tau x) = 0$ if and only if $\tau' g(x) = g(\tau x) = 0$. Since $\tau' g$ belongs to the z-ideal P, so also does $\tau' h$; thus, $h \in Q$. This shows that Q is a z-ideal.

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[4] $\mathrm{t}^{\leftarrow}$ denotes the inverse of the mapping t. Thus, $\mathrm{t}^{\leftarrow}[M] = \{g: \mathrm{t}g \in M\}$.

THEOREM 2.3. *Let Q be an ideal in a ring $C(Y)$. In order that there exist a space X, a maximal ideal M in $C(X)$, and a homomorphism* $\mathfrak{t}: C(Y) \to C(X)$ *such that* $Q = \mathfrak{t}^{\leftarrow}[M]$, *it is necessary and sufficient that Q be a prime z-ideal.*

Proof. Necessity. As was pointed out above, we may suppose that $\mathfrak{t} = \tau'$ for some continuous mapping $\tau: X \to Y$. Since M is a prime z-ideal, so also is $\mathfrak{t}^{\leftarrow}[M]$.

Sufficiency. (This is essentially [1, 14F.*1*].) X will be the set Y with the discrete topology, and τ will be the identity map from X to Y. Let $\mathcal{E}$ be the family of all complements of members of $\boldsymbol{Z}(Y) - \boldsymbol{Z}[Q]$. Since $\boldsymbol{Z}[Q]$ is a prime z-filter, the family $\mathcal{E}$ is closed under finite intersection. Moreover, each member of $\mathcal{E}$ meets every member of $\boldsymbol{Z}[Q]$. It follows that $\mathcal{E} \cup \boldsymbol{Z}[Q]$ has the finite intersection property and so may be embedded in an ultrafilter. Like any ultrafilter on X, this has the form $\boldsymbol{Z}[M]$ for some maximal ideal M in $C(X)$. We claim that $Q = \tau'^{\leftarrow}[M]$—or, expressed in terms of the corresponding z-filters: $\boldsymbol{Z}[Q] = \boldsymbol{Z}(Y) \cap \boldsymbol{Z}[M]$. Clearly, $\boldsymbol{Z}[Q] \subset \boldsymbol{Z}(Y) \cap \boldsymbol{Z}[M]$. And if $Z \in \boldsymbol{Z}(Y) - \boldsymbol{Z}[Q]$, then $X - Z \in \mathcal{E} \subset \boldsymbol{Z}[M]$, so that $Z \notin \boldsymbol{Z}[M]$.

3. Sufficient conditions for realization of quotient field

We have just ascertained that in order to realize the quotient field of $C(Y)/Q$, it is necessary that the prime ideal Q be a z-ideal. It will be seen in §4 that this condition is not sufficient. Here, we present a condition that is sufficient (Theorem 3.2). Although it may appear to be somewhat forced, many of its specializations are quite natural. The condition leads to all examples that we have been able to discover.

The space X that provides the residue class field turns out to be a subspace of Y, and the relevant mapping is the identity map of X into Y, which we shall denote by ι. Thus, ι' is simply the restriction mapping $g \to g \mid X$ from $C(Y)$ into $C(X)$, and

$$\iota'^{\leftarrow}[M] = \{g \in C(Y): g \mid X \in M\}.$$

We recall that a subspace S of a space T is said to be *C^*-embedded* in T if every bounded continuous function on S has a continuous extension to T.

LEMMA 3.1.[5] *Let $S \subset Y$, and let $A = \boldsymbol{Z}(h)$ be a zero-set in Y containing $Y - S$. Then for every $f \in C^*(S)$, $f \cdot (h \mid S)$ has an extension to a function $g \in C(Y)$ such that $\boldsymbol{Z}(g) = A \cup \boldsymbol{Z}(f)$.*

Hence, for every $Z \in \boldsymbol{Z}(S)$, $A \cup Z \in \boldsymbol{Z}(Y)$.

Proof. Define

$$g(s) = f(s)h(s) \quad \text{for} \quad s \in S,$$
$$g(y) = 0 \qquad \text{for} \quad y \in Y - S.$$

Then g is an extension of $f \cdot (h \mid S)$.

[5] This result is a generalization of [3, Lemma 2.1] and [1, 3C].

Clearly, g is continuous at each point of the open set $Y - \boldsymbol{Z}(h)$.

For $a \in \boldsymbol{Z}(h)$, we have $g(a) = 0 = h(a)$. Given $\varepsilon > 0$, choose a neighborhood V of a such that $| h(y) | < \varepsilon/n$ for all $y \in V$, where n is a bound for $| f |$. Then for $y \in V - S$, $g(y) = 0$, and for $s \in V \cap S$,

$$| g(s) | = | f(s) | \cdot | h(s) | < \varepsilon.$$

Remark. In the special case $A = Y - S$, we have: Every zero-set in S is the intersection with S of a zero-set in Y:

$$Z = (A \cup Z) \cap S.$$

THEOREM 3.2. *Let X be a subspace of Y, and let M be a maximal ideal in $C(X)$. If some member of $\boldsymbol{Z}_X[M]$ is C^*-embedded in the complement of a zero-set in Y, then $C(X)/M$ is a realization of the quotient field of $C(Y)/\iota'^{\leftarrow}[M]$.*

Proof. According to Lemma 2.1, the mapping ι'^* is an isomorphism from $C(Y)/\iota'^{\leftarrow}[M]$ into $C(X)/M$. We are to prove that every element of $C(X)/M$ is the quotient of two elements in the image under the mapping. Stated in terms of the behavior of functions, the problem is this: given $f \in C(X)$, to find g_1, $g_2 \in C(Y)$ such that

$$M(f) = M(g_1 | X)/M(g_2 | X).$$

In case $| M(f) |$ is not infinitely large, we may assume that f is bounded on X. By hypothesis, there exist $Z \in \boldsymbol{Z}_X[M]$ and $h \in C(Y)$ such that Z is C^*-embedded in $Y - \boldsymbol{Z}(h)$. The restriction $f | Z$ then has a bounded, continuous extension f_1 defined on $Y - \boldsymbol{Z}(h)$. By the lemma, there exists $g \in C(Y)$ such that $f_1 \cdot (h | [Y - \boldsymbol{Z}(h)]) = g | [Y - \boldsymbol{Z}(h)]$. It follows that

$$(f | Z) \cdot (h | Z) = g | Z.$$

Now, equality of functions in $C(X)$ on the zero-set Z of M implies congruence modulo M; therefore

$$M(f) \cdot M(h | X) = M(g | X).$$

Since $Z \subset Y - \boldsymbol{Z}_Y(h)$, we have $Z \cap \boldsymbol{Z}_X(h | X) = \emptyset$, so that $\boldsymbol{Z}_X(h | X) \notin \boldsymbol{Z}[M]$. Thus, $M(h | X) \neq 0$, and therefore $M(f) = M(g | X)/M(h | X)$.

In case $| M(f) |$ is infinitely large, we first represent $1/M(f)$ as such a quotient, and then take its reciprocal.

Remark. The hypothesis that Z is C^*-embedded in $Y - \boldsymbol{Z}(h)$ was not used in full force: we needed only the fact that those bounded functions on Z that have continuous extensions to X also have continuous extensions to $Y - \boldsymbol{Z}(h)$.

The most noteworthy application of the foregoing theorem is to prime z-ideals that have immediate successors (under set inclusion) in the class of all (prime) z-ideals.

LEMMA 3.3. *Let $\mathfrak{Q}$ be a prime z-filter on Y having an immediate successor in the class of all z-filters. Let A belong to this immediate successor but not to $\mathfrak{Q}$ itself, and define $X = Y - A$. Then the trace of $\mathfrak{Q}$ on X is a z-ultrafilter on X.*

Proof. Since $A \notin \mathfrak{Q}$, each member of $\mathfrak{Q}$ meets X; hence the trace $\mathfrak{Q} \mid X$ of $\mathfrak{Q}$ on X is closed under finite intersection. To prove that $\mathfrak{Q} \mid X$ is a z-ultrafilter, we show that if E belongs to $\boldsymbol{Z}(X)$ but not to $\mathfrak{Q} \mid X$, then E fails to meet some member of $\mathfrak{Q} \mid X$. By Lemma 3.1, $E \cup A \in \boldsymbol{Z}(Y)$; and by hypothesis, $E \cup A \notin \mathfrak{Q}$. Now A does belong to every z-filter that contains $\mathfrak{Q}$ properly; in particular, it belongs to the z-filter generated by $\mathfrak{Q}$ together with $E \cup A$. Hence, there exists $F \in \mathfrak{Q}$ such that $A \supset F \cap (E \cup A)$. Thus, $F \cap E \subset A \cap E = \emptyset$. The set $F \cap X$ is then a member of $\mathfrak{Q} \mid X$ that does not meet E.

THEOREM 3.4. *Let Q be a prime z-ideal in $C(Y)$ having an immediate successor in the class of all z-ideals. Then there exist a subspace X of Y and a maximal ideal M in $C(X)$ such that $C(X)/M$ is a realization of the quotient field of $C(Y)/Q$.*

Proof. Applying the lemma to $\boldsymbol{Z}[Q]$, let M be the maximal ideal in $C(X)$ for which $\boldsymbol{Z}[Q] \mid X = \boldsymbol{Z}_X[M]$. Since Q and M are z-ideals, we have $Q = \iota'^{\leftarrow}[M]$. Theorem 3.2 is now applicable because the member X of $\boldsymbol{Z}_X[M]$ is C^*-embedded in $X = Y - A$.

Many particular instances of a prime z-ideal with an immediate successor are known. Most of them are found by reversing the procedure of Lemma 3.3, as follows.

THEOREM 3.5. *Given $A \in \boldsymbol{Z}(Y)$, let $X = Y - A$, and let $\mathfrak{M}$ be a z-ultrafilter on X each member of which has a limit point in A. Then*

$$\mathfrak{Q} = \{Z \in \boldsymbol{Z}(Y) : Z \cap X \in \mathfrak{M}\}$$

is a prime z-filter on Y, and $\mathfrak{Q}$ has an immediate successor, namely, the z-filter generated by $\mathfrak{Q}$ together with A.

Furthermore, if

$$M = \{f \in C(X) : \boldsymbol{Z}(f) \in \mathfrak{M}\}$$

and

$$Q = \iota'^{\leftarrow}[M] = \{g \in C(Y) : \boldsymbol{Z}(g) \in \mathfrak{Q}\},$$

then $C(X)/M$ is a realization of the quotient field of $C(Y)/Q$.

Proof. It is easy to check that $\mathfrak{Q}$ is a prime z-filter. (In the notation of [1, 4.12], $\mathfrak{Q} = \iota^{\#}\, \mathfrak{M}$.)

The hypothesis implies that $\mathfrak{Q} \cup \{A\}$ has the finite intersection property, and so it generates a z-filter. To see that this is the immediate successor of $\mathfrak{Q}$, consider any $F \in \boldsymbol{Z}(Y)$ that meets every member of $\mathfrak{Q}$ but does not belong to $\mathfrak{Q}$. Then $F \cap X \notin \mathfrak{M}$, and since $\mathfrak{M}$ is a z-ultrafilter, there exists $E \in \mathfrak{M}$ such that $E \cap F = \emptyset$. Now $E \cup A$, which is a zero-set in Y (Lemma 3.1), belongs to $\mathfrak{Q}$. Therefore, $F \cap (E \cup A) = F \cap A$ belongs to the z-filter generated by $\mathfrak{Q}$ together with F. Consequently, A also belongs to this z-filter.

The final statement in the theorem follows from Theorem 3.2.

Remark. A z-ultrafilter $\mathfrak{M}$ satisfying the hypothesis of the theorem will exist precisely when X and A are not separated.

COROLLARY 3.6. *Let $\mathfrak{Q}_0$ be a prime z-filter on Y, let $A \in \boldsymbol{Z}(Y) - \mathfrak{Q}_0$ meet every member of $\mathfrak{Q}_0$, and define $X = Y - A$. Then $\mathfrak{Q}_0$ has an immediate successor, namely, the z-filter generated by $\mathfrak{Q}_0$ together with A, if and only if the trace of $\mathfrak{Q}_0$ on X is a z-ultrafilter on X.*

Proof. The necessity is stated in Lemma 3.3. For the sufficiency, define $\mathfrak{Q}$ as in Theorem 3.5: the largest z-filter on Y whose trace on X is $\mathfrak{M} = \mathfrak{Q}_0 \mid X$. Then $\mathfrak{Q} \supset \mathfrak{Q}_0$. On the other hand, for given $Z \in \mathfrak{Q}$, there exists $Z_0 \in \mathfrak{Q}_0$ such that $Z \cap X = Z_0 \cap X$; then $Z \cup A = Z_0 \cup A \in \mathfrak{Q}_0$; since $A \notin \mathfrak{Q}_0$ and $\mathfrak{Q}_0$ is prime (3.5), we have $Z \in \mathfrak{Q}_0$. Thus, $\mathfrak{Q}_0 = \mathfrak{Q}$. Theorem 3.5 now yields the desired result.

This corollary is a generalization of [3, Theorem 2.2(d)]. In the case treated there, A consists of a single point, so that the immediate successor of $\mathfrak{Q}_0$, when it exists, is a *z-ultrafilter* on Y.

The common occurrence of prime z-ideals having an immediate successor shows up from another point of view.

THEOREM 3.7. *If Q_1 and Q_2 are prime z-ideals, with Q_1 contained properly in Q_2, then there exists a prime z-ideal Q having an immediate successor in the class of all z-ideals and satisfying $Q_1 \subset Q \subset Q_2$.*

Proof. Choose any $h \in Q_2 - Q_1$, and let Q be the union of the chain of all z-ideals containing Q_1 but not h. The immediate successor of Q is the intersection of the chain of all z-ideals containing both Q_1 and h.

Another special case of Theorem 3.2 that has noteworthy applications arises when X itself is C^*-embedded in Y, and, in particular, when $Y = \beta X$.

THEOREM 3.8. *If M is a maximal ideal in $C(X)$, then $C(X)/M$ is a realization of the quotient field of $C(\beta X)/\iota'^{\leftarrow}[M]$.*

Because of the well-known isomorphism between $C(\beta X)$ and the ring $C^*(X)$ of all *bounded* continuous functions on X, this theorem may be restated in terms of C^*. The isomorphism in question is simply the restriction mapping ι'; and for any subset B of $C(X)$, the subset of $C^*(X)$ corresponding to $\iota'^{\leftarrow}[B]$ is $B \cap C^*(X)$. The next theorem includes the restatement of the preceding one along with a theorem of Kohls [2, Theorem 2.5].

THEOREM 3.9. *The mapping*

$$P \to P \cap C^*(X)$$

is one-one from the prime ideals in $C(X)$ *into* (*in general, not onto*) *the prime ideals in* $C^*(X)$. *The residue class ring* $C^*/(P \cap C^*)$ *is isomorphic with the subring of* C/P *obtained by discarding all elements with infinitely large absolute value. The rings* C/P *and* $C^*/(P \cap C^*)$ *have isomorphic quotient fields. In particular, for any maximal ideal* M *in* C, C/M *is a realization of the quotient field of* $C^*/(M \cap C^*)$.

Proof. For any $f \in C$, the function $(-1 \vee f) \wedge 1$ belongs to precisely the same ideals as does f. This implies that an ideal in C is determined by its bounded members. Thus, $P_1 \cap C^* = P_2 \cap C^*$ if and only if $P_1 = P_2$.

The description of the copy of $C^*/(P \cap C^*)$ in C/P is due to Kohls. It reflects the fact that if $|P(f)| < r$ (where $r \in \mathbf{R}$), then the bounded function $(-\boldsymbol{r} \vee f) \wedge \boldsymbol{r}$ is congruent to f modulo P.

To see that the quotient field of the copy of $C^*/(P \cap C^*)$ in C/P is the same as that of C/P, recall first that the unique maximal ideal in C/P is a symmetric interval [1, 14.3] that excludes the element 1. It follows that if $a \in C/P$ and $|a|$ is infinitely large, then a^{-1} exists in C/P, and it belongs to the subring. The quotient field of the latter contains $a = 1/a^{-1}$, and hence contains all of C/P.

We return, now, to the ring $C(\beta X)$. If $\boldsymbol{M}^p$ (see (1a)) is a maximal ideal in $C(X)$, with $p \in \beta X - X$, then Theorem 3.8 asserts that $C(X)/\boldsymbol{M}^p$ is a realization of the quotient field of $C(\beta X)/\iota'^{\leftarrow}[\boldsymbol{M}^p]$. Sometimes, the same conclusion can be inferred from Theorem 3.5. To do so requires the additional hypothesis that $\beta X - X$ be a zero-set in βX—which is equivalent to the requirement that X be locally compact and σ-compact. When this requirement is met, we can conclude further that the prime z-ideal $\iota'^{\leftarrow}[\boldsymbol{M}^p]$ has an immediate successor in the class of all z-ideals in $C(\beta X)$. Otherwise, the question of existence of an immediate successor is left open. The simplest possibility of such a prime z-ideal without an immediate successor appears to be the case where X is an uncountable discrete space and every member of $\boldsymbol{Z}_X[\boldsymbol{M}^p]$ is uncountable.

In case the prime z-ideal $\iota'^{\leftarrow}[\boldsymbol{M}^p]$ does have an immediate successor, there arises the question of whether this immediate successor is the maximal ideal containing $\iota'^{\leftarrow}[\boldsymbol{M}^p]$. That maximal ideal can be denoted unambiguously by $\boldsymbol{M}_p$: it consists of all functions in $C(\beta X)$ that vanish at p. For the space $X = \mathbf{N}$, we can give a complete answer to the question.

We recall that a point $x \in X$ is a *P-point* of X provided that every function in $C(X)$ is constant on a neighborhood of x.

THEOREM 3.10. *For any point* $p \in \beta\mathbf{N} - \mathbf{N}$, *the prime z-ideal* $Q = \iota'^{\leftarrow}[\boldsymbol{M}^p]$ *in* $C(\beta\mathbf{N})$ *has an immediate successor. The latter is the maximal ideal* $\boldsymbol{M}_p$ *in* $C(\beta\mathbf{N})$ *if and only if* p *is a P-point of* $\beta\mathbf{N} - \mathbf{N}$.

Proof. Because $\mathbf{N}$ is a discrete space, the prime z-filter $\boldsymbol{Z}[Q]$ consists of all zero-sets in $\beta\mathbf{N}$ that are neighborhoods of p.[6] Since $\beta\mathbf{N} - \mathbf{N}$ is a zero-set in $\beta\mathbf{N}$, Theorem 3.5 asserts that $\boldsymbol{Z}[Q]$ has an immediate successor; it is, in fact, the z-filter generated by $\boldsymbol{Z}[Q]$ together with $\beta\mathbf{N} - \mathbf{N}$ and consists of all zero-sets in $\beta\mathbf{N}$ that meet $\beta\mathbf{N} - \mathbf{N}$ in a relative neighborhood of p. By definition of P-point, this family of zero-sets is all of $\boldsymbol{Z}[\boldsymbol{M}_p]$ if and only if p is a P-point of $\beta\mathbf{N} - \mathbf{N}$.

It is known that some points of $\beta\mathbf{N} - \mathbf{N}$ are not P-points and, under the continuum hypothesis, that some are [1, 6T, V]. Thus, the maximal ideal $\boldsymbol{M}_p$ is the immediate successor of the prime ideal $\iota'^{\leftarrow}[\boldsymbol{M}^p]$ for some, but not all p in $\beta\mathbf{N} - \mathbf{N}$. We remark that $\boldsymbol{M}^p$ is hyper-real for every $p \in \beta\mathbf{N} - \mathbf{N}$ (see, e.g., [1, 5.10]). In contrast, suppose that we form Y by adding to any space X just *one* point p from $\beta X - X$ for which $\boldsymbol{M}^p$ is hyper-real. Then $\{p\}$ is a zero-set in Y,[3] and so, by Theorem 3.5, $\iota'^{\leftarrow}[\boldsymbol{M}^p]$ necessarily has an immediate successor among the z-ideals in $C(Y)$ (namely, the maximal ideal $\boldsymbol{M}_p$).

4. Quotient fields not isomorphic with any residue class field

We have succeeded in finding prime z-ideals Q for which the quotient field of C/Q can be realized as a residue class field. Now we turn to prime z-ideals Q for which the quotient field of C/Q cannot be isomorphic with *any* residue class field of a ring of functions.

First, we review some facts about such a residue class field. It is totally ordered;[2] moreover, the order is determined by the algebraic structure of the field, because an element is nonnegative if and only if it is a square.[7] The field contains a copy of the reals—specifically, the residue classes of the constants. When the field does not coincide with this copy, it is said to be *hyper-real* (and the corresponding maximal ideal is also termed *hyper-real*). Every hyper-real field is nonarchimedean; moreover, no element is the infimum of any countable set not containing it.[8]

LEMMA 4.1. *Suppose that a ring A has an ordered quotient field F, and that $a \in A$ and $a > 1$ implies $a^{-1} \in A$. Then the set of positive elements in A is coinitial in the set of positive elements of F.*

Proof. Given $c > 0$ in F, we have $c = b/a$ for suitable $a, b \in A$, with $a > 0$. If $a \geqq 1$, then $c = ba^{-1} \in A$; if $a < 1$, then $0 < b = ac < c$.

THEOREM 4.2. *The following conditions on a nonmaximal prime z-ideal Q in $C(Y)$ are equivalent:*

(1) *Q is a countable intersection of prime z-ideals different from Q.*
(2) *Q is a countable intersection of prime ideals different from Q.*
(3) *The set of positive elements of C/Q has a countable coinitial subset.*

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6 In the notation of [1, 7.12], $Q = \boldsymbol{O}_p$.
7 In fact, the field is real-closed [1, 13.4].
8 In fact, the field is an η_1-set [1, 13.8].

Furthermore, if Q satisfies these conditions, then the quotient field of C/Q is not isomorphic with any residue class field of any ring of continuous functions.

Proof. Evidently, (1) implies (2).

To show that (2) implies (3), let (Q_n) be a sequence of prime ideals whose intersection is Q. Choose g_n satisfying $\mathbf{0} \leqq g_n \in Q_n - Q$; we claim that the sequence $(Q(g_n))$ is coinitial in the set of positive elements of C/Q. Indeed, if $0 < a \in C/Q$, then for some n, a does not belong to the ideal Q_n/Q (in the ring C/Q). Since Q_n/Q is an interval in C/Q [1, 14.3], a exceeds every member of Q_n/Q. Thus, $0 < Q(g_n) < a$.

Next, we prove that if Q satisfies (3), then the quotient field of C/Q is not isomorphic with any residue class field of a ring of functions. First of all, since Q is not maximal, C/Q contains elements that are infinitely small [1, 7.16], and so its quotient field cannot be the real field. Now, by (3) and the lemma, the set of positive elements in the quotient field of C/Q has a countable coinitial subset. This implies that no hyper-real residue class field can be order-isomorphic with the quotient field of C/Q. Since the order in a residue class field is determined by its algebraic structure, a hyper-real field cannot be (ring) isomorphic with the quotient field either.

Finally, we prove that (3) implies (1). In view of what has just been proved, Theorem 3.4 shows that Q has no immediate successor in the class of prime z-ideals in C. Hence if $(Q(g_n))$ is a coinitial sequence of positive elements of C/Q, then for each n, there exists a prime z-ideal Q_n containing Q properly, but not containing g_n. Since the prime z-ideal $\bigcap_n Q_n$ contains none of the g_n, it must be Q.

Remark. The contrast between chains of prime z-ideals and chains of ordinary prime ideals shows up sharply in connection with condition (3). This condition is satisfied by *every* prime ideal with an immediate successor in the class of all *prime* ideals [1, Theorem 14.6], but by *no* prime *z-ideal* having an immediate successor in the class of all *z-ideals.*

The following example shows that the theorem is not vacuous.[9]

EXAMPLE 4.3. *A decreasing sequence of prime z-filters.* Let

$$Y = [0, 1]^{\mathbf{N}},$$

that is, a point y of Y is a sequence $(y_i)_{i \in \mathbf{N}}$, with $0 \leqq y_i \leqq 1$, and Y carries the product topology. We shall make use of the fact that Y is a metrizable space, so that, for example, every closed set in Y is a zero-set.

A strictly decreasing sequence of prime z-filters on Y will be defined inductively. First, we define $S_{-1} = \emptyset$ and

$$S_k = \{y \in Y\colon y_i = 0 \text{ for all } i > k\} \quad (k = 0, 1, 2, \cdots).$$

[9] As noted in Mathematical Reviews, the example suggested in [2, Example 4.2] is in error.

Let $\mathfrak{F}_0$ be the z-ultrafilter of all zero-sets containing the point (0). Fix $n \in \mathbf{N}$, and suppose that $\mathfrak{F}_k$ have been defined for $0 \leqq k < n$, satisfying

(a) $\mathfrak{F}_k$ is a prime z-filter on Y.
(b) $S_k \in \mathfrak{F}_k$, while $S_{k-1} \notin \mathfrak{F}_k$.
(c) The trace of $\mathfrak{F}_k$ on $S_k - S_{k-1}$ is a z-ultrafilter $\mathfrak{A}_k$ on $S_k - S_{k-1}$.
(d) $\mathfrak{F}_k \subset \mathfrak{F}_{k-1}$ for $k \geqq 1$.

Let $\mathcal{E}_n$ denote the family of all zero-sets in S_n whose interiors relative to S_n contain members of $\mathfrak{A}_{n-1}$. Then $S_n \in \mathcal{E}_n$, since S_n contains the member $S_{n-1} - S_{n-2}$ of $\mathfrak{A}_{n-1}$, and therefore $\mathcal{E}_n$ is not vacuous. In fact, it is easy to verify that $\mathcal{E}_n$ is a z-filter on S_n. Since every member of $\mathcal{E}_n$ has nonempty interior, and $S_n - S_{n-1}$ is dense in S_n, the trace of $\mathcal{E}_n$ on $S_n - S_{n-1}$ is again a z-filter. We embed the latter in a z-ultrafilter $\mathfrak{A}_n$ on $S_n - S_{n-1}$, and we then define $\mathfrak{F}_n$ to be the family of all zero-sets in Y that contain members of $\mathfrak{A}_n$.

(a). *$\mathfrak{F}_n$ is a prime z-filter on Y.* This is easy to verify (cf. Theorem 3.5).
(b). *$S_n \in \mathfrak{F}_n$, while $S_{n-1} \notin \mathfrak{F}_n$*, obviously.
(c). *The trace of $\mathfrak{F}_n$ on $S_n - S_{n-1}$ is $\mathfrak{A}_n$.* By definition of $\mathfrak{F}_n$, its trace is contained in $\mathfrak{A}_n$. On the other hand, if $A \in \mathfrak{A}_n$, then $\mathrm{cl}_Y A \in \mathfrak{F}_n$; and since A is closed in $S_n - S_{n-1}$, we have $A = \mathrm{cl}_Y A \cap (S_n - S_{n-1})$.
(d). $\mathfrak{F}_n \subset \mathfrak{F}_{n-1}$. For any $Z \in \boldsymbol{Z}(Y) - \mathfrak{F}_{n-1}$, we are to prove that $Z \notin \mathfrak{F}_n$. Since $Z \cap (S_{n-1} - S_{n-2})$ does not belong to the z-ultrafilter $\mathfrak{A}_{n-1}$, there exists $A \in \mathfrak{A}_{n-1}$ such that $Z \cap A = \emptyset$. Now A is closed in $S_{n-1} - S_{n-2}$, which in turn is closed in $S_n - S_{n-2}$. Hence A is closed in $S_n - S_{n-2}$; and it is disjoint from the relatively closed set $Z \cap (S_n - S_{n-2})$. Therefore, there exists a relatively open set G in $S_n - S_{n-2}$ such that G and its closure

$$F = \mathrm{cl}_{S_n - S_{n-2}} G$$

satisfy

$$A \subset G \subset F \subset S_n - Z.$$

Since G is open in the relative topology of S_n, the set $\mathrm{cl}_{S_n} F = \mathrm{cl}_{S_n} G$ belongs to the z-filter $\mathcal{E}_n$. Consequently,

$$F \cap (S_n - S_{n-1}) = \mathrm{cl}_{S_n} F \cap (S_n - S_{n-1}) \in \mathfrak{A}_n .$$

But $F \cap Z = \emptyset$. Therefore $Z \cap (S_n - S_{n-1}) \notin \mathfrak{A}_n$, and so $Z \notin \mathfrak{F}_n$.

This completes the induction.

The family $\mathfrak{F} = \bigcap_n \mathfrak{F}_n$ is a prime z-filter on Y having no immediate successor. The prime z-ideal $Q = \{g \in C(Y) : \boldsymbol{Z}(g) \in \mathfrak{F}\}$ satisfies the conditions of Theorem 4.2.

Remark. A possible method for constructing an uncountable, well-ordered, decreasing family of prime z-ideals is suggested by Theorem 4.2, as follows. Find a space X and a maximal ideal M in $C(X)$ such that the prime z-ideal $Q = \iota'^{\leftarrow}[M]$ is not maximal and has no immediate successor in the class of all

z-ideals in $C(\beta X)$. By Theorems 3.8 and 4.2, Q cannot be a countable intersection of z-ideals different from Q; an uncountable decreasing family of z-ideals containing Q properly can then be found inductively.

References

1. L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton, D. Van Nostrand Co., 1960.
2. C. W. Kohls, *Prime ideals in rings of continuous functions*, Illinois J. Math., vol. 2 (1958), pp. 505–536.
3. ———, *Prime ideals in rings of continuous functions. II*, Duke Math. J., vol. 25 (1958), pp. 447–458.

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