C^*-ALGEBRAS WITH SMOOTH DUAL

BY

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1. Introduction

One of the useful tools for studying the structure of a locally compact group or Banach *-algebra A is the "dual space" $\hat{A}$ of all its irreducible representations in Hilbert space (unitary in the case of groups, and involution-preserving in the case of *-algebras). This dual space has a natural "hull-kernel" topology, first defined by Jacobson [6] in a general algebraic context, and later studied for C^*-algebras by Kaplansky in [7], and for groups by Godement in [3]. It is shown in [2] that, in the case of groups and of C^*-algebras, there are two equivalent definitions of this topology having quite different appearance—one in terms of the kernels of the representations, and the other in terms of positive functionals (or functions of positive type, in the case of groups). One of the objects of this paper is to obtain yet another characterization of the hull-kernel topology of the dual of a C^*-algebra A. This is done roughly as follows: Let H be a Hilbert space of dimension large enough so that every element of $\hat{A}$ can be realized as a concrete representation acting in H (with perhaps a null space). One can introduce a natural topology (derived from the weak operator topology) into the set $\mathfrak{J}^0$ of all concrete irreducible representations of A in H; and this in turn induces a quotient topology in $\hat{A}$, regarded as the quotient space of $\mathfrak{J}^0$ modulo the relation of equivalence. We prove that the quotient topology and the hull-kernel topology of $\hat{A}$ are the same (Theorem 3.1).

But the main result of this paper is an application to Mackey's concept of a "smooth dual" (see [8]). Mackey has shown that there is a wide class of separable groups and C^*-algebras A whose representations can be decomposed as direct integrals with respect to a measure over $\hat{A}$ in the same natural and unique manner which we find in the case of compact and Abelian groups; this class consists of those groups and algebras which are (a) of Type I, and (b) have "smooth duals". The latter property means, roughly speaking, that $\hat{A}$ has a reasonably well-behaved structure of Borel sets. Now there are important classes of groups and algebras—for example, the CCR C^*-algebras (see [7]), the semisimple connected Lie groups (see [5]), and the nilpotent connected Lie groups (see [1])—which are known to be of Type I. But the questions of which groups and algebras have smooth duals is less well understood. We show in this paper (Theorem 4.1) that a separable C^*-algebra A has a smooth dual if $\hat{A}$ is a T_0-space, i.e., if no two distinct irreducible representations of A have the same kernel. It follows that all

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CCR C^*-algebras, and hence (see [4]) all semisimple connected Lie groups, have smooth duals.

Finally, we want to say here that it was Mackey's definition of the quotient Borel structure on the dual space in [8] that suggested to us the corresponding "quotient" definition of the hull-kernel topology.

2. The topology of concrete representations

Throughout this and the next section, A will be an arbitrary fixed C^*-algebra and H a fixed Hilbert space (of unrestricted dimension). By a *concrete representation* (*of A in H*) we mean a homomorphism T of A into the bounded operators on H which carries involution into the adjoint operation, and whose range contains something other than the zero operator. Such a representation is well known to be norm-continuous; in fact $\| T \| \leqq 1$. Denote by $\mathfrak{I}$ the family of all concrete representations.

If $T \in \mathfrak{I}$, let H^T be the closed linear span of the set of all $T_a \xi$ $(a \in A, \xi \in H)$; equivalently, H^T is the orthogonal complement of $\{\xi \in H \mid T_a \xi = 0$ for all a in $A\}$. We call H^T the *essential space* of T, and denote by P^T the projection onto H^T. If $H^T = H$, T is *nowhere trivial.*

Two concrete representations T and T' will be called *equivalent* $(T \sim T')$ if they are equivalent in the usual sense when restricted to their essential spaces, that is, if there exists a linear isometry S of $H^{T'}$ onto H^T such that $S^{-1}T_a S = T'_a$ for all a. They are *unitarily equivalent* $(T \cong T')$ if there exists a unitary U on H such that $U^{-1}T_a U = T'_a$ for all a. A concrete representation T is *irreducible* if there is no closed linear subspace of H, invariant under T, lying properly between H^T and $\{0\}$.

$\mathfrak{I}$ will now receive a topology. For each T in $\mathfrak{I}$, let $N(T)$ be the family of all intersections of finitely many sets each of which is either of the form

$$M_{a,\xi,\eta,\delta}(T) = \{S \in \mathfrak{I} \mid |(S_a \xi, \eta) - (T_a \xi, \eta)| < \delta\}$$

or of the form

$$P_{\xi,\delta}(T) = \{S \in \mathfrak{I} \mid \| P^S \xi - \xi \| < \delta\},$$

where $a \in A$, $\xi, \eta \in H^T$, and $\delta > 0$. The $N(T)$ satisfy the axioms for a system of neighborhoods in a topology for $\mathfrak{I}$; the verification of this is easy and will be omitted. We always consider $\mathfrak{I}$ as equipped with this topology. A net $\{T^i\}$ of elements of $\mathfrak{I}$ converges to an element T of $\mathfrak{I}$ if and only if $(T^i_a \xi, \eta) \to_i (T_a \xi, \eta)$ and $\| P^{T^i}\xi - \xi \| \to_i 0$ for all a in A, and all ξ, η in H^T. In case A has a unit element 1, then $P_{\xi,\delta}(T) = M_{1,\xi,\xi,\delta^2}(T)$, so that the $P_{\xi,\delta}(T)$ can be omitted from the description of the topology.

It is easy to see that $\mathfrak{I}$ is a T_0-space (i.e., open sets separate points) but need not be a T_1-space (i.e., points need not be closed). However, the subspace of nowhere trivial representations, with the relativized topology, is a Hausdorff space; in fact, its topology is the smallest for which all the functions $T \to (T_a \xi, \eta)$ $(a \in A, \ \xi, \eta \in H)$ are continuous.

How small a cardinality can a base for the open subsets of $\mathfrak{T}$ have? Suppose that D_A and D_H are dense subsets of A and H respectively, that B is a countable base for the open subsets of the complex plane, and that E is the set of all positive rationals. We verify without difficulty that finite intersections of sets of the form $\{T \in \mathfrak{T} \mid (T_a\,\xi, \eta) \in W\}$ and $\{T \in \mathfrak{T} \mid \| P^T\xi - \xi \| < \delta\}$ (where $a \in D_A$, $\xi, \eta \in D_H$, $W \in B$, and $\delta \in E$) form a base for the open sets in $\mathfrak{T}$. From a simple reckoning with cardinals we now get the following lemma:

LEMMA 2.1. *Let α be the smallest cardinal number of a dense subset of A, β the smallest cardinal number of a dense subset of H, and γ the larger of α and β. Then $\mathfrak{T}$ has a base for its open sets of cardinality no greater than γ.*

Our next concern is the connection between the topology of $\mathfrak{T}$ and the relations of equivalence and unitary equivalence. If $\mathcal{S} \subset \mathfrak{T}$, let

$$\mathcal{S}^e = \{T \in \mathfrak{T} \mid T \sim S \quad \text{for some } S \text{ in } \mathcal{S}\},$$

$$\mathcal{S}^u = \{T \in \mathfrak{T} \mid T \cong S \quad \text{for some } S \text{ in } \mathcal{S}\},$$

and let $\mathrm{Cl}\,\mathcal{S}$ be the closure of $\mathcal{S}$ in $\mathfrak{T}$. Evidently $\mathcal{S} \subset \mathcal{S}^u \subset \mathcal{S}^e$.

LEMMA 2.2. $(\mathrm{Cl}\,\mathcal{S})^e \subset \mathrm{Cl}\,\mathcal{S}^u$.

Proof. Let T be in $(\mathrm{Cl}\,\mathcal{S})^e$. Pick $a_1, \cdots, a_n$ in A, $\xi_1', \cdots, \xi_n', \eta_1', \cdots, \eta_n'$ in H^T, and $\delta > 0$. We shall prove that there exists an R in $\mathcal{S}^u$ such that, for all i,

$$(1) \qquad R \in M_{a_i,\xi_i',\eta_i',\delta}(T), \qquad R \in P_{\xi_i',\delta}(T).$$

Let S be an element of $\mathrm{Cl}\,\mathcal{S}$ such that $T \sim S$. Then there is a linear isometry F of H^T onto H^S such that

$$(2) \qquad FT_a\,\xi = S_a\,F\xi \qquad (a \in A,\ \xi \in H^T).$$

Define $\xi_i = F\xi_i'$, $\eta_i = F\eta_i'$. Since $S \in \mathrm{Cl}\,\mathcal{S}$, there is a Q in $\mathcal{S}$ belonging to all $M_{a_i,\xi_i,\eta_i,\delta}(S)$ and $P_{\xi_i,\delta}(S)$. Now there exists a unitary operator U on H which coincides with F on the (finite-dimensional) space spanned by the ξ_i', η_i', and $T_{a_i}\,\xi_i'$ ($i = 1, \cdots, n$). By (2),

$$(3) \qquad U\xi_i' = \xi_i, \qquad U\eta_i' = \eta_i, \qquad UT_{a_i}\,\xi_i' = S_{a_i}\,\xi_i.$$

Let us define R by $R_a = U^{-1}Q_a\,U$; then $R \in \mathcal{S}^u$. We have by (2) and (3)

$$|(R_{a_i}\,\xi_i', \eta_i') - (T_{a_i}\,\xi_i', \eta_i')| = |(Q_{a_i}\,\xi_i, \eta_i) - (S_{a_i}\,\xi_i, \eta_i)| < \delta,$$

and

$$\| P^R\xi_i' - \xi_i' \| = \| P^Q\xi_i - \xi_i \| < \delta,$$

so that (1) is proved.

Now (1), together with the arbitrariness of the a_i, ξ_i', η_i', δ, shows that $T \in \mathrm{Cl}\,\mathcal{S}^u$. Since T was an arbitrary element of $(\mathrm{Cl}\,\mathcal{S})^e$, the lemma is proved.

COROLLARY 1. *If* $\mathcal{S} \subset \mathfrak{I}$, $(\mathrm{Cl}\,\mathcal{S})^e \subset \mathrm{Cl}\,\mathcal{S}^e$, *and* $(\mathrm{Cl}\,\mathcal{S})^u \subset \mathrm{Cl}\,\mathcal{S}^u$.

COROLLARY 2. *If* $\mathcal{S} \subset \mathfrak{I}$, $\mathrm{Cl}\,\mathcal{S}^e = \mathrm{Cl}\,\mathcal{S}^u$.

Proof. Replacing $\mathcal{S}$ by $\mathcal{S}^u$ in the lemma, we get

$$\mathcal{S}^e \subset (\mathrm{Cl}\,\mathcal{S}^u)^e \subset \mathrm{Cl}\,(\mathcal{S}^u)^u = \mathrm{Cl}\,\mathcal{S}^u.$$

Hence, by taking closures, $\mathrm{Cl}\,\mathcal{S}^e \subset \mathrm{Cl}\,\mathcal{S}^u$. The reverse inclusion is obvious.

COROLLARY 3. *Let* $\mathcal{S}' \subset \mathcal{S} \subset \mathfrak{I}$, *where* $\mathcal{S}'$ *is open relative to* $\mathcal{S}$. *If* $\mathcal{S} = \mathcal{S}^u$, *then* $\mathcal{S}'^u$ *is open relative to* $\mathcal{S}$; *if* $\mathcal{S} = \mathcal{S}^e$, *then* $\mathcal{S}'^e$ *is open relative to* $\mathcal{S}$.

Proof. Assume $\mathcal{S} = \mathcal{S}^u$. By Corollary 1,

$$(\mathrm{Cl}\,(\mathcal{S} - \mathcal{S}'^u))^u \subset \mathrm{Cl}\,(\mathcal{S} - \mathcal{S}'^u)^u = \mathrm{Cl}\,(\mathcal{S} - \mathcal{S}'^u) \subset \mathrm{Cl}\,(\mathcal{S} - \mathcal{S}').$$

Since $\mathrm{Cl}(\mathcal{S} - \mathcal{S}') \cap \mathcal{S}' = \Lambda$, this gives

$$(\mathrm{Cl}\,(\mathcal{S} - \mathcal{S}'^u))^u \cap \mathcal{S}' = \Lambda.$$

But this implies that

$$\mathrm{Cl}(\mathcal{S} - \mathcal{S}'^u) \cap \mathcal{S}'^u = \Lambda;$$

that is, $\mathcal{S}'^u$ is open relative to $\mathcal{S}$.

The other part of the corollary is proved similarly.

COROLLARY 4. *If* $\mathcal{S}$ *is an open subset of* $\mathfrak{I}$, *then* $\mathcal{S}^e$ *and* $\mathcal{S}^u$ *are open.*

COROLLARY 5. *If* $\mathcal{S} \subset \mathfrak{I}$ *and* $\mathcal{S} = \mathcal{S}^u$, *then*

$$(\mathrm{Cl}\,\mathcal{S})^u = (\mathrm{Cl}\,\mathcal{S})^e = \mathrm{Cl}\,\mathcal{S}.$$

Proof. Using the lemma yields

$$\mathrm{Cl}\,\mathcal{S} \subset (\mathrm{Cl}\,\mathcal{S})^u \subset (\mathrm{Cl}\,\mathcal{S})^e \subset \mathrm{Cl}\,\mathcal{S}^u = \mathrm{Cl}\,\mathcal{S}.$$

3. The quotient topology

If $T \in \mathfrak{I}$, let Eq T be the equivalence class of the relation $\sim$ to which T belongs. If $\mathcal{S} \subset \mathfrak{I}$, Eq $\mathcal{S}$ will be the set of all Eq T, where $T \in \mathcal{S}$; in particular, Eq $\mathfrak{I}$ is the set of all equivalence classes Eq T. We topologize Eq $\mathfrak{I}$ with the quotient topology obtained from $\mathfrak{I}$: A subset W of Eq $\mathfrak{I}$ is *open* if and only if $\{T \in \mathfrak{I} \mid \mathrm{Eq}\,T \in W\}$ is open. By Corollary 4 of Lemma 2.2, the natural map $T \to \mathrm{Eq}\,T$ is open and continuous on $\mathfrak{I}$.

If $\mathcal{S} \subset \mathfrak{I}$, we can equip Eq $\mathcal{S}$ with a topology in two natural ways, either (i) by relativizing the topology of Eq $\mathfrak{I}$ to Eq $\mathcal{S}$, or (ii) by relativizing the topology of $\mathfrak{I}$ to $\mathcal{S}$, then taking the quotient topology of Eq $\mathcal{S}$ obtained from the relativized topology of $\mathcal{S}$. Clearly topology (i) is contained in topology (ii), but in general they are not equal.

LEMMA 3.1. *If* $\mathcal{S} = \mathcal{S}^u$, *topologies* (i) *and* (ii) *for* Eq $\mathcal{S}$ *coincide.*

Proof. We will show that in this case topology (ii) is contained in topology (i). Let W be a subset of Eq $\mathcal{S}$ closed in topology (ii); and put

$W' = \{T \in \mathcal{S} \mid \text{Eq } T \in W\}$. Then W' is closed relative to $\mathcal{S}$, i.e.,

$$(4) \qquad (\text{Cl } W') \cap \mathcal{S} = W'.$$

Put $Z = (\text{Cl } W')^e$. Since $\mathcal{S} = \mathcal{S}^u$, we have also $W' = W'^u$; so, by Corollary 4 of Lemma 2.2, $Z = \text{Cl } W'$. This and (4) give $Z \cap \mathcal{S} = W'$; whence

$$Z \cap \mathcal{S}^e = (W')^e.$$

This equation, together with the fact that Z is closed and $Z = Z^e$, shows that W ($= \text{Eq } W'$) is closed in Eq $\mathcal{S}$ in topology (i). Thus topology (i) contains topology (ii).

If $\mathcal{S} \subset \mathfrak{I}$, $\mathcal{S} = \mathcal{S}^u$, either of the two identical topologies (i) or (ii) will be called the *quotient topology* of Eq $\mathcal{S}$.

COROLLARY. *If $\mathcal{S} \subset \mathfrak{I}$, $\mathcal{S} = \mathcal{S}^u$, and $T \cong T'$ whenever T and T' are in $\mathcal{S}$ and $T \sim T'$, then the map $T \to$ Eq T is open on $\mathcal{S}$; that is, if $\mathcal{S}'$ is a subset of $\mathcal{S}$ open relative to $\mathcal{S}$, then* Eq $\mathcal{S}'$ *is open relative to* Eq $\mathcal{S}$.

Proof. Let $\mathcal{S}'$ be a subset of $\mathcal{S}$ open relative to $\mathcal{S}$. By Lemma 3.1, we need only show that $\mathcal{S}'^e \cap \mathcal{S}$ is open relative to $\mathcal{S}$. But, by hypothesis, $\mathcal{S}'^e \cap \mathcal{S} = \mathcal{S}'^u$; and this is open relative to $\mathcal{S}$ by Corollary 3 of Lemma 2.2.

In a recent paper [2] we discussed the so-called hull-kernel topology for the dual space $\hat{A}$ (i.e., the set of all unitary equivalence classes of irreducible *-representations) of A. In order to describe it, we remind the reader that a nonzero positive functional ϕ on A is *associated with* a *-representation T of A if, for some ξ in the space of T, $\phi(a) = (T_a \xi, \xi)$ for all a in A. Let $W \subset \hat{A}$, $T \in \hat{A}$. It is proved in [2] that T belongs to the hull-kernel closure of W if and only if some positive functional ϕ on A associated with T is a weak* limit of positive functionals ϕ_ν, each of which is associated with some S in W, and such that $\| \phi_\nu \| = \| \phi \|$. For the purpose of the present paper, this may be taken as the definition of the hull-kernel topology of $\hat{A}$.

Let us denote by $\mathfrak{I}^0$ the family of all irreducible concrete representations of A in H. Then Eq $\mathfrak{I}^0$ can be identified with the subset of $\hat{A}$ consisting of those elements whose dimensions are equal to or less than that of H; and the hull-kernel topology relativized to Eq $\mathfrak{I}^0$ will be called the *hull-kernel topology* of Eq $\mathfrak{I}^0$.

THEOREM 3.1. *The hull-kernel and quotient topologies of* Eq $\mathfrak{I}^0$ *coincide.*

Proof. Part I. Let $\mathcal{S} \subset \mathfrak{I}^0$, $\mathcal{S} = \mathcal{S}^e$, Eq $\mathcal{S} = W$, and assume W closed in the hull-kernel topology of Eq $\mathfrak{I}^0$. If $T \in (\text{Cl } \mathcal{S}) \cap \mathfrak{I}^0$, there exists a net of elements $\{T^i\}$ of $\mathcal{S}$ with $T^i \to T$; so, if $0 \neq \xi \in H^T$,

$$\phi_i(a) = (T^i_a \xi, \xi) \to (T_a \xi, \xi) = \phi(a)$$

for all a, where ϕ and ϕ_i are associated with T and T^i respectively. But this implies that Eq $T^i \to$ Eq T in the hull-kernel topology (see [2]), so that Eq $T \in W$, $T \in \mathcal{S}$. Thus $\mathcal{S}$ is closed relative to $\mathfrak{I}^0$, which implies that W is closed in the quotient topology of Eq $\mathfrak{I}^0$. Since W was an arbitrary subset of Eq $\mathfrak{I}^0$ closed in the hull-kernel topology, we have shown that the hull-kernel topology is contained in the quotient topology of Eq $\mathfrak{I}^0$.

Part II. Let W be a subset of Eq $\mathfrak{I}^0$ closed in the quotient topology; we will show that W is closed in the hull-kernel topology of Eq $\mathfrak{I}^0$. Define $\mathcal{S} = \{T \in \mathfrak{I}^0 \mid \text{Eq } T \in W\}$.

If A has a unit, let $A_1 = A$; if not, let A_1 be the C^*-algebra obtained by adjoining a unit 1 to A. Each T in $\mathfrak{I}$ can be extended to a concrete representation T^1 of A_1 by setting $T_1^1 = P^T$. Let S be an element of $\mathfrak{I}^0$ such that Eq S belongs to the hull-kernel closure of W. Then it is known (see for example Lemma 1.8 of [2]) that the equivalence class of S^1 belongs to the hull-kernel closure of $W^1 = \{\text{Eq } T^1 \mid \text{Eq } T \in W\}$. So, by our description of the hull-kernel topology, there is a net $\{T^\lambda\}$ of elements of $\mathcal{S}$, a vector ξ in H^T, and, for each λ, a vector ξ^λ in H^{T^λ} such that

$$\| \xi \| = \| \xi^\lambda \| = 1,$$

$$(T_a^\lambda \xi^\lambda, \xi^\lambda) \to (S_a \xi, \xi) \qquad \text{for all } a \text{ in } A_1. \tag{5}$$

(Here, for simplicity of notation, we have written S, T^λ instead of S^1, $(T^\lambda)^1$).

Let F be a fixed finite subset of A_1 containing 1; and put

$$m(F, \lambda) = \inf_U \max_{a \in F} \| T_a^\lambda \xi^\lambda - U S_a \xi \|,$$

where the infimum is taken over all unitary operators U on H. We shall prove that, for fixed F,

$$\lim_\lambda m(F, \lambda) = 0. \tag{6}$$

Indeed, let $\eta_1, \cdots, \eta_n$ be an orthonormal set of vectors spanning the same space as the $S_a \xi$, $a \in F$; and let r_i^a, s_a^i be complex numbers such that

$$\eta_i = \textstyle\sum_{a \in F} r_i^a S_a \xi, \qquad S_a \xi = \sum_{i=1}^n s_a^i \eta_i . \tag{7}$$

We define for each λ and each $i = 1, \cdots, n$

$$\zeta_i^\lambda = \textstyle\sum_{a \in F} r_i^a T_a^\lambda \xi^\lambda.$$

Then by (5) and (7)

$$\begin{aligned} (\zeta_i^\lambda, \zeta_j^\lambda) &= \textstyle\sum_{a,b \in F} r_i^a \overline{r_j^b} (T_{b^*a}^\lambda \xi^\lambda, \xi^\lambda) \\ &\to \textstyle\sum_{a,b \in F} r_i^a \overline{r_j^b} (S_{b^*a} \xi, \xi) \\ &= (\eta_i, \eta_j) = \delta_{ij} . \end{aligned} \tag{8}$$

Now

$$\begin{aligned} &\| \textstyle\sum_{i=1}^n s_a^i \zeta_i^\lambda - T_a^\lambda \xi^\lambda \|^2 \\ &\quad = \| \textstyle\sum_{i=1}^n s_a^i \zeta_i^\lambda \|^2 + \| T_a^\lambda \xi^\lambda \|^2 - 2 \operatorname{Re} \left(\sum_{i=1}^n s_a^i \zeta_i^\lambda, T_a^\lambda \xi^\lambda \right). \end{aligned}$$

Applying (5), (7), and (8), we see that the first, second, and third terms of the right side of this equation approach $\| S_a \xi \|^2$, $\| S_a \xi \|^2$, and $-2 \| S_a \xi \|^2$

respectively; so that, for any a in F,

$$(9)\qquad \| \textstyle\sum_{i=1}^{n} s_a^i \zeta_i^\lambda - T_a^\lambda \xi^\lambda \| \to_\lambda 0.$$

Now, in virtue of (8), the $\zeta_1^\lambda, \cdots, \zeta_n^\lambda$ are linearly independent for all large enough λ; in fact, one can choose for all large enough λ an orthonormal set of vectors $\pi_1^\lambda, \cdots, \pi_n^\lambda$ and complex coefficients w_{ij}^λ such that

$$\pi_i^\lambda = \textstyle\sum_{j=1}^{n} w_{ij}^\lambda \zeta_j^\lambda ,$$

and

$$(10)\qquad w_{ij}^\lambda \to_\lambda \delta_{ij} \qquad \text{for } i, j = 1, \cdots, n.$$

Let us define U^λ to be a unitary operator on H for which $U^\lambda \eta_i = \pi_i^\lambda$ for $i = 1, \cdots, n$. Then, by (7) and (10), for a in F,

$$\textstyle\sum_{i=1}^{n} s_a^i \zeta_i^\lambda - U^\lambda S_a \xi = \sum_{i=1}^{n} s_a^i(\zeta_i^\lambda - \pi_i^\lambda) \to_\lambda 0.$$

Combining this with (9), we have for a in F

$$(11)\qquad \| T_a^\lambda \xi^\lambda - U^\lambda S_a \xi \| \to_\lambda 0.$$

Putting $a = 1$ in (11), and combining the result with (11) again, we obtain

$$\|T_a^\lambda U^\lambda \xi - U^\lambda S_a \xi \| \to_\lambda 0,$$

or

$$(12)\qquad \| S_a^\lambda \xi - S_a \xi \| \to_\lambda 0,$$

where $S^\lambda \in \mathcal{S}$, $S_a^\lambda = (U^\lambda)^{-1} T_a^\lambda U^\lambda$.

Equation (12) shows that, for each finite subset F of A_1—and, in particular, of A—and each $\varepsilon > 0$, there is a T in $\mathcal{S}$ such that

$$(13)\qquad \| T_a \xi - S_a \xi \| < \varepsilon \qquad \text{for all } a \text{ in } F.$$

To show that $S \in \text{Cl}\,\mathcal{S}$, it must be proved that, for each $\xi_1, \cdots, \xi_r$ in H^S, each $a_1, \cdots, a_r$ in A_1, and each $\varepsilon > 0$, there is a T in $\mathcal{S}$ such that

$$(14)\qquad |(S_{a_i} \xi_i, \xi_i) - (T_{a_i} \xi_i, \xi_i)| < \varepsilon.$$

Since the $S_b \xi$ are dense in H^S, it is sufficient to show (14) under the assumption that $\xi_i = S_{b_i} \xi$, $b_i \in A$. But then (14) becomes

$$(15)\qquad | S_{b_i^* a_i b_i} \xi, \xi) - (T_{a_i} S_{b_i} \xi, S_{b_i} \xi)| < \varepsilon.$$

It is easy to see that (15) will be satisfied if we choose T in $\mathcal{S}$ so that the $\| T_{b_i^* a_i b_i} \xi - S_{b_i^* a_i b_i} \xi \|$ and $\| T_{b_i} \xi - S_{b_i} \xi \|$ are sufficiently small; and this is possible by (13).

Thus $S \in \text{Cl}\,\mathcal{S}$. Since $W = \text{Eq}\,\mathcal{S}$ is closed in the quotient topology of $\text{Eq}\,\mathfrak{J}^0$, $\mathcal{S}$ must be closed in $\mathfrak{J}^0$. Therefore $S \in \mathcal{S}$, or $\text{Eq}\,S \in W$. But $\text{Eq}\,S$ was an arbitrary element of the hull-kernel closure of W. Therefore W is closed in the hull-kernel topology of $\text{Eq}\,\mathfrak{J}^0$. This completes the proof.

Let us denote by $\mathfrak{J}^p$ the family of nowhere trivial representations in $\mathfrak{J}$. If

T, $T' \in \mathfrak{I}^p$, then $T \sim T'$ implies $T \cong T'$. Hence, combining Theorem 3.1 with Lemma 3.1 and its corollary, we get

COROLLARY. *Let $\hat{A}_H$ denote the family of those T in $\hat{A}$ whose dimension is equal to that of H; and equip $\hat{A}_H$ with the relativized hull-kernel topology. Then the natural map of $\mathfrak{I}^p \cap \mathfrak{I}^0$ onto $\hat{A}_H$ is continuous and open.*

THEOREM 3.2. *If α is the smallest cardinal number of a dense subset of A, then $\hat{A}$ has a base for its open sets which is of cardinality no greater than α.*

Proof. First we remark that every T in $\hat{A}$ has dimension equal to or less than α. Indeed, let D be a dense subset of A of cardinality α, and T an irreducible representation of A acting in K. If $0 \neq \xi \in K$, the set $\{T_a \xi \mid a \in D\}$ is dense in H and of cardinality no greater than α.

Let H be of infinite dimension α. Then by the preceding remark $\hat{A} = \mathrm{Eq}\, \mathfrak{I}^0$; so that $\hat{A}$ is a topological subspace of $\mathrm{Eq}\, \mathfrak{I}$. By Lemma 2.1, $\mathfrak{I}$ has a base B of open sets of cardinality $\leqq \alpha$. Since $T \to \mathrm{Eq}\, T$ is a continuous open map of $\mathfrak{I}$ onto $\mathrm{Eq}\, \mathfrak{I}$ (Corollary 4 of Lemma 2.2), the set of $\mathrm{Eq}\, W$, where $W \in B$, is a base for the open subsets of $\mathrm{Eq}\, \mathfrak{I}$, and of cardinality $\leqq \alpha$. Since $\mathrm{Eq}\, \mathfrak{I}$ has a base of open sets of cardinality $\leqq \alpha$, so does its subspace $\hat{A}$.

COROLLARY. *If A is separable, $\hat{A}$ has a countable base for its open sets.*

4. The Mackey Borel structure

We recall some definitions from [8] about Borel structures. Let X be a set. A *Borel structure B on X* is a nonvoid family of subsets of X closed under countable unions and complementation with respect to X. A set X and a Borel structure B on X define a *Borel space*; the elements of B are the *Borel subsets* of X.

Let X, B be a Borel space. A subfamily B' of B is a *separating family* if, for any two distinct points x and y in X, there is an A in B' such that $x \in A$, $y \notin A$; it is a *generating family* if B is the smallest Borel structure on X containing B'. We say X, B is *separated* if B is a separating family; it is *countably separated* if there is a countable separating subfamily of B; it is *countably generated* if it is separated and there is a countable generating subfamily of B.

If X is a topological space, the smallest Borel structure B on X containing all the open sets is said to be *generated* by the topology. A complex-valued function f on a Borel space X, B is a *Borel function* if $f^{-1}(A) \in B$ for each open subset A of the complex number system.

We now fix a separable C^*-algebra A. The Borel structure of $\hat{A}$ generated by the hull-kernel topology of $\hat{A}$ will be called the *topological Borel structure* of $\hat{A}$. Mackey in [8] has also defined a Borel structure on $\hat{A}$ independently of the topology, as follows:

Let $\hat{A}_n$ be the subset of $\hat{A}$ consisting of those T whose dimension is

n ($n = 1, 2, \cdots, \aleph_0$); and, for each such n, fix a Hilbert space H_n of dimension n. Denote by $\mathfrak{I}_n$ the family of all concrete irreducible representations acting nowhere trivially in H_n. We give to $\mathfrak{I}_n$ the smallest Borel structure in which all the functions

$$T \to (T_a \xi, \eta)$$

($a \in A$, $\xi, \eta \in H_n$) are Borel functions; or, equivalently, the Borel structure generated by the topology defined on $\mathfrak{I}_n$ in §2. Now the natural map which assigns to each T in $\mathfrak{I}_n$ its equivalence class Eq T under unitary equivalence carries $\mathfrak{I}_n$ onto $\hat{A}_r$. We give to $\hat{A}_n$ the quotient Borel structure; i.e., a subset W of $\hat{A}_n$ is a Borel set if $\{T \in \mathfrak{I}_n \mid \text{Eq } T \in W\}$ is a Borel subset of $\mathfrak{I}_n$.

By the argument of the proof of Theorem 3.2,

$$\hat{A} = \cup_n \hat{A}_n \qquad (n = 1, 2, \cdots, \aleph_0).$$

We define the *Mackey Borel structure* of $\hat{A}$ as the family of all subsets W of $\hat{A}$ such that, for all $n = 1, 2, \cdots, \aleph_0$, $W \cap \hat{A}_n$ is a Borel subset of $\hat{A}_n$. (This is equivalent to Mackey's definition by Theorem 8.3 of [8].)

LEMMA 4.1. *The Mackey Borel structure of $\hat{A}$ contains the topological Borel structure.*

Proof. It is sufficient to show that for each open subset W of $\hat{A}$, and each $n = 1, 2, \cdots, \aleph_0$, $W \cap \hat{A}_n$ is a Mackey Borel set. But this follows from the continuity statement in the Corollary to Theorem 3.1.

We shall say that $\hat{A}$ is *smooth*, or A *has a smooth dual*, if the Mackey Borel structure of $\hat{A}$ is countably separated. This concept has important implications in the theory of representations of groups and algebras (see [8]).

THEOREM 4.1. *If A is a separable C^*-algebra, and $\hat{A}$ is a T_0-space with the hull-kernel topology (or, equivalently, if no two distinct elements of $\hat{A}$ have the same kernel), then $\hat{A}$ is smooth, and the topological and Mackey Borel structures of $\hat{A}$ coincide.*

Proof. By the Corollary to Theorem 3.2, $\hat{A}$ has a countable base C for its open sets; thus C is a countable generating family for the topological Borel structure. Since $\hat{A}$ is a T_0-space, its topological Borel structure is separating; hence C is a countable separating family. From this it follows by Lemma 4.1 that C is a countable separating family of Mackey Borel sets. Therefore $\hat{A}$ is smooth.

Since $\hat{A}$ is smooth, its Mackey Borel structure is *analytic* (see Theorem 8.4 of [8]). We have seen that the topological Borel structure of $\hat{A}$ is a countably generated and separated sub-Borel structure of the Mackey Borel structure. Now apply Theorem 4.3 of [8] to conclude the identity of the two structures.

A C^*-algebra A is said to be CCR if all the elements of $\hat{A}$ are completely continuous. In that case $\hat{A}$ is a T_0-space—in fact, even a T_1-space (see Lemma 1.11 of [2]). We therefore conclude:

COROLLARY. *A separable CCR C^*-algebra has a smooth dual.*

Added June 23, 1959. It has been pointed out to the author by J. Dixmier that this corollary is valid for GCR algebras. A GCR algebra (see [7]) is a C^*-algebra A in which there exists an ascending well-ordered set $\{I_\alpha\}$ (α running over all ordinals equal to or less than α_0) of closed two-sided ideals of A such that (i) if α is a limit ordinal, then I_α is the closure of $\bigcup_{\beta<\alpha} I_\beta$; (ii) the quotient $I_{\alpha+1}/I_\alpha$ is CCR for each $\alpha < \alpha_0$; (iii) $I_0 = \{0\}$, $I_{\alpha_0} = A$.

LEMMA 4.2. *The dual space of any GCR algebra is T_0 in the hull-kernel topology.*

Proof. Let A be a GCR algebra, and $\{I_\alpha\}$ the appropriate well-ordered set of ideals. Suppose that T and T' are two irreducible representations of A having the same kernel K; and let α be the smallest ordinal such that I_α is not contained in K. Evidently $\alpha = \beta + 1$, where $I_\beta \subset K$. Thus T and T' induce irreducible representations (also called T and T') of A/I_β, whose restrictions S and S' to I_α/I_β do not vanish. Now S and S' are irreducible and have the same kernel. Since I_α/I_β is CCR, this implies that S and S' are equivalent. Therefore T and T' are also equivalent; and the lemma is proved.

COROLLARY. *A separable GCR algebra has a smooth dual.*

Proof. Combine Theorem 4.1 with Lemma 4.2.

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