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A RESTRICTION THEOREM FOR FLAT MANIFOLDS OF CODIMENSION TWO

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Introduction

Let M denote a submanifold of $\mathbf{R}^{n+2}$ of codimension 2. Let $\mathscr{R}$ denote a restriction operator

$$(1.1) \qquad \mathscr{R}f(\eta) = \int e^{-i\langle x, \eta\rangle} f(x)\, dx, \quad \eta \in M, \quad f \in \mathscr{S}(\mathbf{R}^{n+2}).$$

We wish to find an optimal range of exponents p such that

$$(1.2) \qquad \|\mathscr{R}f\|_{L^2(M, d\sigma)} \leq C_p \|f\|_{L^p(\mathbf{R}^{n+2})},$$

where $d\sigma$ is a compactly supported measure on M.

Let $\mathscr{F}[d\sigma]$ denote the Fourier transform of $d\sigma$. By a theorem of Greenleaf (see [G]), the inequality (1.2) holds for

$$p = \frac{2(2+\gamma)}{4+\gamma}$$

if

$$(1.3) \qquad |\mathscr{F}[d\sigma](R\zeta)| \leq C(1+R)^{-\gamma}, \quad \zeta \in S^{n+1}.$$

The purpose of this paper is to use Greenleaf's result to establish a restriction theorem for a class of degenerate submanifolds of $\mathbf{R}^{n+2}$ of codimension 2. We shall assume that our manifold is given as a joint graph of two homogeneous functions, where the first graphing function is homogeneous of degree 1 and the second graphing function is homogeneous of degree m. Under the appropriate curvature assumption we will show that (1.3) holds with $\gamma = n/m$.

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An application of Greenleaf's result yields a restriction theorem with

$$p = \frac{2(2m+n)}{4m+n}.$$

We shall need the following definitions.

Nonvanishing Gaussian curvature. Let Σ be a submanifold of $\mathbf{R}^{N+1}$ of codimension 1 equipped with a smooth compactly supported measure $d\mu$. Let $J\colon \Sigma \to S^N$ be the usual Gauss map taking each point on Σ to the outward unit normal at that point. We say that Σ has everywhere nonvanishing Gaussian curvature if the differential of the Gauss map dJ is always nonsingular.

Strong curvature condition. Let S be a submanifold of $\mathbf{R}^{N+2}$ of codimension 2 equipped with a smooth compactly supported measure $d\mu$. Suppose that S is a joint graph of smooth functions g_1 and g_2, where $g_j\colon \mathbf{R}^N \to \mathbf{R}$. Let $\mathscr{N}_{x_0}(S)$ denote the two dimensional space of normals to S at a point x_0. We say that S satisfies the strong curvature condition (SCC) if for all $x_0 \in S$ in some neighborhood of support$(d\mu)$,

$$\det D^2(\nu_1 g_1(x) + \nu_2 g_2(x)) \neq 0, \quad \forall \nu \in \mathscr{N}_{x_0},$$

where D^2 denotes the Hessian matrix.

One can check that the above definitions are independent of the parametrization. Our main result is the following:

MAIN THEOREM. *Let* $M = \{(x, x_{n+1}, x_{n+2}) \in \mathbf{R}^{n+2}\colon\ x_{n+1} = \phi_1(x),\ x_{n+2} = \phi_2(x)\}$, $n \geq 2$, *where* $\phi_i \in \mathscr{C}^\infty(\mathbf{R}^n \setminus \{0\})$, ϕ_1 *is homogeneous of degree* 1, *and* ϕ_2 *is homogeneous of degree* $m \geq 2$. *Let* $\Sigma_j = \{x\colon\ \phi_j(x) = 1\}$. *Assume also that* ϕ_2 *only vanishes at the origin and that* Σ_2 *has everywhere nonvanishing Gaussian curvature. Let*

$$F(\xi, \lambda_1, \lambda_2) = \int_{\mathbf{R}^n} e^{i(\langle \xi, x\rangle + \lambda_1\phi_1(x) + \lambda_2\phi_2(x))}\chi(x)\, dx,$$

where $\chi \in \mathscr{C}_0^\infty(\mathbf{R}^n)$.

(a) *Suppose that the restriction of* ϕ_1 *to the set where* $\phi_2 = 1$, $\phi_1|_{\Sigma_2}$, *is constant. Then*

$$(1.4) \qquad |F(\xi, \lambda_1, \lambda_2)| \leq C(|\xi| + |\lambda|_1 + |\lambda|_2)^{-n/m}$$

when $m \geq 2n$.

(b) *Let* $M|_{\{x_{n+2}=1\}}$ *denote the restriction of* M *to the hyperplane* $\{x_{n+2} = 1\}$. *If* $M|_{\{x_{n+2}=1\}}$ *(viewed as a submanifold of codimension* 2 *of* $\{x_{n+2} = 1\}$*) satisfies the strong curvature condition, then* (1.4) *holds for* $m \geq 2$.

The conclusions of part (a) do not in general hold if $m < 2n$. Let $\phi_1(x) = |x|$, $\phi_2(x) = |x|^m$. Let $\xi = (0, 0, \ldots, 0)$. Then, in polar coordinates,

$$F(0, \lambda_1, \lambda_2) = C\int_0^\infty e^{i(\lambda_1 r + \lambda_2 r^m)} r^{n-1}\chi(r)\, dr.$$

It is not hard to see that the best isotropic decay for this integral cannot exceed

$$O\left(\left(\sqrt{\lambda_1^2 + \lambda_2^2}\right)^{-1/2}\right).$$

Hence the restriction $m \geq 2n$ is necessary.

Remarks. (1) It is known that isotropic decay estimates for the Fourier transform of the surface-carried measure cannot be expected to yield an optimal restriction theorem (see e.g., [C]). We shall apply a homogeneity argument due to Knapp to the class of manifolds considered in the theorem above.

Let $\mathscr{R}$ denote the restriction operator defined above. Let $\hat{f}_\delta(x, x_{n+1}, x_{n+2}) = h(\delta^{-1}x, \delta^{-1}x_{n+1}, \delta^{-m}x_{n+2})$, where h is the characteristic function of a rectangle in $\mathbf{R}^{n+2}$ with sides of lengths $(1, 1, \ldots, 1, C, C)$, C large.

Then

$$\|f_\delta\|_p \approx \delta^{(1-1/p)(n+m+1)} \quad \text{and} \quad \|\mathscr{R}f_\delta\|_p \approx \delta^{n/2}.$$

Hence (1.2) can only hold if

$$p \leq \frac{2(n+m+1)}{n+2(m+1)}.$$

If we apply Greenleaf's result (1.3) to the Main Theorem, we see that (1.2) holds for

$$p \leq \frac{2(2m+n)}{4m+n}.$$

The gap between this exponent and the exponent given by Knapp's homogeneity argument suggests that the restriction theorem (1.2) may hold for a wider range of exponents. The result obtained using the Main Theorem is not sharp. In order to obtain a sharp result one would probably have to obtain precise non-isotropic estimates for the Fourier transform of the surface carried measure using the techniques of M. Christ (see [C]).

(2) The curvature conditions of the Main Theorem are not entirely satisfying because there is no natural transition between parts (a) and (b).

We hope to address these difficulties in a subsequent paper.

Proof of the main result

Notation. (1) Given $a, b > 0$ we say that $a \approx b$ (a comparable to b) if there exist $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. We say that $a \gg b$ (a much larger than b) if the inequality $a \leq Cb$ is not satisfied for any $C > 0$. The notion $a \ll b$ is defined similarly.

(2) We denote by C a generic constant which may change from line to line.

Proof of part (a) *of the Main Theorem.* Let $\Psi(x) = \langle \xi, x \rangle + \lambda_1 \phi_1(x) + \lambda_2 \phi_2(x)$. Then

$$\nabla\Psi(x) = \xi + \lambda_1 \nabla\phi_1(x) + \lambda_2 \nabla\phi_2(x).$$

Since $\phi_1|_{\Sigma_2}$ is constant by assumption, then $\phi_1 \neq 0$ away form the origin. Hence, $\nabla\phi_1(x) \neq 0$ away from the origin by the Euler homogeneity relation, and since every component of $\nabla\phi_1(x)$ is homogeneous of degree zero, we have $|\nabla\phi_1(x)| \geq C$ for all $x \in \text{support}(\phi_1)$.

Suppose that $|\xi| \ll |\lambda_2| \ll |\lambda_1|$ or $|\lambda_2| \ll |\xi| \ll |\lambda_1|$. Then $|\nabla\Psi(x)| \geq C|\lambda_1|$ and so an integration by parts argument (see Theorem 1 in the appendix) shows that

$$|F(\xi, \lambda_1, \lambda_2)| \leq C(1 + |\lambda_1|)^{-N} \quad \forall N > 0.$$

Similarly, if $|\lambda_1| \ll |\lambda_2| \ll |\xi|$, or $|\lambda_1| \approx |\lambda_2| \ll |\xi|$, then

$$|F(\xi, \lambda_1, \lambda_2)| \leq C(1 + |\xi|)^{-N} \quad \forall N > 0.$$

If we rewrite F using polar coordinates with respect to Σ_2 and assume that χ is radial with respect to Σ_2, we get

$$F(\xi, \lambda_1, \lambda_2) = \int_0^{+\infty} r^{n-1} \chi(r) \int_{\Sigma_2} e^{i(r\langle \xi, \omega \rangle + r\lambda_1 + r^m \lambda_2)} \, d\sigma(\omega) \, dr,$$

where $d\sigma$ is the Lebesgue measure carried by Σ_2. Let $I(\xi)$ denote the

Fourier transform of the surface-carried measure on Σ_2,

$$I(\xi) = \int_{\Sigma_2} e^{i\langle \xi, \omega\rangle}\, d\sigma(\omega).$$

Since the Gaussian curvature on Σ_2 never vanishes, we can use the method of stationary phase (see theorem (3) in the appendix) to write $I(\xi) = b(\xi)e^{iq(\xi)}$, where ξ belongs to a cone Γ containing the normal directions to Σ_2 on the support of $d\sigma$, and where $b(\xi)$ is a symbol of order $-(n-1)/2$, $q(\xi)$ is homogeneous of degree 1, and $q(\xi) \approx |\xi|$. Away from Γ, $I(\xi)$ decays rapidly in $|\xi|$.

Suppose that we are in one of the cases where $|\xi|$ dominates:

(1) $|\lambda_2| \ll |\lambda_1| \approx |\xi|$,
(2) $|\lambda_1| \ll |\lambda_2| \approx |\xi|$,
(3) $|\lambda_1| \ll |\lambda_2| \ll |\xi|$,
(4) $|\lambda_2| \ll |\lambda_1| \ll |\xi|$,
(5) $|\lambda_1| \approx |\lambda_2| \approx |\xi|$.

Using our observation about $I(\xi)$, we write

$$F(\xi, \lambda_1, \lambda_2) = \int_0^{+\infty} r^{n-1} e^{i(rq(\xi)+r\lambda_1+r^m\lambda_2)} b(r\xi)\chi(r)\, dr.$$

Then

$$|F(\xi, \lambda_1, \lambda_2)| \le C\int_0^2 r^{n-1}|b(r\xi)|\, dr.$$

Let $s = r|\xi|$, and define $\tilde{\xi} = \xi|\xi|^{-1}$. The integral above is bounded by

$$C|\xi|^{-n}\int_0^{2|\xi|} s^{n-1}\left|b(s\tilde{\xi}\,)\right| ds$$
$$= C|\xi|^{-n}\int_0^{N} s^{n-1}\left|b(s\tilde{\xi}\,)\right| ds + C|\xi|^{-n}\int_N^{2|\xi|} s^{n-1}\left|b(s\tilde{\xi}\,)\right| ds,$$

where N is large. The first integral is $O(|\xi|^{-n})$ and the second integral is bounded by

$$C|\xi|^{-n}\int_N^{2|\xi|} s^{(n-1)/2}\, ds \le C(1+|\xi|)^{-(n-1)/2}.$$

Note that $(n-1)/2 \ge n/m$ when $m \ge 2n/(n-1)$.

We are left to consider the cases where λ_2 dominates:

(1) $|\xi| \approx |\lambda_1| \ll |\lambda_2|$,
(2) $|\xi| \ll |\lambda_1| \approx |\lambda_2|$,
(3) $|\xi| \ll |\lambda_1| \ll |\lambda_2|$,
(4) $|\lambda_1| \ll |\xi| \ll |\lambda_2|$.

As before, let

$$F(\xi, \lambda_1, \lambda_2) = \int_0^{+\infty} r^{n-1} e^{i(rq(\xi)+r\lambda_1+r^m\lambda_2)} b(r\xi)\chi(r)\, dr.$$

Let $s\lambda_2^{-1/m} = r$. Then

$$F(\xi, \lambda_1, \lambda_2) = \lambda_2^{-n/m} \int_0^{+\infty} s^{n-1} e^{i(q(s\lambda_2^{-1/m}\xi)+s\lambda_2^{-1/m}\lambda_1+s^m)} b(s\lambda_2^{-1/m}\xi)\chi(s\lambda_2^{-1/m})\, ds.$$

Let

$$G(\xi, \lambda_1, \lambda_2) = \int_0^{+\infty} s^{n-1} e^{i(\lambda_2^{-1/m}sq(\xi)+s\lambda_2^{-1/m}\lambda_1+s^m)} b(s\lambda_2^{-1/m}\xi)\chi(s\lambda_2^{-1/m})\, ds.$$

It suffices to show that $|G(\xi, \lambda_1, \lambda_2)|$ is uniformly bounded. When

$$\left|\frac{\lambda_1 + |\xi|}{\lambda_2^{1/m}}\right|$$

is sufficiently small, then $|G|$ is bounded by $C|\int_0^{+\infty} e^{it^m} t^{n-1}\, dt|$. An integration by parts argument shows that this integral converges. In particular the above integral equals

$$e^{2\pi i/m}\frac{1}{m}\Gamma\left(\frac{n}{m}\right).$$

Thus we may assume that

$$\left|\frac{\lambda_1 + |\xi|}{\lambda_2^{1/m}}\right| \geq C.$$

Let

$$\Phi(s) = s\frac{\lambda_1 + q(\xi)}{\lambda_2^{1/m}} + s^m.$$

Then

$$\Phi'(s) = 0 \quad \text{if } s = C\left(\frac{\lambda_1 + q(\xi)}{\lambda_2^{1/m}}\right)^{1/(m-1)},$$

and

$$\Phi''(s) = m(m-1)s^{m-2}.$$

If we apply the van der Corput Lemma (see Theorem 2 in the appendix) in the case $k = 2$, and recall that in particular $|b|$ is uniformly bounded, we see that $|G|$ is bounded by

$$C\left|\frac{\lambda_1 + |\xi|}{\lambda_2^{1/m}}\right|^{-(m-2)/2(m-1)+(n-1)/(m-1)}.$$

The power of $|(\lambda_1 + |\xi|)/\lambda_2^{1/m}|$ in the expression above is non-positive if $m \geq 2n$, and so $G(\xi, \lambda_1, \lambda_2)$ is uniformly bounded. This completes the proof of part (a) of the Main Theorem.

Proof of part (B) *of the Main Theorem.* As before, we rewrite F using polar coordinateds associated to Σ_2. We get

$$F(\xi, \lambda_1, \lambda_2) = \int_0^{+\infty}\int_{\Sigma_2} e^{i(r\langle\omega, \xi\rangle + r\lambda_1\phi_1(\omega) + \lambda_2 r^m)} r^{n-1}\xi(r)\, d\omega\, dr,$$

where, as before, ξ is a smooth cutoff function which is radial with respect to the polar coordinates associated to Σ_2. Let

$$I(\xi, \lambda_1) = \int_{\Sigma_2} e^{i(\langle\omega, \xi\rangle + \lambda_1\phi_1(\omega))}\, d\omega.$$

Using the implicit function theorem we can parametrize Σ_2 near a point s_0 by a smooth function $\psi\colon \mathbf{R}^{n-1} \to \mathbf{R}$. Without loss of generality, we can assume that $\nabla\phi_1(s_0) = 0$ and that $\nabla\phi_2(s_0) = (0, 0, \ldots, 0, 1)$. Thus, we can locally write $\Sigma_2 = \{(\omega', \omega_n)\colon \omega_n = \psi(\omega')\}$. The restriction of M to the hyperplane $\{x_{n+2} = 1\}$ can thus be locally parametrized by the functions $\psi(\omega')$ and $\phi_1(\omega', \psi(\omega'))$. If we let $\xi = (\xi', \xi_n)$, we can write $I(\xi, \lambda_1)$ as a finite sum of terms of the form

$$\int_{\mathbf{R}^{n-1}} e^{i(\langle\omega', \xi'\rangle + \xi_n\psi(\omega') + \lambda_1\phi_1(\omega', \psi(\omega')))}\chi_1(\omega')\, d\omega', \tag{1.5}$$

where χ_1 is a smooth cutoff function supported in a neighborhood of s_0. It was observed by M. Christ (see [C]) that the strong curvature condition (see the introduction) implies the following result.

LEMMA. *Let Ω be a submanifold of $\mathbf{R}^{N+2}$ of codimension 2 locally parametrized by smooth functions g_1 and g_2, where $g_j\colon \mathbf{R}^N \to \mathbf{R}$. Let $d\sigma$ denote a smooth measure on Ω. Suppose that Ω satisfies the strong curvature condition. Then*

$$|\mathscr{F}[d\sigma](R\eta)| \le C(1+R)^{-N/2}.$$

The proof of the lemma shows that the integral in (1.5) can be written as $b(\xi, \lambda_1)e^{iq(\xi, \lambda_1)}$, where (ξ, λ_1) belongs to a cone containing the normal directions to $M|_{\{x_{n+2}=1\}}$ on the support of $d\sigma$, $b(\xi, \lambda_1)$ is a symbol of order $-(n-1)/2$, $q(\xi, \lambda_1)$ is homogeneous of degree 1, and $|q(\xi, \lambda_1)| \approx (|\xi| + |\lambda_1|)$.

We must analyze the integral

$$\int_0^{+\infty} r^{n-1} e^{i(rq(\xi, \lambda_1) + r^m \lambda_2)} b(r\xi, r\lambda_1)\chi(r)\, dr. \tag{1.6}$$

We may assume that $|q(\xi, \lambda_1)| \le C|\lambda_2|$, since if $|q(\xi, \lambda_1)| \ge c|\lambda_2|$ for a sufficiently large $c > 0$, then the integral in (1.6) decays rapidly in $|\xi| + |\lambda_1|$. (See Theorem 1 in the appendix.)

Let $s = r\lambda_2^{1/m}$. Then, the integral in (1.6) can be written as

$$\lambda_2^{-n/m} \int_0^{+\infty} s^{n-q} e^{i(s\lambda_2^{-1/m} q(\xi, \lambda_1) + s^m)} b\left(s\lambda_2^{-1/m}\xi, s\lambda_2^{-1/m}\lambda_1\right)\chi\left(s\lambda_2^{-1/m}\right) dr.$$

Let

$$G(\xi, \lambda_1, \lambda_2) = \int_0^{+\infty} s^{n-1} e^{i(s\lambda_2^{-1/m} q(\xi, \lambda_1) + s^m)} b\left(s\lambda_2^{-1/m}\xi, s\lambda_2^{-1/m}\lambda_1\right)\chi\left(s\lambda_2^{-1/m}\right) dr.$$

As before, it suffices to show that $|G(\xi, \lambda_1, \lambda_2)|$ is uniformly bounded. When $|(|\lambda_1| + |\xi|)/\lambda_2^{1/m}|$ is sufficiently small, then $|G|$ is bounded by $C|\int_0^{+\infty} e^{it^m} t^{n-1}\, dt|$. Hence we can assume

$$\left|\frac{|\lambda_1| + |\xi|}{\lambda_2^{1/m}}\right| \ge C.$$

We can write

$$G(\xi, \lambda_1, \lambda_2) = \int_0^N + \int_N^{C|\lambda_2|^{1/m}}, \quad N \text{ large}.$$

The first integral is uniformly bounded. In order to handle the second integral let

$$\Phi(s) = s\lambda_2^{-1/m}q(\xi, \lambda_1) + s^m.$$

Then

$$\Phi'(s) = 0 \quad \text{if } s = c_m\left(\lambda_2^{-1/m}q(\xi, \lambda_1)\right)^{1/(m-1)},$$

and

$$\Phi''(s) = m(m-1)s^{m-2}.$$

If the critical point is smaller then N the integral has rapid decay, so we may assume that $|\lambda_2^{-1/m}q(\xi, \lambda_1)|$ is large. If we recall that $|q(\xi, \lambda_1)| \approx |\xi| + |\lambda_1|$, then by the van der Corput lemma (see Theorem 2 in the appendix) we get

$$(1.7) \quad \int_N^{C|\lambda_2|^{1/m}} \leq \left|\frac{|\lambda_1| + |\xi|}{\lambda_2^{1/m}}\right|^{-(m-2)/2(m-1)+(n-1)/(m-1)-(n-1)/2(m-1)-(n-1)/2}$$

Note that the third and the fourth terms in the power of $|(|\lambda_1| + |\xi|)/\lambda_2^{1/m}|$ arise from the fact that b is a symbol of order $-(n-1)/2$, and $|(|\lambda_1| + |\xi|)/\lambda_2^{1/m}|$ is large.

The power of $|(|\lambda_1| + |\xi|)/\lambda_2^{1/m}|$ in (1.7) is nonnegative provided that $m \geq 2$. Hence, $|G(\xi, \lambda_1, \lambda_2)|$ is bounded and the proof is complete.

Appendix

In this section we recall a few classical results that we used to prove the Main Theorem. The first two theorems, which deal with oscillatory integrals, can be found for example in [St].

THEOREM 1. *Suppose $\phi \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ and suppose that ψ is a real-valued and smooth function which has no critical point on the support of ϕ. Then*

$$\left|\int_{\mathbf{R}^n} e^{i\lambda\psi(x)}\phi(x)\,dx\right| = O(\lambda^{-N})$$

as $\lambda \to \infty$, for every $N \geq 0$.

THEOREM 2. *Suppose that ψ is real-valued and smooth and that ϕ is complex-valued and smooth in $[a, b]$. If $|\psi^{(k)}(x)| \geq 1$, then*

$$\left|\int_a^b e^{i\lambda\psi(x)}\phi(x)\,dx\right| \leq C_k\lambda^{-1/k}\left[|\phi(b)| + \int_a^b|\phi'(t)|\,dt\right]$$

holds when

(1) $k \geq 2$.

or

(2) $k = 1$, *if in addition it is assumed that* $\psi'(x)$ *is monotonic.*

THEOREM 3. *Let* S *be a smooth hypersurface in* $\mathbf{R}^n$ *with nonvanishing Gaussian curvature, and let* $d\sigma$ *be a* $\mathscr{C}^\infty$ *measure on* S. *Then*

$$|\widehat{d\mu}(\xi)| \leq C(1 + |\xi|)^{-(n-1)/2}.$$

Moreover suppose that $\Gamma \subset \mathbf{R}^n \setminus \{0\}$ *is the cone consisting of all* ξ *which are normal to some point* $x \in S$ *belonging to some compact neighborhood* $\mathscr{N}$ *of support*$(d\mu)$. *Then,*

$$\frac{\partial^\alpha}{\partial \xi} \widehat{d\mu}(\xi) = O\big((1 + |\xi|)^{-N}\big), \quad \forall N, \textit{ if } \xi \notin \Gamma,$$

$$\widehat{d\mu}(\xi) = \Sigma a_j(\xi) e^{i\langle x_j, \xi \rangle}, \quad \textit{if } \xi \in \Gamma,$$

where the finite sum is taken over all points $x_j \in \mathscr{N}$ *having* ξ *as a normal and*

$$\left| \frac{\partial^{(\alpha)}}{\partial \xi} \widehat{d\mu}(\xi) \right| \leq C_\alpha (1 + |\xi|)^{-(n-1)/2 - |\alpha|}.$$

Proof. See [So], pp. 50–51.

REFERENCES

[C] M. CHRIST, *Restriction of the Fourier transform to submanifolds of low codimension*, thesis, University of Chicago, 1982.

[DC] L. DE CARLI, *Unique continuation for higher order elliptic operators*, thesis, UCLA, 1993.

[G] A. GREENLEAF, *Principal curvatures in harmonic analysis*, Indiana Univ. Math. J. **30** (1981), 519–537.

[H] L. HORMANDER, *The analysis of linear partial differential operators I*, Springer-Verlag, New York, 1990.

[I1] A. IOSEVICH, *Maximal operators associated to families of flat curves and hypersurfaces*, thesis, UCLA, 1993.

[I2] ______, *Maximal Operators associated to families of flat curves in the plane*, Duke Math. J. **76** (1994), 533–544.

[L] W. LITTMAN, *Fourier transforms of surface-carried measures and differentiability of surface averages*, Bull. Amer. Math. Soc. **69** (1963), 766–770.

[So] C.D. SOGGE, *Fourier integrals in classical analysis*, Cambridge University Press, Cambridge, 1993.

[St] E.M. STEIN, *Harmonic analysis*, Princeton University Press, Princeton, 1993.

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