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A CHARACTERIZATION OF TWISTED FUNCTION RINGS

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1. Introduction

In this paper all rings have identity and all morphisms preserve the identity. We shall assume familiarity with the Pierce Sheaf (cf. [12] for a detailed treatment and [5, Chapter 1] for a summary) $\mathbf{k}(R)$ of a ring R over $\mathbf{X}(R)$, the Stone space of the Boolean algebra $\mathbf{B}(R)$ of all central idempotents of R. ($\mathbf{X}(R)$ is also called the Boolean spectrum of R.) There is a canonical isomorphism $R \cong \Gamma(\mathbf{X}(R), \mathbf{k}(R))$ that represents any ring R as the ring of all global sections of $\mathbf{k}(R)$ over $\mathbf{X}(R)$. For any Boolean space X, ring F with the discrete topology, and families $\{C_\alpha: \alpha \in \mathscr{I}\}$ and $\{F_\alpha: \alpha \in \mathscr{I}\}$ of closed subsets of X and subrings of F, let $\mathscr{C}(X, F; \{(X_\alpha, F_\alpha): \alpha \in \mathscr{I}\})$ denote the ring of all continuous functions $f: X \to F$ such that $f(C_\alpha) \subseteq F_\alpha$, for all $\alpha \in \mathscr{I}$.

In [3] and [4] results from [1] were generalized to partially classify those rings R that can be written in the above form, where $X = \mathbf{X}(R)$ and F is indecomposable. In particular any commutative ring R has this form if (i) it is von Neumann regular (i.e., it satisfies $(\forall r)(\exists s)(rsr = r)$), (ii) it is an algebra over some field L with algebraic closure F, (iii) it is algebraic over L, and (iv) $\mathbf{B}(R)$ is either countable or is complete as a Boolean algebra. In terms of sheaves, this was equivalent to showing that there was an embedding $\mathbf{k}(R) \subseteq \mathbf{X}(R) \times F$, where $\mathbf{X}(R) \times F$ is viewed as the simple sheaf with the product topology. If (iii) or (iv) is omitted, then, by [5], R can be represented as the ring of certain specified functions $f: U_f \to F$, where each U_f is a dense open subset of $\mathbf{X}(R)$. However such a representation is not as nice as one in which each f has the same compact domain. If only (iv) is omitted, the well known Arens and Kaplansky results [1] establish that R is isomorphic to a ring of certain twisted functions defined on the Boolean space $\mathbf{X}(F \otimes_L R)$. In §2 we shall show that certain ring embeddings give rise to a very general kind of twisting on sections of sheaves. We apply these results in §3 to accomplish our main goal (the characterization of *all* rings of twisted functions given in theorem 3.7) and finish with applications to particular rings. The full details of this work show that many rings which can not be represented as functions

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defined on their Boolean spectra, can be represented using twisted functions defined on some "non-standard" (in the analysts' sense) version of this space. To the best of our knowledge, "non-standard spectra" have not previously been used to represent rings.

In essence, the results in this paper have been deduced from the behaviour of certain sheaf morphisms, with assistance from the model theory of saturated structures. At present few algebraists are familiar with saturated structures. Nonetheless this paper should be accessible to algebraists willing to accept the properties ascribed to saturated structures in Proposition 2.10 and Lemma 3.3. Our results provide (further) evidence that model theoretic methods can be fruitfully applied to some problems that are strictly algebraic.

Convention. Throughout this paper we shall assume that all rings R have *clopen support*. (i.e., $\{x \in \mathbf{X}(R): r(x) \neq 0\}$ is clopen (closed and open) in $\mathbf{X}(R)$, for all $r \in \Gamma(\mathbf{X}(R), \mathbf{k}(R))$.) This assumption involves no loss of generality in §3 and is reasonable in §2 as, by [4, proposition 1.3], any function ring would have this property, even if we did not assume it. (Function rings are formally defined in 3.2.)

2. Representations using twisted sections

In this section we shall define a very general kind of twisting, on *sections of sheaves* and use it to represent rings. Following some necessary topological results, we shall see that the domain space and twisting can (but need not) always be chosen so as to be closely related to certain groups of homeomorphisms of the domain. This symmetry ensures that our results still retain much of the flavour of previous ones, such as [1, Theorem 6.1]

DEFINITION 2.1. Let $\mathscr{L}$ be a sheaf of rings over a Boolean space Y.

(1) A twisting $\mathscr{T}$ on $(Y, \mathscr{L})$ (or more briefly on $\mathscr{L}$) is a pair $\mathscr{T} = (E, \{\Phi_{y,z}: (y, z) \in E\})$ such that;

(i) E is an equivalence relation on Y;

(ii) $\Phi_{y,z}: \mathscr{L}_z \to \mathscr{L}_y$ is an isomorphism, for each $(y, z) \in E$;

(iii) $\Phi_{y,y}$ is the identity map and $\Phi_{y,w} \circ \Phi_{w,z} = \Phi_{y,z}$, whenever (y, z), (y, w), and $(w, z) \in E$; and

(iv) the map $(r(x), x, y) \to \Phi_{y,x}(r(x))$ is a continuous map $D \to \mathscr{L}$, where

$$D = \{(r(x), x, y) : r \in \Gamma(Y, \mathscr{L}), \text{ and } (x, y) \in E\}$$

is topologized as a subset of $\mathscr{L} \times Y \times Y$.

The above twisting is denoted by $\mathscr{T} = (E, \Phi)$.

(2) Let $\Gamma(Y, \mathscr{L}, \mathscr{T}) = \{\sigma \in \Gamma(Y, \mathscr{L}) : \Phi_{y,x}(\sigma(x)) = \sigma(y),$ whenever $(y, x) \in E\}$, where $\mathscr{T} = (E, \Phi)$ is a twisting on $\mathscr{L}$.

(3) Elements of $\Gamma(Y, \mathscr{L}, \mathscr{T})$ may be called $\mathscr{T}$-equivariant sections.

(Analogous notation for function rings is discussed in remark 3.1.)

Clearly there is a unique twisting $\mathscr{T} = (E, \Phi)$ on $\mathscr{L}$ (called the *trivial twisting*) for which $(x, y) \in E$ iff $x = y$. Of course, in this case, $R \cong \Gamma(\mathbf{X}(R), \mathbf{k}(R)) = \Gamma(\mathbf{X}(R), \mathbf{k}(R), \mathscr{T})$. We shall show that the embeddings $R \subseteq S$ described in Definition 2.2 (ii) give rise to a twisting $\mathscr{T}$ on $\mathbf{k}(S)$ such that $R \cong \Gamma(\mathbf{X}(S), \mathbf{k}(S), \mathscr{T})$. In subsequent applications we shall pick S such that the topology on $\mathbf{k}(S)$ is well enough understood that the representation $R \cong \Gamma(\mathbf{X}(S), \mathbf{k}(S), \mathscr{T})$ is more illuminating then the representation $R \cong \Gamma(\mathbf{X}(R), \mathbf{k}(R))$.

DEFINITION 2.2. In this definition let $\bar{}: R \cong \Gamma(\mathbf{X}(R), \mathbf{k}(R))$ and $\hat{}: S \cong \Gamma(\mathbf{X}(S), \mathbf{k}(S))$ be the standard isomorphisms, where R and S are rings.

(i) A monomorphism $f: R \to S$ is *conformal* iff $f(\mathbf{B}(R)) \subseteq \mathbf{B}(S)$. When f is conformal, let $\pi_f: \mathbf{X}(S) \to \mathbf{X}(R)$ be the continuous onto map corresponding to f via Stone's theorem.

(ii) A monomorphism $f: R \to S$ is a *local monomorphism* iff it is conformal and, for each $y \in \mathbf{X}(S)$, the map $\mathbf{k}_{\pi_f(y)}(R) \to \mathbf{k}_y(S)$ given by $\bar{r}(\pi_f(y)) \to \widehat{f(r(y))}$ is a monomorphism, where r varies over R. (These maps are a minor alteration of the morphism of ringed spaces $(\mathbf{X}(S), \mathbf{k}(S)) \to (\mathbf{X}(R), \mathbf{k}(R))$ corresponding to f, that is described in [12, Definition 6.1 and Theorem 6.6]. In general they exist whenever f is conformal, and are merely well defined homomorphisms, even if f is a monomorphism.)

We now characterize the local monomorphisms.

DEFINITION 2.3. Let $\mathbf{id}(e)$, $e < f$, and $\mathbf{nz}(e, r)$ be, respectively, the first order formulae

$$e^2 = e \wedge (\forall r)(re = er),$$

$$\mathbf{id}(e) \wedge \mathbf{id}(f) \wedge e = ef,$$

and

$$\mathbf{id}(e) \wedge (\forall f)[(0 \neq f < e) \Rightarrow rf \neq 0].$$

LEMMA 2.4. *Suppose that $f: R \to S$ is a conformal ring monomorphism and that R has clopen support. Then f is a local monomorphism in each of the following cases.*

(1) $\mathbf{k}_x(R)$ *is a simple ring, for each* $x \in \mathbf{X}(R)$.

(2) *If* $R \vDash \mathbf{nz}(e, r)$ *then* $S \vDash \mathbf{nz}(f(e), f(r))$, *for all* $r \in R$ *and* $e \in \mathbf{B}(R)$.

(3) *$f(\mathbf{B}(R))$ is a dense Boolean subalgebra of* $\mathbf{B}(S)$.
(4) *f: $R \to S$ is a (logical) elementary embedding.*

Proof. Let $\bar{\ }$: $R \cong \Gamma(\mathbf{X}(R), \mathbf{k}(R))$ and $\hat{\ }$: $S \cong \Gamma(\mathbf{X}(S), \mathbf{k}(S))$ be the standard maps, assume without loss of generality that f is the inclusion $R \subseteq S$, and let $\pi = \pi_f$. For each $y \in \mathbf{X}(S)$, let Θ_y: $\mathbf{k}_{\pi(y)}(R) \to \mathbf{k}_y(S)$ be given by $\bar{r}(\pi(y)) \to \hat{r}(y)$.

(2) Suppose that $0 \neq \bar{r}(\pi(y)) \in \mathbf{k}_{\pi(y)}(R)$, for some $r \in R$ and $y \in \mathbf{X}(S)$. As R has clopen support, there exists $e \in \mathbf{B}(R)$ such that $\bar{e}(\pi(y)) \neq 0$, $R \models \mathbf{nz}(e, r)$, and thus $S \models \mathbf{nz}(e, r)$. Thus $\Theta_y(\bar{r}(\pi(y))) = \hat{r}(y) \neq 0$ for, if not, there would exist $f \in \mathbf{B}(S)$ such that $0 \neq f < e$ and $fr = 0$.

Cases (3) and (4) follow from (2). Case (1) is trivial. □

In fact condition (2) (above) holds iff f is a local monomorphism.

We now use the Pierce sheaf to represent rings using twisted sections.

DEFINITION 2.5. Retain the notation from Lemma 2.4 and assume that f: $R \subseteq S$ is a local monomorphism. Let $\mathscr{L}(R, S)$ (or more properly $\mathscr{L}(f)$) be $\{\hat{r}(y) \in \mathbf{k}_y(S)$: $r \in R$ and $y \in \mathbf{X}(S)\}$. Let $\mathscr{T}(R, S)$ ($\mathscr{T}(f)$) be the twisting (E, Φ) on $\mathscr{L}(R, S)$ given by

(i) $(x, y) \in E$ iff $\pi(x) = \pi(y)$, and

(ii) $\Phi_{y,x}$ is the composition $\mathscr{L}_x(R, S) \xrightarrow{\Theta_x^{-1}} \mathbf{k}_{\pi(x)}(R) = \mathbf{k}_{\pi(y)}(R) \xrightarrow{\Theta_y} \mathscr{L}_y(R, S)$.

Note that our definition of Φ (and hence of the twisting $\mathscr{T}(f)$) requires the hypothesis "f is a local monomorphism." Verification that $\mathscr{L}(R, S)$ is a subsheaf of $\mathbf{k}(S)$ and that $\mathscr{T}(R, S)$ actually is a twisting on $\mathscr{L}(R, S)$, presents no difficulty. If R is commutative then all stalks of $\mathbf{k}(R)$ and $\mathscr{L}(R, S)$ are indecomposable, so that (i) $\mathscr{L}(R, S)$ is a reduced sheaf (cf. [12, page 15]). Thus, letting $\bar{R} = \Gamma(\mathbf{X}(S), \mathscr{L}(R, S))$, we get (ii) $\mathbf{X}(S) = \mathbf{X}(\bar{R})$, (iii) $\mathbf{B}(S) = \mathbf{B}(\bar{R})$, and (iv) $\mathbf{k}(\bar{R}) = \mathscr{L}(R, S)$. Conclusions (i)–(iv) hold for different routine reasons when $\mathbf{B}(R)$ is dense in $\mathbf{B}(S)$. However in all cases $\bar{R}$ is the subring of S generated by $R \cup \mathbf{B}(S)$.

THEOREM 2.6. *Retain the notation from Lemma* 2.4 *and Definition* 2.5 *and (crucially) the assumption that f: $R \subseteq S$ is a local monomorphism. Then* $R \cong \Gamma(\mathbf{X}(S), \mathscr{L}(f), \mathscr{T}(f))$.

Proof. Let $\mathscr{L} = \mathscr{L}(f)$, $\mathscr{T} = \mathscr{T}(f)$, and $Y = \mathbf{X}(S)$. The map $\hat{\ }|_R$: $R \to \Gamma(Y, \mathscr{L}, \mathscr{T})$ is an embedding, as $\mathscr{L}$ is a subsheaf of $\mathbf{k}(S)$. Note that $\Theta_y^{-1}(\hat{r}(y)) = \bar{r}(\pi(y))$, for all $r \in R$ and $y \in Y$.

Let $s \in \Gamma(Y, \mathscr{L}, \mathscr{T})$ be arbitrary. To finish the proof, we must find $r \in R$ such that $\hat{\ }|_R(r) = \hat{r} = s$. By a modification of standard arguments, such as [12, Theorem 4.4], it suffices to let $y \in Y$ be arbitrary, and obtain $e_y \in \mathbf{B}(R)$

and $r_y \in R$, such that $\hat{e}_y(y) = 1$ and $s\hat{e}_y = \hat{r}_y \hat{e}_y$. The existence of $r_y \in R$, such that $\hat{s}(y) = \hat{r}_y(y)$, follows from the definition of $\mathscr{L}(f)$. Let $U_y = \{z \in Y: s(z) = \hat{r}_y(z)\}$. Note that $\pi(U_y) \cap \pi(Y - U_y) = \emptyset$ since, if $z_1 \in U_y$, $z_2 \in Y$ and $\pi(z_1) = \pi(z_2)$, then

$$\begin{aligned} s(z_2) &= \Phi_{z_2, z_1}(s(z_1)) = \Phi_{z_2, z_1}(\hat{r}_y(z_1)) = \Theta_{z_2} \circ \Theta_{z_1}^{-1}(\hat{r}_y(z_1)) \\ &= \Theta_{z_2}(\bar{r}(\pi(z_1))) = \hat{r}_y(z_2). \end{aligned}$$

Consequently $\pi(U_y) = \mathbf{X}(R) - \pi(Y - U_y)$, so that $\pi(U_y)$ is open in $\mathbf{X}(R)$. Thus there exists $e_y \in \mathbf{B}(R)$ such that $\bar{e}_y(\pi(y)) = 1$ yet $\bar{e}_y(x) = 0$, whenever $x \notin \pi(U_y)$. It follows routinely that $s\hat{e}_y = \hat{r}_y \hat{e}_y$ and $\hat{e}_y(y) = 1$, as was required. □

Arens and Kaplansky's concept of twisted functions applied to the elements of a ring $T = \mathscr{C}(X, F)$ of continuous functions. They utilized a group G of certain automorphisms of F and a representation $g \to g^*$ of G into the group of homeomorphisms of X. They called an element $t \in T$ twisted iff $g(t(x)) = t(g^*(x))$, for all $g \in G$ and $x \in X$. Clearly the orbits of G^* in X play the role that the equivalence classes of E do in our concept of twisting. We see no way of introducing anything completely analogous to their group G of automorphisms, into our context. However it is still desirable (and sometimes possible) to work directly and analogously with a suitable group of homeomorphisms of X.

DEFINITION 2.7. Let $\mathscr{T} = (E, \Phi)$ be a twisting on a sheaf $\mathscr{S}$ of rings over a Boolean space X and G be a group of homeomorphisms of X. Then:

(1) $\mathscr{T}$ is called *G-symmetric* iff the orbits of G acting on X are just the equivalence classes of E.
(2) (G, Φ) can be used to denote $\mathscr{T}$, when $\mathscr{T}$ is G-symmetric.
(3) If $\mathscr{T}$ is G-symmetric, then the elements of $\Gamma(X, \mathscr{S}, \mathscr{T})$ may be called *(G, Φ)-equivariant* or (ambiguously) *G-equivariant sections.*

If $\mathscr{T}$ and G are both trivial, then $\mathscr{T}$ is G-symmetric. Our next goal is to show that less trivial examples abound. The following concept will be used to construct suitable homeomorphism groups.

DEFINITION 2.8. Suppose that $p \in W$ where W is a Boolean space. Then:
(α) p is *atomless* iff it has a neighbourhood containing no isolated points.
(β) p is *atomic* iff it has a neighbourhood containing no atomless points.
(γ) p is *transitional* iff it is neither atomic nor atomless.

Warning: All elements of the one point compactification of N are atomic. Thus an atomic point need not be isolated.

Remark 2.9. It is easy to find a first order formula **atmc**(z) such that if $W = \mathbf{X}(T)$ for some ring T, and if $e \in \mathbf{B}(T)$ and the clopen set $C \subseteq W$ correspond via Stone's theorem, then all points in C are atomic iff $T \models$ **atmc**(e). Similarly, there are formulae **atmlss**(z) and **trans**(z) such that $T \models$ **atmlss**(e) iff all points in C are atomless, and $T \models$ **trans**(e) iff C contains a transitional point. If ϕ is any of the above formulae assume, without loss of generality, that it has been chosen such that $\vdash (\forall z)(\phi(z) \Rightarrow$ **id**$(z))$.

In the next proposition, saturated (λ-saturated) structures are used in two distinct ways, to construct Boolean spaces with sufficiently intricate homeomorphism groups. Firstly, (iv) follows easily from this consequence of the saturation hypothesis, where $\lambda(z)$ is **trans**(z) and $\bar{\ }$: $R \cong \Gamma(\mathbf{X}(R), \mathbf{k}(R))$ is the standard isomorphism: *Suppose that $x \in \mathbf{X}(R)$ is fixed but arbitrary and that $\lambda(z)$ is any first order formula such that whenever $f \in \mathbf{B}(R)$ satisfies $\bar{f}(x) \neq 0$, then there exists $g \in \mathbf{B}(R)$ such that $0 \neq g < f$ and $R \models \lambda(g)$. Then there exists $e \in \mathbf{B}(S)$ such that $S \models \lambda(e)$ and, whenever $f \in \mathbf{B}(R)$ and $\bar{f}(x) \neq 0, 0 \neq e < f$.* Conclusions (i) and (ii) follow similarly, when $\lambda(z)$ is $(z = z)$. Secondly, in conclusion (v), there exists a homeomorphism g: $C_1 \cong C_2$, since the Boolean algebras corresponding to C_1 and C_2 are isomorphic. (They are isomorphic as they are saturated, elementarily equivalent, and have the same cardinality.) Trivially, g can be extended to a homeomorphism of $\mathbf{X}(S)$. We omit detailed proofs, as similar arguments have been routine to model theorists since saturated structures were first defined. (cf. [7, p. 524], for historical background material.) Conclusion (iii) follows from definition 2.8 and the hypothesis "$\mathbf{B}(R) \prec \mathbf{B}(S)$", without recourse to arguments involving saturation.

PROPOSITION 2.10. *Suppose that f: $R \subseteq S$ is a conformal ring monomorphism such that $\mathbf{B}(R) \prec \mathbf{B}(S)$ and that $\mathbf{B}(S)$ is $|\mathbf{B}(R)|^+$-saturated. Let $Y = \mathbf{X}(S)$, $X = \mathbf{X}(R)$, and let π be the map π_f: $Y \to X$. Then*:

(i) $(\pi^{-1}(x))^{\circ -} = \pi^{-1}(x)$, *for all $x \in X$ ($^\circ$ and $^-$ are the topological interior and closure operators respectively.)*

(ii) $\bigcup\{(\pi^{-1}(x))^\circ$: $x \in X\}$ *is a dense open subset of Y.*

(iii) *If $x \in X$ is atomic, then $\pi^{-1}(x)$ consists only of atomic points. The corresponding result applies if x is atomless.*

(iv) *If $x \in X$ is transitional, then $(\pi^{-1}(x))^\circ$ contains a transitional point.*

Suppose in addition that $\mathbf{B}(S)$ is saturated. Then:

(v) *Suppose that C_1 and C_2 are disjoint clopen subsets of Y containing only atomless points. Then there is a homeomorphism g: $Y \to Y$ such that $g|_{C_1}$: $C_1 \to C_2$ and $g|_{C_2}$: $C_2 \to C_1$ are homeomorphisms, and $g(y) = y$, whenever $y \notin C_1 \cup C_2$.*

THEOREM 2.11. *Suppose that* $f\colon R \subseteq S$ *is a local monomorphism such that* $\mathbf{B}(R) \prec \mathbf{B}(S)$, $|\mathbf{B}(R)| < |\mathbf{B}(S)|$, *and* $\mathbf{B}(S)$ *is saturated. Then there is a (G-symmetric of course) twisting* $\mathscr{T}_G = (G, \Psi)$ *on* $\mathscr{L}(f)$, *for some group* G *of homeomorphisms of* $\mathbf{X}(S)$, *such that* $R \cong \Gamma(\mathbf{X}(S), \mathscr{L}(f), \mathscr{T}_G)$. *Thus* R *is the ring of all* $\mathscr{T}_G$*-equivariant sections from* $\Gamma(\mathbf{X}(S), \mathscr{L}(f))$.

Proof. Let $Y = \mathbf{X}(S)$, $X = \mathbf{X}(R)$, $\pi = \pi_f\colon Y \to X$, $\mathscr{L} = \mathscr{L}(f)$, and $(E, \Phi) = \mathscr{T}(f)$. The twisting $\mathscr{T}_G$ is constructed from $\mathscr{T}$ as follows:

(i) $G = \{g\colon Y \to Y$ is a homeomorphism such that for each $x \in X$ there exists a clopen set $C_{x,g} \subseteq \pi^{-1}(x)$ such that $g(C_{x,g}) = C_{x,g}$ and $g(y) = y$, for all $y \in \pi^{-1}(x) - C_{x,g}\}$.

(ii) $\Psi = \{\Phi_{y,z}\colon y$ and $z(\varepsilon Y)$ are in the same orbit of $G.\}$.

This theorem asserts that $\Gamma(Y, \mathscr{L}, \mathscr{T}) = \Gamma(Y, \mathscr{L}, \mathscr{T}_G)$. To see that $\Gamma(Y, \mathscr{L}, \mathscr{T}) \subseteq \Gamma(Y, \mathscr{L}, \mathscr{T}_G)$, let $\sigma \in \Gamma(Y, \mathscr{L}, \mathscr{T})$. Thus $\Phi_{y,z}(\sigma(z)) = \sigma(y)$, whenever $\pi(y) = \pi(z)$ (i.e., whenever $(x, y) \in E$). Note that each orbit of G is contained in an equivalence class of E. We now have $\Psi_{y,z}(\sigma(z)) = \Phi_{y,z}(\sigma(z)) = \sigma(y)$, whenever $g \in G$ and $g(z) = y$. Hence $\sigma \in \Gamma(Y, \mathscr{L}, \mathscr{T}_G)$. To prove the reverse inclusion let $\sigma \in \Gamma(Y, \mathscr{L}, \mathscr{T}_G)$. By the construction of $\mathscr{T} = \mathscr{T}(f)$ (in Definition 2.5) it suffices to show that, where $x \in X$ is fixed but arbitrary, there exists $r \in \Gamma(Y, \mathscr{L}, \mathscr{T})$ such that $(\sigma - r)|_{\pi^{-1}(x)} = 0$. We only give the proof when x is transitional, as easier versions of our argument apply to the other two cases. The reader should clarify the following steps with a diagram. Use Proposition 2.10 (iv) to choose a transitional point $y \in \pi^{-1}(x)^{\circ}$. By the construction of $\mathscr{L}$ and $\mathscr{T}$ there exists $r \in \Gamma(Y, \mathscr{L}, \mathscr{T})$ $(\cong R)$ such that $r(y) = \sigma(y)$. Chosen clopen C such that $y \in C$, $C \subseteq \pi^{-1}(x)^{\circ}$, and $(\sigma - r)|_C = 0$. As y is transitional, there exist points y_s and $y_p \in C$ such that y_p has a clopen neighbourhood $U \subseteq C$ containing no isolated points and y_s is isolated. Suppose contrary to our wishes that $(\sigma - r)|_{\pi^{-1}(x)} \neq 0$. By Proposition 2.10 (i), there exists clopen non empty $D \subseteq \pi^{-1}(x)$ such that $\sigma(z) \neq r(z)$, for all $z \in D$. Note that $C \cap D = \emptyset$. We shall now establish:

(∗) There exists $d \in D$, $c \in C$, and $g \in G$, such that $g(d) = c$ and $g(c) = d$.

If D contains an isolated point z_s, then $g \in G$ where $g(z_s) = y_s$, $g(y_s) = z_s$, and $g(y) = y$ whenever $y \notin \{y_s, z_s\}$. Otherwise Proposition 2.10(v) yields $g \in G$ such that $g(D) = U$ and $g(U) = D$. As $\sigma \in \Gamma(Y, \mathscr{L}, \mathscr{T}_G)$ and $r \in R \cong \Gamma(Y, \mathscr{L}, \mathscr{T}) \subseteq \Gamma(Y, \mathscr{L}, \mathscr{T}_G)$, (∗) establishes the theorem via the following contradiction: $\sigma(d) = \sigma(g(c)) = \Psi_{g(c),c}(\sigma(c)) = \Phi_{g(c),c}(r(c)) = r(g(c)) = r(d)$. □

3. Representing rings via twisted functions

In this section we apply Theorems 2.6 and 2.11 when $R \subseteq S$ is a local monomorphism and S is a ring of continuous functions defined on its Boolean spectrum, to represent R as a ring of twisted functions (cf. Remark 3.1 and Theorem 3.5.) In Theorem 3.7 we obtain a characterization of those rings that can be represented using various kinds of twisted functions. Some particular rings are so represented in application 3.8. To begin, we clarify (and define) the relationship between twisted sections and twisted functions.

Remark 3.1 (Including some definitions). Suppose that f: $R \subseteq S$ is a conformal monomorphism, where $\hat{}: S \cong \mathscr{C}(\mathbf{X}(S), F)$, for some indecomposable F and isomorphism $\hat{}$. Let

$$\bar{}: R \cong \Gamma(\mathbf{X}(R), \mathbf{k}(R))$$

be the standard isomorphism. Note that:

(1) $\mathbf{B}(R) \subseteq \mathbf{B}(S)$ and there is a continuous onto map π_f: $\mathbf{X}(S) \to \mathbf{X}(R)$, which we denote by π in this remark.

(2) $\mathbf{k}(S)$ is the simple sheaf $\mathbf{X}(S) \times F$ with the product topology, where $\mathbf{k}_x(S) = \{x\} \times F \cong F$, for each $x \in \mathbf{X}(S)$ (cf. [12, Definition 11.2]).

(3) There is an isomorphism Λ^{-1}: $\mathscr{C}(\mathbf{X}(S), F) \cong \Gamma(\mathbf{X}(S), \mathbf{k}(S))$ given by $\Lambda^{-1}(\sigma)(x) = (x, \sigma(x))$. Thus Λ: $\Gamma(\mathbf{X}(S), \mathbf{X}(S) \times F) \cong \mathscr{C}(\mathbf{X}(S), F)$ is an isomorphism.

(4) Using (3), f is a local monomorphism iff the map Θ_y: $\mathbf{k}_{\pi(y)}(R) \to F$ given by $\Theta_y(\bar{r}(\pi(y))) = \hat{r}(y)$ (where r varies over R) is a monomorphism, for each $y \in \mathbf{X}(S)$.

Now assume that f: $R \subseteq S$ is a local monomorphism.

(5) For each $y \in \mathbf{X}(S)$, let $F_y = \Theta_y(\mathbf{k}_{\pi(y)}(R))$. Thus

$$\mathscr{L}_y(f) \cong \mathbf{k}_{\pi(y)}(R) \overset{\Theta_y}{\cong} F_y \subseteq F.$$

Consequently

(6) $\Gamma(\mathbf{X}(S), \mathscr{L}(f)) \cong \mathscr{C}(\mathbf{X}(S), F; \{(\{y\}, F_y)\text{: } y \in \mathbf{X}(S)\}$.

It is traditional (although not necessary) to avoid repetitions of the F_y in situations similar to (6) as follows: Let $\{F_\alpha\text{: } \alpha \in \mathscr{I}\}$ be an indexing of $\{F_y\text{: } y \in \mathbf{X}(S)\}$, with $\mathscr{I} \cap \mathbf{X}(S) = \emptyset$ and $F_\alpha \neq F_\beta$ whenever $\alpha \neq \beta$ and $\alpha, \beta \in \mathscr{I}$. Fix arbitrary $\alpha \in \mathscr{I}$ and let $C_\alpha = \{y \in \mathbf{X}(S)\text{: } F_y \subseteq F_\alpha\}$. As elements of $\mathscr{C}(\mathbf{X}(S), F)$ are locally constant and $C_\alpha = \cap_{r \in R}(\{y \in \mathbf{X}(S)\text{: } \hat{r}(y) \in F_\alpha\})$, we see that C_α is closed in $\mathbf{X}(S)$.

By modifying arguments from [1] we have $C_\alpha \subseteq C_\beta$ iff $F_\alpha \subseteq F_\beta$, for all α and $\beta \in \mathscr{I}$, and

(7) $\Gamma(\mathbf{X}(S), \mathscr{L}(f)) \cong \mathscr{C}(\mathbf{X}(S), F; \{(C_\alpha, F_\alpha): \alpha \in \mathscr{I}\})$.

Note that F_y can be recovered as $F_y = \cap\{F_\alpha: y \in C_\alpha\}$, whenever $\alpha \in \mathscr{I}$.
Let $(E, \Phi) = \mathscr{T}(f)$ (cf. Definition 2.5). By (5), Φ can also be viewed as a collection of isomorphisms $\Phi_{z,y}$: $F_y \cong F_z$, where y and $z \in \mathbf{X}(S)$ and $\pi(y) = \pi(z)$. By modifying Definition 2.5 to this setting we obtain:

(8) $\mathscr{C}(\mathbf{X}(S), F: \{(C_\alpha, F_\alpha): \alpha \in \mathscr{I}\}; \mathscr{T}(f))$ $(= \mathscr{C}(\mathbf{X}(S), F; \{(\{y\}, F_y): y \in X(S)\}; \mathscr{T}(f))$
$= \{\sigma \in \mathscr{C}(\mathbf{X}(S), F; \{(C_\alpha, F_\alpha): \alpha \in \mathscr{I}\}): \sigma(z) = \Phi_{z,y}(\sigma(y))$, whenever $(y, z) \in E\}$.

By Theorem 2.6 we now have:

(9) $R \cong \mathscr{C}(\mathbf{X}(S), F; \{(C_\alpha, F_\alpha): \alpha \in \mathscr{I}\}; \mathscr{T}(f))$.

(10) The natural analogues of (8) and (9) apply to the symmetric twisting $\mathscr{T}_G$ which exists whenever the hypothesis of Theorem 2.11 are satisfied.

(11) Elements of the rings from (8)–(10) are called twisted functions or $\mathscr{T}(f)$ equivariant functions or (ambiguously in (10)) G-equivariant functions. □

We have already spoken loosely of various kinds of function rings, in this paper. A formal definition of these concepts follows.

Definition 3.2. In this definition R always denotes a ring, $\mathscr{M}$ a class of indecomposable rings, F an element of $\mathscr{M}$, $\{F_\alpha: \alpha \in \mathscr{I}\}$ a set of subrings of F, Y a Boolean space, $\{C_\alpha: \alpha \in \mathscr{I}\}$ a collection of closed subsets of Y such that $C_\alpha \subseteq C_\beta$ iff $F_\alpha \subseteq F_\beta$ (where $\alpha, \beta \in \mathscr{I}$), and $\mathscr{T}$ a twisting on $\{F_y$: $y \in Y\}$, as described in Remark 3.1.

(1) R is an *$\mathscr{M}$-function ring* iff there is a local monomorphism of the form f: $R \subseteq \mathscr{C}(Y, F)$.

(2) R is a *standard $\mathscr{M}$-function ring* iff it has the form

$$R = \mathscr{C}(\mathbf{X}(R), F; \{(C_\alpha, F_\alpha); \alpha \in \mathscr{I}\}).$$

(3) R is a *twisted $\mathscr{M}$-function ring* iff it has the form

$$(*) \qquad R = \mathscr{C}(Y, F; \{(C_\alpha, F_\alpha): \alpha \in \mathscr{I}\}; \mathscr{T}),$$

where Y is the Stone space of some Boolean algebra $A \supseteq \mathbf{B}(R)$, and the canonical monomorphism $R \subseteq \mathscr{C}(Y, F)$ is local.

(4) R is a *non-standard $\mathscr{M}$-function ring* iff it is a twisted $\mathscr{M}$-function ring as in (3), $A \succ \mathbf{B}(R)$, $A \neq \mathbf{B}(R)$, and $\mathscr{T}$ is G-symmetric for some group G of homeomorphisms of Y.

Note that if $\mathscr{T} = (E, \Phi)$ and each $\Phi_{y,z}$ were an automorphism of some fixed $H \supseteq F$ rather than merely an isomorphism $F_z \cong F_y$, then the ring R in definition 3.2 (3) $(*)$ would remain the same. Classical formulations of twisting use such automorphisms rather than our "partial automorphisms" $\Phi_{y,z}$. Our formulation of twisting, via the partial automorphism $\Phi_{y,z}\colon F_z \cong F_y$, can not always be reduced to the classical one. (To find a counterexample example, let $\mathscr{M}$ be the class of real closed fields and find $\mathscr{T} = (E, \Phi)$ and $y, z \in Y$ such that $\Phi_{z,y}\colon F_y = Q(\sqrt{2}) \to Q(\sqrt{2}) = F_z$ satisfies $\Phi_{z,y}(\sqrt{2}) = -\sqrt{2}$.) However the following modification of [10, Chapter 4, §2, example 14, (p. 153] (with hints on page 124) shows that there are important situations in which this is possible.

LEMMA 3.3. *Retain the notation of Definition* 3.2 *and suppose that* $\mathscr{T} = (E, \Phi)$. *Suppose in addition that* $\mathscr{M}$ *has the amalgamation property and that it is the class of models of a set of universal sentences. Then there exists* $H \in \mathscr{M}$ *such that* $H \supseteq F$ *and each* $\Phi_{z,y}$ *can be extended to an automorphism of* H.

(In fact, H is simply a suitably chosen saturated element of $\mathscr{M}$.)

That the class of all fields satisfies the amalgamation property is both basic and well known. In the 1960's P. M. Cohn established the far deeper fact that the class of all division rings also satisfies the amalgamation property, as a consequence of his innovative work on free ideal rings. A very nice outline of Cohn's lengthly proof is given in [10, pp. 108–124].

COROLLARY 3.4. *Suppose that R has the form* $(*)$ *from Definition* 3.2 (3), *that* $\mathscr{T} = (E, \Phi)$, *and that*

(i) $\mathscr{M}$ *is the class of fields, or*

(ii) $\mathscr{M}$ *is the class of division rings and R is a von Neumann regular ring having no nilpotent elements.*

Then R admits a representation as in $(*)$, *such that each* $\Phi_{z,y}$ *is actually an automorphism of* F, *for all* $y, z \in Y$.

Proof. In case (i) the corollary follows from Lemma 3.3 as the class of commutative integral domains satisfies the state hypothesis. In case (ii) note that $\mathscr{M}$ is the class of all rings satisfying

$$(\forall r)(r \neq 0 \Rightarrow rr^{-1} = 1).$$

Thus $\mathscr{M}$ can be axiomatized using universal sentences in a language $\bar{\mathscr{L}}$ that is suitable for rings and includes an additional unary operation symbol, corresponding to $^{-1}$. The corollary would now follow from Lemma 3.3 so long as each F_y were an $\mathscr{L}$-substructure of F (i.e., were also a division ring.) However this is so as, under our hypothesis, $\mathbf{k}(R)$ is a sheaf of division rings and $F_y \cong \mathbf{k}_{\pi(y)}(R)$, for all $y \in Y$. (cf. [3, p. 256, Remark (a)].) □

In Definition 3.2 we may, when it is helpful to do so, specify the domain of R to be Y. When convenient we shall also say that R is any of the above kinds of function rings iff it is isomorphic to one of that kind. Of course, with this usage, the domain of R is no longer unique, and R might be both standard and non-standard. Note that the domain of a standard function ring R (in the strict sense of our definitions) must be $\mathbf{X}(R)$.

For the rest of this paper, R denotes a ring, Y and Z denote Boolean spaces, and $\mathscr{M}$ denotes a class of indecomposable rings. An easy translation (va Remark 3.1 and Definition 3.2) of theorem 2.6 to function rings yields:

THEOREM 3.5. *R is an $\mathscr{M}$-function ring with domain Y iff it is a twisted $\mathscr{M}$-function ring with domain Y.*

When R is not a standard function ring, it can sometimes be represented as a twisted or even non-standard function ring with a domain Y that is closely related to $\mathbf{X}(R)$. In fact it is the inclusion $\mathbf{B}(R) \subseteq A$ in Definition 3.2 (3) and (4) that yields the familiar continuous onto map $\pi\colon Y \to \mathbf{X}(R)$. Whereas an *intended* embedding $R \xrightarrow{*} \mathscr{C}(\mathbf{X}(R), F)$ may be defective in that $r^*\colon \mathbf{X}(R) \to F$ may fail to be continuous, for some $r \in R$, Y can often be chosen such that r^* can be lifted (through π) to a continuous map $Y \to F$, for each $r \in R$. The full description of R then follows via twisting. Our requirement that $R \subseteq \mathscr{C}(Y, F)$ be a local monomorphism guarantees that the twisting functions $\Phi_{z,y}\colon F_y \cong F_z$, act on copies of the stalks of $\mathbf{k}(R)$, as they do in previous work such as [1], although the conceptual framework used there differs from ours. Later in this paper we shall, in essence, give more details about the epimorphism $\pi\colon Y \to \mathbf{X}(R)$, by giving more details about the inclusion $\mathbf{B}(R) \subseteq A$. Specifically we shall seek representations of R for which $\mathbf{B}(R)$ is dense in A (i.e., each non empty open subset of Y contains a subset of the form $\pi^{-1}(V)$ where $V \neq \emptyset$ is clopen in $\mathbf{X}(R)$) or for which $\mathbf{B}(R) \prec A$. In the latter case Y may be viewed as a "non-standard" version of $\mathbf{X}(R)$, much as certain real closed fields with infinitesimals are viewed as non-standard versions of the real numbers. Elements of $\pi^{-1}(x)$, where $x \in \mathbf{X}(R)$, may then be viewed as being infinitely close in Y. Thus there is an interplay between our concept of non-standard, and the concept already used by analysts.

Even if an *intended* map $*\colon R \to \mathscr{C}(\mathbf{X}(R), F)$ is defective as described above, it will still be an embedding $*\colon R \to F^{\mathbf{X}(R)}$. Such maps arise naturally as follows:

Observation 3.6. Suppose that F is a ring such that, for each $x \in \mathbf{X}(R)$, there is an embedding $\mathbf{k}_x(R) \to F$. Then these embeddings induce

(α) an embedding $\prod_{x \in \mathbf{X}(R)}(\mathbf{k}_x(R)) \to F^{\mathbf{X}(R)}$,

and thus (as $R \cong \Gamma(\mathbf{X}(R), \mathbf{k}(R)) \subseteq F^{\mathbf{X}(R)}$),

(β) an embedding $*$: $R \to F^{\mathbf{X}(R)}$.

If in fact $*$ is an embedding

$$R \xrightarrow{*} \mathscr{C}(\mathbf{X}(R), F),$$

then range(r^*) is finite, for each $r \in R$. Surprisingly, although the converse fails, we do have:

THEOREM 3.7. *Suppose that F is an indecomposable ring and that $\mathscr{M} = \{F\}$. Then the following are equivalent*:

(i) *There exists an embedding $*$: $R \to F^{\mathbf{X}(R)}$, arising as in observation* 3.6, *such that range(r^*) is finite, for all $r \in R$.*

(ii) *R is an $\mathscr{M}$-function ring.*

(iii) *R is a twisted $\mathscr{M}$-function ring.*

(iv) *R is a non-standard $\mathscr{M}$-function ring.*

Proof. Identify $R = \Gamma(\mathbf{X}(R), \mathbf{k}(R))$. Clearly (iv) $\to$ (iii) and (iii) $\to$ (ii).

(ii) $\to$ (i). Suppose that f: $R \to \mathscr{C}(Y, F)$ is a local monomorphism, and recall that π_f: $Y \to \mathbf{X}(R)$ is continuous and onto. For each $x \in \mathbf{X}(R)$ choose $y_x \in Y$ such that $\pi_f(y_x) = x$, and note that we have an embedding $\mathbf{k}_x(R) \to F$ given by $r(x) \to f(r)(y_x)$, where r varies over $R = \Gamma(\mathbf{X}(R), \mathbf{k}(R))$. Thus, letting $*$ arise as in Observation 3.6, we have range(r^*) $\subseteq$ range($f(r)$), for all $r \in R$. Thus (i) holds as, since Y is compact and F is discrete, range($f(r)$) is finite, for each $r \in R$.

(i) $\to$ (iv). Assume (i). By Proposition 3.5 and Theorem 2.11, it suffices to obtain a local monomorphism $R \to \mathscr{C}(\mathbf{X}(A_3), F)$, for some saturated Boolean algebra A_3 such that $|A_3| > |\mathbf{B}(R)|$ and $A_3 \succ \mathbf{B}(R)$. Choose $A_1 \succ \mathbf{B}(R)$ such that A_1 is $|\mathbf{B}(R)|^+$-saturated. Let A_2 be the Boolean completion of A_1, Y be the Stone space of A_2, and let π: $Y \to \mathbf{X}(R)$ be the continuous onto map given by Stone's theorem. For each $x \in \mathbf{X}(R)$, let $U_x = \pi^{-1}(x)^0$, and let

$$U = \bigcup\{U_x\colon x \in \mathbf{X}(R)\}.$$

Proposition 2.10 guarantees that $U_x \neq \emptyset$, for all $x \in \mathbf{X}(R)$, and (as A_1 is dense in A_2) that $U^- = Y$. Define $\hat{}$: $R \to \mathscr{C}(U, F)$ by $\hat{r}(y) = r^*(x)$, whenever $y \in U_x$ and $x \in \mathbf{X}(R)$. As range ($\hat{r}$) = range(r^*), we see that range ($\hat{r}$) is finite, for each $r \in R$. Thus, as A_2 is complete, [5, Proposition 5.32(A)] asserts that each $\hat{r}$ can be extended to an element $\tilde{r}$ of $\mathscr{C}(Y, F)$. By our definitions we have (for all $y \in U$):

(†) The map $\mathbf{k}_{\pi(y)}(R) \to F$ given by $r(\pi(y)) \to \tilde{r}(y)$ (where r varies over $R = \Gamma(\mathbf{X}(R), \mathbf{k}(R))$) is a monomorphism.

As $U^{-} = Y$, (†) also holds for all $y \in Y$. Use this fact to show first that $\tilde{}$: $R \to \mathscr{C}(Y, F)$ is a well defined monomorphism and (trivially) that $\tilde{}$ is local. Note that, whenever π: $Z \to Y$ is a continuous onto map and Z is a Boolean space, the function $\check{}$: $R \to \mathscr{C}(Z, F)$ given by $\check{r}(z) = \tilde{r}(\pi(z))$ (for all $z \in Z$) is a local monomorphism. Thus, to finish our proof, it suffices to obtain a Boolean algebra $A_3 \supseteq A_2$, such that $\mathbf{B}(R) \prec A_3$. (Routinely, when A_3 exists, it may be chosen so as to also be saturated and satisfy $|A_3| > |\mathbf{B}(R)|$.) By [2, Lemma 9.3.9], the required Boolean algebra A_3 exists iff $\mathbf{B}(R) \subseteq_{\forall} A_2$. We are now done, as $\mathbf{B}(R) \prec A_1$, and as [6, Lemma 2.3] guarantees that $A_1 \subseteq_{\forall} A_2$. □

APPLICATION 3.8. *Suppose that R is a commutative semi-simple algebraic algebra over a field L. Moreover, suppose that:*

(1) *F is the algebraic closure of L, or*

(2) *R and (hence L) are formally real and that F is the real closure of L. Then R is a non-standard $\{F\}$-function ring.*

Proof. Routinely (cf. [1]) R is a von Neumann regular ring, so that its Jacobson radical is zero and its spectrum is homeomorphic to $\mathbf{X}(R)$. Thus there is an embedding $*$: $R \to F^{\mathbf{X}(R)}$ of L-algebras. Moreover range(r^*) is finite, for all $r \in R$, as R is algebraic over L. The conclusion now follows from Theorem 3.7. □

We finish this paper with some remarks.

Remarks. (1) Any of the rings from [3, p. 250], [5, example 5.31′], or [13, p. 139] satisfy (i)–(iv) from Theorem 3.7, yet do not have an elementary extension that is an $\mathscr{M}$-standard function ring, where $\mathscr{M}$ is the class of ordered fields.

(2) If R has clopen elementary support (cf. [5, definition 3.4]), then $R \prec S$ whenever $R \prec T$ and S is the subring of T generated by $R \cup \mathbf{B}(T)$. Consequently if any of (i)–(iv) from Theorem 3.7 apply to such an R, then our current proofs guarantee that there does exist a standard $\mathscr{M}$-function ring $S \succ R$.

(3) The results from [1] showed that a commutative semi-simple algebraic algebra over a field L with algebraic closure F could be represented using twisted functions defined on $\mathbf{X}(R \otimes_L F)$. The methods used there were entirely different, and did not involve anything analogous to our concept of a

twisted function ring with a non-standard domain. New examples of twisted function rings can easily be produced using Theorem 3.7.

(4) Naturally occurring examples of rings to which our work applies can be found in [8] and [9]. Both [1] and [9] establish that there exist twisted function rings that are not standard function rings.

(5) Application 3.8 can be strengthened to include the following conclusion: R is an $\{F\}$-function ring with domain Y, where Y is the Stone space of the Boolean completion of $\mathbf{B}(R)$. This follows from Theorem 3.5 since, in both cases, [3, Theorem 3.4], [5, Lemma 5.27] and [13, Lemma 6.1] ensure that there is an embedding $R \to \mathscr{C}(Y, F)$.

(6) We believe (but have not shown) that there is a function ring whose domain can not be chosen to be the completion of its Boolean spectrum.

(7) The rings characterized by [5, Theorem 5.17 and Lemma 5.7(D)], can be represented using Theorems 2.6 and 2.11, but usually not using Theorem 3.7.

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