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ON THE MEAN SQUARE VALUE OF THE HURWITZ ZETA-FUNCTION

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1. Introduction

For a complex number $s = \sigma + it$ and a real number $0 < \alpha < 1$ let $\xi(s, \alpha)$ be the Hurwitz zeta-function defined by

$$\xi(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}$$

for $\mathrm{Re}(s) > 1$, and its analytic continuation for $\mathrm{Re}(s) \leq 1$, and let $\xi_1(s, \alpha) = \xi(s, \alpha) - \alpha^{-s}$.

The main purpose of this paper is to study the asymptotic properties of the mean square value

$$\int_0^1 \xi_1(\sigma_1 + it, \alpha)\xi_1(\sigma_2 - it, \alpha)\, d\alpha \tag{1}$$

where $0 < \sigma_1, \sigma_2 < 1$ and t is an arbitrary real number.

V. V. Rane [1] proved that

$$\int_0^1 \left|\zeta_1\left(\tfrac{1}{2} + it, \alpha\right)\right|^2 d\alpha = \ln t + O(1)$$

holds uniformly in $t > 2$.

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The author [2] obtained a sharper asymptotic formula, namely

$$\int_0^1 \left|\xi_1\left(\frac{1}{2} + it, \alpha\right)\right|^2 d\alpha = \ln\left(\frac{t}{2\pi}\right) + \gamma + \Delta(t),$$

where $\Delta(t) \ll t^{-3/16}\ln^{11/8}t$, γ is Euler's constant and $t > 2$.
In [3] he improved $3/16$ to $7/36$.
In this paper, we shall derive more precise results.

THEOREM 1. *For any real number $t > 2$, the asymptotic formula*

$$\int_0^1 \xi_1(\sigma + it, \alpha)\xi_1(1 - \sigma - it, \alpha)\, d\alpha$$
$$= \ln\left(\frac{t}{2\pi}\right) + \gamma - \frac{\xi(\sigma + it)}{\sigma + it} - \frac{\xi(1 - \sigma - it)}{1 - \sigma - it} + O\left(\frac{1}{t}\right)$$

holds uniformly in $0 < \sigma < 1$, where $\xi(s)$ is the Riemann zeta-function.

THEOREM 2. *For real numbers $0 < \sigma_1, \sigma_2 < 1$ with $\sigma_1 + \sigma_2 \neq 1$, and for any real number $t > 2$, the asymptotic formula*

$$\int_0^1 \xi_1(\sigma_1 + it, \alpha)\xi_1(\sigma_2 - it, \alpha)\, d\alpha$$
$$= \frac{2\Gamma(1 - \sigma_1 - it)\Gamma(1 - \sigma_2 + it)}{(2\pi)^{2-\sigma_1-\sigma_2}}$$
$$\times \xi(2 - \sigma_1 - \sigma_2)\cos\left(\frac{\pi}{2}(\sigma_1 - \sigma_2) + \pi it\right)$$
$$- \frac{1}{1 - \sigma_1 - \sigma_2} - \frac{\xi(\sigma_2 - it)}{1 - \sigma_1 - it} - \frac{\xi(\sigma_1 + it)}{1 - \sigma_2 + it} + O\left(\frac{1}{t}\right)$$

holds uniformly in σ_1 and σ_2.

THEOREM 3. *We have*

$$\int_0^1 \xi_1^2\left(\frac{1}{2}, \alpha\right) d\alpha = 4 - 4\xi\left(\frac{1}{2}\right) - \ln(8\pi) - 2\sum_{n=1}^{\infty} \frac{\xi\left(n + \frac{1}{2}\right) - 1}{n + \frac{1}{2}}.$$

For the Hurwitz zeta-function $\xi(s, \alpha)$ we can also prove the following.

THEOREM 4. *Let t be a real number, $0 < \sigma_1 + \sigma_2 < 1$. Then we have the identity*

$$\begin{aligned}\int_0^1 \xi(\sigma_1 + it, \alpha)\xi(\sigma_2 - it, \alpha)\, d\alpha \\ = 2\frac{\Gamma(1-\sigma_1 - it)\Gamma(1-\sigma_2 + it)}{(2\pi)^{2-\sigma_1-\sigma_2}} \\ \times \xi(2 - \sigma_1 - \sigma_2)\cos\left(\frac{\pi}{2}(\sigma_1 - \sigma_2) + \pi i t\right).\end{aligned}$$

From the theorems we may immediately deduce the following corollaries.

COROLLARY 1. *For any real number $t > 2$ we have*

$$\int_0^1 \left|\xi_1\left(\frac{1}{2} + it, \alpha\right)\right|^2 d\alpha = \ln\left(\frac{t}{2\pi}\right) + \gamma - \frac{\xi\left(\frac{1}{2} + it\right)}{\frac{1}{2} + it} - \frac{\xi\left(\frac{1}{2} - it\right)}{\frac{1}{2} - it} + O\left(\frac{1}{t}\right)$$

COROLLARY 2. *For $0 < \sigma < 1$ and $\sigma \neq 1/2$ we have*

$$\begin{aligned}\int_0^1 |\xi_1(\sigma + it, \alpha)|^2 d\alpha = \left|\frac{t}{2\pi}\right|^{1-2\sigma} \xi(2 - 2\sigma) - \frac{1}{1-2\sigma} - \frac{\xi(\sigma + it)}{1 - \sigma + it} \\ - \frac{\xi(\sigma - it)}{1 - \sigma - it} + O\left(\frac{1}{t}\right) + O\left(\frac{1}{t^{2\sigma}}\right)\end{aligned}$$

where t is any real number satisfying $|t| > 1$.

COROLLARY 3. *For any fixed $\sigma > 1/2$ we have*

$$\lim_{t\to\infty} \int_0^1 |\xi_1(\sigma + it, \alpha)|^2 d\alpha = \frac{1}{2\sigma - 1}.$$

2. Some lemmas

To complete the proof of the theorems, we need several lemmas.

LEMMA 1. *Let $0 < \sigma_1, \sigma_2 < 1$, $t > 2$. Then we have*

$$\int_0^1 \alpha^{-\sigma_1 - it}\xi_1(\sigma_2 - it, \alpha)\, d\alpha = \frac{\xi(\sigma_2 - it)}{1 - \sigma_1 - it} + O\left(\frac{1}{t}\right)$$

Proof. Using integration by parts repeatedly and noticing that

$$\frac{\partial}{\partial\alpha}\xi_1(s,\alpha) = -s\xi_1(s+1,\alpha),$$

we have

$$\begin{aligned}
&\int_0^1 \alpha^{-\sigma_1-it}\xi_1(\sigma_2-it,\alpha)\,d\alpha \\
&\quad= \frac{\alpha^{1-\sigma_1-it}\xi_1(\sigma_2-it,\alpha)}{1-\sigma_1-it}\Bigg|_0^1 \\
&\qquad+ \int_0^1 \frac{\sigma_2-it}{1-\sigma_1-it}\alpha^{1-\sigma_1-it}\xi_1(1+\sigma_2-it,\alpha)\,d\alpha \\
&\quad= \frac{\xi(\sigma_2-it)-1}{1-\sigma_1-it} + \frac{\sigma_2-it}{1-\sigma_1-it}\int_0^1 \alpha^{1-\sigma_1-it}\xi_1(1+\sigma_2-it,\alpha)\,d\alpha \\
&\quad= \frac{\xi(\sigma_2-it)-1}{1-\sigma_1-it} + \frac{(\sigma_2-it)(\xi(1+\sigma_2-it)-1)}{(1-\sigma_1-it)(2-\sigma_1-it)} \\
&\qquad+ \cdots + \frac{(\sigma_2-it)\cdots(n+\sigma_2-it)(\xi(n+1+\sigma_2-it)-1)}{(1-\sigma_1-it)(2-\sigma_1-it)\cdots(n+2-\sigma_1-it)} \\
&\qquad+ \frac{(\sigma_2-it)\cdots(n+1+\sigma_2-it)}{(1-\sigma_1-it)\cdots(n+2-\sigma_1-it)} \\
&\qquad\quad\times\int_0^1 \alpha^{n+2-\sigma_1-it}\xi_1(n+2+\sigma_2-it,\alpha)\,d\alpha \\
&\quad= \frac{\xi(\sigma_2-it)-1}{1-\sigma_1-it} + O\left[\frac{1}{t}\left(\frac{1}{2^{1+\sigma_2}} + \frac{1}{2^{2+\sigma_2}} + \cdots + \frac{1}{2^{n+2+\sigma_2}}\right)\right] \\
&\qquad+ O\left(\int_0^{\frac{1}{2}} \alpha^{n+1}\,d\alpha\right) + O\left(\int_{\frac{1}{2}}^1 \xi_1\left(n+2+\sigma_2,\frac{1}{2}\right)d\alpha\right) \\
&\quad= \frac{\xi(\sigma_2-it)}{1-\sigma_1-it} + O\left(\frac{1}{2^n}\right) + O\left(\frac{1}{(3/2)^n}\right) + O\left(\frac{1}{t}\right) \\
&\quad= \frac{\xi(\sigma_2-it)}{1-\sigma_1-it} + O\left(\frac{1}{t}\right),\ n \ge 3\ln t,
\end{aligned}$$

where we have used the estimates

$$\frac{(\sigma_2 - it)\cdots(n+\sigma_2 - it)}{(1-\sigma_1 - it)\cdots(n+2-\sigma_1-it)}\xi_1(n+1+\sigma_2 - it, 1)$$
$$\ll \frac{\xi_1(n+1+\sigma_2, 1)}{t} \ll \frac{1}{t2^{n+1+\sigma_2}}.$$

LEMMA 2. *Let* $0 < \sigma_1, \sigma_2 < 1$ *and let* t *be a real number. Then for* $\mathrm{Re}(w) < 0$ *we have*

$$\int_0^1 \xi_1(\sigma_1 + it + w, \alpha)\xi_1(\sigma_2 - it + w, \alpha)\, d\alpha$$
$$= \frac{1}{2w+\sigma_1+\sigma_2-1} - \int_0^1 \alpha^{-(\sigma_1+it+w)}\xi_1(\sigma_2 - it + w, \alpha)\, d\alpha$$
$$- \int_0^1 \alpha^{-(\sigma_2-it+w)}\xi_1(\sigma_1 + it + w, \alpha)\, d\alpha$$
$$+ 2\frac{\Gamma(1-\sigma_1-it-w)\Gamma(1-\sigma_2+it-w)}{(2\pi)^{2-\sigma_1-\sigma_2-2w}}\xi(2-\sigma_1-\sigma_2-2w)$$
$$\times\cos\left(\frac{\pi}{2}(\sigma_1-\sigma_2)+\pi it\right).$$

Proof. For complex w, we define $T(w)$ by

$$T(w) = \int_0^1 \xi(\sigma_1 + it + w, \alpha)\xi(\sigma_2 - it + w, \alpha)\, d\alpha$$

From [4], Theorem 12.6, we know that for $\mathrm{Re}(s) > 1$,

$$\xi(1-s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s}\left\{e^{-\frac{\pi is}{2}}F(\alpha, s) + e^{\frac{\pi is}{2}}F(-\alpha, s)\right\}, \qquad (2)$$

where

$$F(x, s) = \sum_{n=1}^{\infty}\frac{e^{2\pi inx}}{n^s}$$

is an absolutely convergent series. Using

$$\int_0^1 e^{2\pi in\alpha}\, d\alpha = \begin{cases}1, & n = 0,\\ 0, & n \neq 0,\end{cases}$$

we get, from (2),

$$T(w) = \frac{\Gamma(1-\sigma_1-it-w)\Gamma(1-\sigma_2+it-w)}{(2\pi)^{2-2w-\sigma_1-\sigma_2}}$$

$$\cdot\int_0^1 \{e^{-\pi i(1-\sigma_1-it-w)/2}F(\alpha, 1-\sigma_1-it-w)$$

$$+e^{\pi i(1-\sigma_1-it-w)/2}F(-\alpha, 1-\sigma_1-it-w)\}$$

$$\cdot\{e^{-\pi i(1-\sigma_2+it-w)/2}F(\alpha, 1-\sigma_2+it-w)$$

$$+e^{\pi i(1-\sigma_2+it-w)/2}F(-\alpha, 1-\sigma_2+it-w)\}\,d\alpha$$

$$= \frac{\Gamma(1-\sigma_1-it-w)\Gamma(1-\sigma_2+it-w)}{(2\pi)^{2-\sigma_1-\sigma_2-2w}}$$

$$\times\{e^{\pi i(\sigma_1-\sigma_2+it)/2}\xi(2-\sigma_1-\sigma_2-2w)$$

$$+e^{-\pi i(\sigma_1-\sigma_2+2it)/2}\xi(2-\sigma_1-\sigma_2-2w)\}$$

$$= 2\frac{\Gamma(1-\sigma_1-it-w)\Gamma(1-\sigma_2+it-w)}{\cdot(2\pi)^{2-\sigma_1-\sigma_2-2w}}\xi(2-\sigma_1-\sigma_2-2w)$$

$$\times\cos\left(\frac{\pi}{2}(\sigma_1-\sigma_2)+\pi it\right). \quad (3)$$

On the other hand, we have

$$T(w) = \int_0^1 \xi_1(\sigma_1+it+w,\alpha)\xi_1(\sigma_2-it+w,\alpha)\,d\alpha$$

$$+\frac{1}{1-\sigma_1-\sigma_2-2w}+\int_0^1 \alpha^{-(\sigma_1+it+w)}\xi_1(\sigma_2-it+w,\alpha)\,d\alpha$$

$$+\int_0^1 \alpha^{-(\sigma_2-it+w)}\xi_1(\sigma_1+it+w,\alpha)\,d\alpha. \quad (4)$$

From (3) and (4) we know that Lemma 2 holds for $\mathrm{Re}(s) < -1$. The lemma remains valid for $\mathrm{Re}(w) < 0$ by analytic continuation.

LEMMA 3. *For any fixed* $0 < \sigma < 1$ *and any real number* $t > 2$, *we have the asymptotic formula*

$$\frac{\Gamma'(\sigma+it)}{\Gamma(\sigma+it)} = \ln t + \frac{1}{2}\pi i + O\left(\frac{1}{t}\right)$$

Proof. See [5], Lemma 3.

3. Proof of the theorems

In this section, we shall complete the proof of the theorems. We first prove Theorem 1. For $\sigma_1 + \sigma_2 = 1$, we get, on letting $w \to 0$ in Lemma 2,

$$\begin{aligned}
&\int_0^1 \xi_1(\sigma + it, \alpha)\xi_1(1 - \sigma - it, \alpha)\, d\alpha \\
&= -\int_0^1 \alpha^{-(\sigma+it)}\xi_1(1 - \sigma - it, \alpha)\, d\alpha - \int_0^1 \alpha^{-(1-\sigma-it)}\xi_1(\sigma + it, \alpha)\, d\alpha \\
&\quad + \lim_{w\to 0} \frac{2\Gamma(1 - \sigma - it - w)\Gamma(\sigma + it - w)}{(2\pi)^{1-2w}}\left(\xi(1 - 2w) + \frac{1}{2w}\right) \\
&\quad \times \cos\left(\frac{\pi}{2}(2\sigma - 1) + \pi it\right) \\
&\quad + \lim_{w\to 0} \frac{1}{2w}\left[1 - \frac{2\Gamma(1 - \sigma - it - w)\Gamma(\sigma + it - w)}{(2\pi)^{1-2w}} \right. \\
&\qquad \left. \times \cos\left(\frac{\pi}{2}(2\sigma - 1) + \pi it\right)\right] \\
&= -\int_0^1 \alpha^{-(\sigma+it)}\xi_1(1 - \sigma - it, \alpha)\, d\alpha - \int_0^1 \alpha^{-(1-\sigma-it)}\xi_1(\sigma + it, \alpha)\, d\alpha \\
&\quad + \frac{1}{2}\left[\frac{\Gamma'(\sigma + it)}{\Gamma(\sigma + it)} + \frac{\Gamma'(1 - \sigma - it)}{\Gamma(1 - \sigma - it)} - 2\ln(2\pi)\right] + \gamma, \qquad (5)
\end{aligned}$$

where we have used the identity $\Gamma(s)\Gamma(1 - s) = \pi/\sin(\pi s)$ and the relation

$$\lim_{s\to 1}\left(\xi(s) + \frac{1}{1 - s}\right) = \gamma.$$

Theorem 1 follows from (5), Lemma 1 and Lemma 3.

For $\sigma_1 + \sigma_2 \neq 1$, we see by analytic continuation that Lemma 2 holds for $\mathrm{Re}(w) \leq 0$. Taking $w = 0$ in Lemma 2 and combining the result with Lemma 1, we immediately deduce Theorem 2.

Taking $\sigma = 1/2$ and $t = 0$ in (5), noting that $(\Gamma'(1/2))/(\Gamma(1/2)) = -\gamma - 2\ln 2$, and proceeding as in the proof of Lemma 1, we get Theorem 3.

Theorem 4 follows from (3) and its analytic continuation. This completes the proof of the theorems.

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