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ON THE INTEGRATION OF VECTOR-VALUED FUNCTIONS

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Introduction

A large number of different methods of integration of Banach-space-valued functions have been introduced, based on the various possible constructions of the Lebesgue integral. They commonly run fairly closely together when the range space is separable (or has w^*-separable dual) and diverge more or less sharply for general range spaces. The McShane integral as described by [7] is derived from the 'gauge-limit' integral of [11]. Here we answer some questions left open in [7] concerning the relationship between the McShane and Pettis integrals. Our original objectives were simply to confirm that McShane integrable functions are Pettis integrable (2C) and to find a Pettis integrable function which is not McShane integrable. Seeking interesting examples of such functions we were led to investigate the connections between the McShane and Talagrand integrals (2L, 2M, 3A, 3C). As far as we know, the 'Talagrand integral' of 1Ab below is explicitly described here for the first time, although all the significant facts we use are given in [13] and [14].

Perhaps we should make some remarks on the context of our results. The ordinary functional analyst is naturally impatient with the multiplicity of definitions of 'integral' which have been proposed for vector-valued functions, and would much prefer to have a single canonical one for general use. In our view the only integral with any claim to such pre-eminence is the Bochner integral (1Ac). But elementary examples (3D below is a classic) show that the Bochner integral is highly restrictive, in that it integrates few functions; and quite simple problems lead us to demand extensions. In this paper we work with three such extensions. We hope that our positive results (e.g., McShane integrable functions are Pettis integrable) will make the jungle seem a little less impenetrable, while our negative results (e.g., the domains of the McShane and Talagrand integrals are incomparable) will at least clarify the irreducible difficulties of the subject. In passing, we prove a weak convergence theorem for the McShane integral (2I-2J), showing that

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this integral is at least an option to consider when faced with a question in vector integration.

There is a survey of methods of vector integration in [9]. Among the integrals discussed there the closest to the work of this paper is the Birkhoff integral [2]. We believe that we can show that the Birkhoff and McShane integrals are closely related, though not identical; we hope to publish our findings elsewhere. Two more recent papers to which our ideas may be relevant are [6], [8].

1A Definitions. We recall the following definitions. Let (S, Σ, μ) be a probability space and X a Banach space, with dual X^*.

(a) A function ϕ: $S \to X$ is *Pettis integrable* if for every $E \in \Sigma$ there is a $w_E \in X$ such that $\int_E f(\phi(x))\mu(dx)$ exists and is equal to $f(w_E)$ for every $f \in X^*$; in this case w_S is the *Pettis integral* of ϕ, and the map $E \mapsto w_E$: $\Sigma \to X$ is the *indefinite Pettis integral* of ϕ.

(b) A function ϕ: $S \to X$ is *Talagrand integrable*, with *Talagrand integral* w, if $w = \lim_{n\to\infty}(1/n)\Sigma_{i<n}\phi(s_i)$ for almost all sequences $\langle s_i\rangle_{i\in\mathbf{N}} \in S^{\mathbf{N}}$, where $S^{\mathbf{N}}$ is given its product probability. (See [14], Theorem 8.)

(c) A function ϕ: $S \to X$ is *Bochner integrable*, with *Bochner integral* w, if for every $\varepsilon > 0$ we can find a partition $E_0, \dots, E_n$ of S into measurable sets and vectors $x_0, \dots, x_n \in X$ and an integrable function h: $S \to \mathbf{R}$ such that

$$\int h \leq \varepsilon, \|\phi(t) - x_i\| \leq h(t)$$

for $t \in E_i$, $i \leq n$ and $\|w - \Sigma_{i\leq n}\mu E_i.x_i\| \leq \varepsilon$.

1B. Now we come to an integral which has been defined for functions with domains which are intervals in $\mathbf{R}$. In fact it can be satisfactorily generalized to very much wider contexts; but as the extension involves ideas from topological measure theory unnecessary for the chief results of this paper, we confine ourselves here to the original special case.

Definitions A *McShane partition* of $[0, 1]$ is a finite sequence

$$\langle([a_i, b_i], t_i)\rangle_{i\leq n}$$

such that $\langle[a_i, b_i]\rangle_{i\leq n}$ is a non-overlapping family of intervals covering $[0, 1]$ and $t_i \in [0, 1]$ for each i. A *gauge* on $[0, 1]$ is a function δ: $[0, 1] \to]0, \infty[$. A McShane partition $\langle([a_i, b_i], t_i)\rangle_{i\leq n}$ is *subordinate* to a gauge δ if $t_i - \delta(t_i) \leq a_i \leq b_i \leq t_i + \delta(t_i)$ for every $i \leq n$.

Now let X be a Banach space. Following [7], we say that a function ϕ: $[0, 1] \to X$ is *McShane integrable*, with *McShane integral* w, if for every $\varepsilon > 0$ there is a gauge δ: $[0, 1] \to]0, \infty[$ such that $\|w - \Sigma_{i\leq n}(b_i - a_i)\phi(t_i)\| \leq \varepsilon$ for every McShane partition $\langle([a_i, b_i], t_i)\rangle_{i\leq n}$ of $[0, 1]$ subordinate to δ.

1C *Summary of results.* With four integrals to play with, a good many questions can be asked; and the situation is complicated by the fact that certain natural restrictions which may be put on the space X and the function ϕ change the answers. We therefore set out the facts in a semi-tabular form. We give references to the literature for those which are already known, and references to paragraphs below for those which we believe to be new.

(a) Consider first the situation in which no restriction is placed on the Banach space X nor on the function $\phi: [0, 1] \to X$. In this context, a Bochner integrable function is Talagrand integrable, a Talagrand integrable function is Pettis integrable, and the integrals coincide whenever defined [14, Theorem 8]. A Bochner integrable function is McShane integrable [7, Theorem 16] (in fact, a measurable Pettis integrable function is McShane integrable—see [7], Theorem 17); a McShane integrable function is Pettis integrable (2C below).

None of the implications here can be reversed. To see this, it is enough to find a McShane integrable function which is not Talagrand integrable (3A, 3E) and a Talagrand integrable function which is not McShane integrable (3C).

(b) Now suppose that the unit ball $B_1(X^*)$ of the dual X^* of X is separable (that is, that X is isometric to a subspace of $l^\infty(\mathbf{N})$) and that ϕ is bounded. In this case, a McShane integrable function must be Talagrand integrable (2M). We ought to observe at this point that in these circumstances the continuum hypothesis, for instance, is enough to make any *Pettis* integrable function Talagrand integrable [13, 6-1-3], and that it remains conceivable that this is a theorem of ZFC (see [12]). But our result in 2M does not depend on any special axiom.

In this context it is still true that a McShane integrable function need not be Bochner integrable (3D) and that a Talagrand integrable function need not be McShane integrable (3C).

(c) If we take X to be separable, but allow ϕ to be unbounded, then the Bochner and Talagrand integrals coincide (see 2K below), and the McShane and Pettis integrals coincide (2D).

There is still a McShane integrable function which is not Talagrand integrable (3E).

(d) For separable X and bounded ϕ, the Bochner and Pettis integrals coincide (2K), so all four integrals here coincide.

(e) Finally, the same is true, for unbounded ϕ, if X is finite-dimensional.

2. Positive results

In this section we give our principal positive results. A McShane integrable function is Pettis integrable (2C); using this we are able to prove a convergence theorem for McShane integrable functions (2I) with some corollaries (2J). We conclude by showing that a bounded McShane integrable function from $[0,1]$ to a space with w^*-separable dual unit ball is Talagrand integrable (2M).

2A. We approach the first result by means of the 'Dunford integral'. Recall that a function ϕ: $[0,1] \to X$ is *Dunford integrable* if $h\phi$: $[0,1] \to \mathbf{R}$ is integrable for every $h \in X^*$; in this case we have an *indefinite Dunford integral* ν: $\Sigma \to X^{**}$, where Σ is the algebra of Lebesgue measurable subsets of $[0,1]$, given by the formula $(\nu E)(h) = \int_E h\phi$ for every $h \in X^*$, $E \in \Sigma$ [13], 4-4-1 or [4], p. 52, Lemma 1). Thus a Pettis integrable function is just a Dunford integrable function whose indefinite integral takes values in X (identified, of course, with its canonical image in X^{**}). Now we have the following general characterization of Pettis integrable functions on $[0,1]$ (which in part, at least, is already known; cf. [5], 4.2).

2B Proposition. *Let X be a Banach space and ϕ: $[0,1] \to X$ a Dunford integrable function with indefinite integral ν: $\Sigma \to X^{**}$. Suppose that $\nu([a,b]) \in X$ for every subinterval $[a,b]$ of $[0,1]$. Then the following are*

(i) *ϕ is Pettis integrable*;

(ii) *for every sequence $\langle [a_i, b_i] \rangle_{i \in \mathbf{N}}$ of non-overlapping subintervals of $[0,1]$, $\sum_{i \in \mathbf{N}} \nu([a_i, b_i])$ exists in X (for the norm of X)*;

(iii) *for every $\varepsilon > 0$ there is an $\eta > 0$ such that $\|\nu E\| \leq \varepsilon$ whenever $\mu E \leq \eta$*;

(iv) *ν is countably additive.*

Proof. (i) $\Rightarrow$ (ii). This is a theorem of Pettis (see [4], II.3.5).

(ii) $\Rightarrow$ (iii). The point is that $\{h\phi\colon h \in B_1(X^*)\}$ is uniformly integrable. To see this, it is enough to show that

$$\lim_{n \to \infty} \sup\left\{ \int_{G_n} |h\phi| : h \in B_1(X^*) \right\} = 0$$

for every disjoint sequence $\langle G_n \rangle_{n \in \mathbf{N}}$ of open sets in $[0,1]$ (see [3], VII.14). But given such a sequence, and a sequence $\langle h_n \rangle_{n \in \mathbf{N}}$ in $B_1(X^*)$, set $\alpha_n = \int_{G_n} |h_n \phi|$ for each n. Then we can find for each n a set $F_n \subseteq G_n$, a finite union of closed intervals, such that $\int_{F_n} |h_n \phi| \geq \alpha_n - 2^{-n}$. Now we can find a sequence $\langle [a_i, b_i] \rangle_{i \in \mathbf{N}}$ of non-overlapping intervals, and an increasing sequence $\langle k_n \rangle_{n \in \mathbf{N}}$ of integers, such that $F_n = \bigcup_{k_n \leq i < k_{n+1}} [a_i, b_i]$ for each n. So

$$\alpha_n - 2^{-n} \leq \left\| \sum_{k_n \leq i < k_{n+1}} \nu([a_i, b_1]) \right\|$$

for each n. But as $\sum_{i \in \mathbf{N}} \nu([a_i, b_i])$ exists in X, $\lim_{n \to \infty} \alpha_n = 0$, as required.

Now it follows that for every $\varepsilon > 0$ there is an $\eta > 0$ such that $\int_E |h\phi| \le \varepsilon$ whenever $h \in B_1(X^*)$, $E \in \Sigma$ and $\mu E \le \eta$, writing μ for Lebesgue measure; so that $\|\nu E\| \le \varepsilon$ whenever $\mu E \le \eta$.

(iii) $\Rightarrow$ (iv). Elementary.

(iv) $\Rightarrow$ (i). Our original hypothesis was that $\nu E \in X$ for intervals E; it follows that $\nu E \in X$ whenever E is a finite union of intervals. Because ν is countably additive, $\nu E \in X$ whenever E is open, and therefore whenever E is G_δ; but also of course $\nu E = 0 \in X$ whenever $\mu E = 0$, so $\nu E \in X$ for every $E \in \Sigma$, and ϕ is Pettis integrable.

2C Theorem. *Let X be a Banach space and ϕ: $[0,1] \to X$ a McShane integrable function. Then ϕ is Pettis integrable.*

Proof. As remarked in [7], Theorem 8, ϕ is Dunford integrable; let ν: $\Sigma \to X^{**}$ be its indefinite Dunford integral. We know also that $\nu([a,b]) \in X$ for every subinterval $[a,b]$ of $[0,1]$ ([7], Theorem 4). So we seek to show that (ii) of 2B above holds true.

Let $\varepsilon > 0$. Let δ: $[0,1] \to]0,\infty[$ be a gauge such that

$$\left\|\nu([0,1]) - \sum_{i\le n}(b_i - a_i)\phi(t_i)\right\| \le \varepsilon$$

whenever $\langle([a_i,b_i],t_i)\rangle_{i\le n}$ is a McShane partition of $[0,1]$ subordinate to δ. Fix a particular McShane partition $\langle([a_i,b_i],t_i)\rangle_{i\le n}$ of $[0,1]$ subordinate to δ, and set $M = \sup_{i\le n}\|\phi(t_i)\|$. We claim that if $E \subseteq [0,1]$ is a finite union of closed intervals then $\|\nu E\| \le M\mu E + 2\varepsilon$. To see this, express E as $\bigcup_{j\le m}[c_j,d_j]$ where the $[c_j,d_j]$ are non-overlapping, and let $\eta > 0$. For each $i \le n$, we can express $[a_i,b_i]\setminus \operatorname{int} E$ as a (possibly empty) finite union of non-overlapping intervals $[a_{ik},b_{ik}]$ for $k < r(i)$; write $t_{ik} = t_i$ for $i \le n$, $k < r()$. Then we see that

$$\left\|\sum_{i\le n}(b_i - a_i)\phi(t_i) - \sum_{i\le n}\sum_{k<r(i)}(b_{ik} - a_{ik})\phi(t_{ik})\right\| \le M\mu E.$$

Next, for each $j \le m$ we can find a McShane partition $\langle([c_{jk},d_{jk}],u_{jk})\rangle_{k\le q(j)}$ of $[c_j,d_j]$, subordinate to δ, such that

$$\left\|\nu([c_j,d_j]) - \sum_{k\le q(j)}(d_{jk} - c_{jk})\phi(u_{jk})\right\| \le \eta.$$

Assembling these, we see that

$$\langle([a_{ik},b_{ik}],t_{ik})\rangle_{i\le n,\,k<r(i)},\ \langle([c_{jk},d_{jk}],u_{jk})\rangle_{j\le m,\,k\le q(j)}$$

together form a McShane partition of $[0,1]$ subordinate to δ, so that

$$\Bigl\|\nu([0,1]) - \sum_{i\le n,\, k<r(i)} (b_{ik} - a_{ik})\phi(t_{ik}) - \sum_{j\le m,\, k\le q(j)} (c_{jk} - d_{jk})\phi(t_{jk})\Bigr\| \le \varepsilon.$$

Also, of course,

$$\Bigl\|\nu([0,1]) - \sum_{i\le n} (b_i - a_i)\phi(t_i)\Bigr\| \le \varepsilon.$$

Putting these formulae together, we get

$$\Bigl\|\sum_{j\le m} \nu\bigl([c_j, d_j]\bigr)\Bigr\| \le M\mu E + 2\varepsilon + m\eta.$$

But η was arbitrary, so $\|\nu E\| \le M\mu E + 2\varepsilon$, as claimed.

Now this means that if $\langle [c_j, d_j]\rangle_{j\in\mathbf{N}}$ is any sequence of non-overlapping intervals,

$$\limsup_{m\to\infty} \sup_{l\ge m} \Bigl\|\sum_{m\le j\le l} \nu\bigl([c_j, d_j]\bigr)\Bigr\| \le 2\varepsilon.$$

But of course ε was arbitrary, so the limit must actually be 0; accordingly $\sum_{j\in\mathbf{N}}\nu([c_j, d_j])$ is defined.

Thus (ii) of 2B is satisfied, and ϕ is Pettis integrable.

2D COROLLARY. *Let X be a separable Banach space. Then a function ϕ: $[0,1] \to X$ is McShane integrable iff it is Pettis integrable.*

Proof. If ϕ is Pettis integrable, it is measurable, because X is separable; so it is McShane integrable by Theorem 17 of [7]. Now 2C gives the reverse implication.

2E. As a further consequence of 2C we have the following.

THEOREM. *Let X be a Banach space and ϕ: $[0,1] \to X$ a McShane integrable function. Then for any measurable $E \subseteq [0,1]$ the function $\phi_E = \phi \times \chi(E)$: $[0,1] \to X$, defined by writing $\phi_E(t) = \phi(t)$ if $t \in E$ and 0 otherwise, is McShane integrable.*

Proof. Again write ν for the indefinite integral of ϕ; we now know that ν takes all its values in X. Let $\varepsilon > 0$. By 2B, we can find an $\eta > 0$ such that

$\|\nu H\| \leq \varepsilon$ whenever $\mu H \leq \eta$. Let $F_1 \subseteq E$, $F_2 \subseteq [0,1] \setminus E$ be closed sets such that $\mu F_1 + \mu F_2 \geq 1 - \eta$. Let $\delta\colon [0,1] \to]0,\infty[$ be a gauge such that

$$\left\| \nu([0,1]) - \sum_{i \leq n} (b_i - a_i)\phi(t_i) \right\| \leq \varepsilon$$

whenever $\langle([a_i, b_i], t_i)\rangle_{i \leq n}$ is a McShane partition subordinate to δ. Let $\delta_1\colon [0,1] \to]0,\infty[$ be such that $\delta_1(t) \leq \delta(t)$ for every t and $[t - \delta_1(t), t + \delta_1(t)] \cap F_j = \emptyset$ whenever $t \notin F_j$, for both $j \in \{1,2\}$.

Suppose that $\langle([a_i, b_i], t_i)\rangle_{i \leq n}$ is a McShane partition of $[0,1]$ subordinate to δ_1. We seek to estimate $\|\nu E - \sum_{i \leq n}(b_i - a_i)\phi_E(t)\|$. Set

$$I = \{i\colon i \leq n,\, t_i \in E\},\ H = \bigcup_{i \in I} [a_i, b_i],$$

Then we must have $F_1 \subseteq H \subseteq [0,1] \setminus F_2$. Consequently $\mu(H \Delta E) \leq \eta$ and $\|\nu E - \nu H\| \leq 2\varepsilon$. But now recall that by Theorem 5 of [7] we know that $\|\nu H - \sum_{i \in I}(b_i - a_i)\phi(t_i)\| \leq \varepsilon$. So

$$\left\| \nu E - \sum_{i \leq n} (b_i - a_i)\phi_E(t_i) \right\| = \left\| \nu E - \sum_{i \in I} (b_i - a_i)\phi(t_i) \right\| \leq 2\varepsilon.$$

As ε is arbitrary, ϕ_E is McShane integrable.

2F. For the next theorem of this section, we need to recall some well-known facts concerning vector measures. Suppose that Σ is a σ-algebra of sets and X a Banach space.

(a) Let us say that a function $\nu\colon \Sigma \to X$ is 'weakly countably additive' if

$$f\left(\nu\left(\bigcup_{i \in \mathbf{N}} E_i\right)\right) = \sum_{i \in \mathbf{N}} f(\nu E_i)$$

for every disjoint sequence $\langle E_i \rangle_{i \in \mathbf{N}}$ in Σ and every $f \in X^*$. The first fact is that in this case ν is countably additive, that is, $\sum_{i \in \mathbf{N}} \nu E_i$ is unconditionally summable to $\nu(\bigcup_{i \in \mathbf{N}} E_i)$ for the norm topology whenever $\langle E_i \rangle_{i \in \mathbf{N}}$ is a disjoint sequence of measurable sets with union E ('Orlicz-Pettis theorem', [13], 2-6-1 or [4], p. 22, Cor. 4).

(b) Now, if μ is a measure with domain Σ such that $\nu E = 0$ whenever $\mu E = 0$, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\nu E\| \leq \varepsilon$ whenever $\mu E \leq \delta$ [4, p. 10, Theorem 1].

(c) Thirdly, suppose that $\langle \nu_n \rangle_{n \in \mathbf{N}}$ is a sequence of countably additive functions from Σ to X such that $\nu E = \lim_{n \to \infty} \nu_n E$ exists in X, for the weak topology of X, for every $E \in \Sigma$; then ν is countably additive. (Use Nikodým's theorem [3, p. 90] to see that ν is weakly countably additive.)

2G Lemma. *Let X be a Banach space. If ϕ: $[0,1] \to X$ is McShane integrable with McShane integral w, then*

$$\|w\| \leq \underline{\int} \|\phi(t)\| \mu(dt).$$

Proof. Take any f in the unit ball of X^*. By [7], Theorem 8, $f(w)$ is the McShane integral of $f\phi$: $[0,1] \to \mathbf{R}$, and by 6-4 and 6-5 of [11] this is the ordinary integral of $f\phi$. So we have

$$|f(w)| = \left|\int f\phi\right| \leq \int |f\phi| \leq \underline{\int} \|\phi\|.$$

As f is arbitrary, $\|w\| \leq \underline{\int} \|\phi\|$.

2H Lemma. *Let X be a Banach space and Φ: $[0,1] \to X$ a McShane integrable function; let $\varepsilon > 0$. Then there is a gauge δ: $[0,1] \to]0,\infty[$ such that $\|\int_E \phi - \sum_{i \leq n} \mu E_i \phi(t_i)\| \leq \varepsilon$ whenever $E_0, \ldots, E_n$ are disjoint measurable subsets of $[0,1]$ with union $E, t_0, \ldots, t_n \in [0,1]$ and $E_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$ for every i.*

Proof. Let δ be a gauge such that $\|\int \phi - \sum_{i \leq n}(b_i - a_i)\phi(t_i)\| \leq \varepsilon$ whenever $\langle([a_i, b_i], t_i)\rangle_{i \leq n}$ is a McShane partition of $[0,1]$ subordinate to δ. Let $E_0, \ldots, t_n, E$ be as in the statement of the lemma; set $M = \max_{i \leq n}\|\phi(t_i)\|$. Take $\eta > 0$; let $\eta' > 0$ be such that $(n+1)M\eta' \leq \eta$ and $\|\int_H \phi\| \leq \eta$ whenever $\mu H \leq (n+1)\eta'$ (see 2B(iii)). Then we can find a family $\langle [a_{ij}, b_{ij}]\rangle_{i \leq n,\ j \leq r(i)}$ of non-overlapping closed intervals such that $\mu(E_i \,\Delta \bigcup_{j \leq r(i)}[a_{ij}, b_{ij}]) \leq \eta'$ and $t_i - \delta(t_i) \leq a_{ij} \leq b_{ij} \leq t_i + \delta(t_i)$ for each $i \leq n$, $j \leq r(i)$. Write $t_{ij} = t_i$ for $i \leq n$, $j \leq r(i)$. Then $\langle([a_{ij}, b_{ij}], t_{ij})\rangle_{i \leq n,\ j \leq r(i)}$ can be extended to an McShane partition of $[0,1]$ subordinate to δ. So writing $F_i = \bigcup_{j \leq r(i)}[a_{ij}, b_{ij}]$ for each i, $F = \bigcup_{i \leq n} F_i$, we have

$$\left\|\int_F \phi - \sum_{i \leq n,\ j \leq r(i)} (b_{ij} - a_{ij})\phi(t_{ij})\right\| \leq \varepsilon$$

by [7], Theorem 5; that is,

$$\left\|\int_F \phi - \sum_{i \leq n} \mu F_i \phi(t_i)\right\| \leq \varepsilon.$$

Next,

$$\left\| \int_E \phi - \int_F \phi \right\| \leq 2\eta$$

because $\mu(E \Delta F) \leq (n+1)\eta'$. Also

$$\left\| \sum_{i \leq n} \mu F_i \phi(t_i) - \sum_{i \leq n} \mu E_i \phi(t_i) \right\| \leq M(n+1)\eta' \leq \eta.$$

Putting these together,

$$\left\| \int_E \phi - \sum_{i \leq n} \mu E_i \phi(t_i) \right\| \leq \varepsilon + 3\eta;$$

as η is arbitrary we have the result.

2I THEOREM. *Let X be a Banach space. Let $\langle \phi_n \rangle_{n \in \mathbf{N}}$ be a sequence of McShane integrable functions from* $[0,1]$ *to X, and suppose that $\phi(t) = \lim_{n \to \infty} \phi_n(t)$ exists in X for every $t \in [0,1]$. If moreover the limit*

$$\nu E = \lim_{n \to \infty} \int_E \phi_n$$

exists in X, for the weak topology, for every measurable $E \subseteq [0,1]$, ϕ is McShane integrable and $\int \phi = \nu([0,1])$.

Proof. Fix $\varepsilon > 0$. Write μ for Lebesgue measure, Σ for the algebra of Lebesgue measurable subsets of $[0,1]$.

(a) For $t \in [0,1]$, $n \in \mathbf{N}$ set

$$q_n(t) = \sup_{j \geq i \geq n} \left\| \phi_j(t) - \phi_i(t) \right\|.$$

For each t, write

$$r(t) = \min\{n : q_n(t) \leq \varepsilon, |\phi(t)| \leq n\};$$

set $A_k = \{t : r(t) = k\}$ for each k. For each $k \in \mathbf{N}$, let $W_k \supseteq A_k$ be a measurable set with $\mu_*(W_k \setminus A_k) = 0$; set $V_k = W_k \setminus \bigcup_{j<k} W_j$ for each k, so that $\langle V_k \rangle_{k \in \mathbf{N}}$ is a disjoint cover of $[0,1]$ by measurable sets, and $A_k \subseteq \bigcup_{j \leq k} V_j$ and $\mu_*(V_k \setminus A_k) = 0$ for each k. For each k, write $V_k^* = \bigcup_{j \leq k} V_j = \bigcup_{j \leq k} W_j$; take $\eta_k > 0$ such that $\|\nu E\| \leq 2^{-k}\varepsilon$ whenever $\mu E \leq \eta_k$ (see

(b) and (c) of 2F above); let $G_k \supseteq V_k^*$ be an open set such that $\mu(G_k \setminus V_k^*) \le \min(\eta_k, 2^{-k}\varepsilon)$.

(b) If $k \in \mathbf{N}$ and $E \subseteq V_k^*$ is measurable, then $\|\nu E - \int_E \phi_k\| \le \varepsilon\mu E$. To see this, it is enough to consider the case $E \subseteq V_j$ where $j \le k$. In this case, observe that

$$\left\|\nu E - \int_E \phi_k\right\| \le \limsup_{n\to\infty}\left\|\int_E \phi_n - \int_E \phi_k\right\| \le \sup_{n\ge k} \underline{\int}_E \|\phi_n(t) - \phi_k(t)\|\mu(dt)$$

by Lemma 2G. Now $\mu_*(E \setminus A_j) = 0$ and for $t \in A_j$ we have $\|\phi_n(t) - \phi_k(t)\| \le q_j(t) \le \varepsilon$ for every $n \ge k$, so

$$\underline{\int}_E \|\phi_n(t) - \phi_k(t)\|\mu(dt) \le \varepsilon\mu E$$

for every $n \ge k$, giving the result.

(c) For each $k \in \mathbf{N}$ let δ_k: $[0,1] \to]0,\infty[$ be a gauge such that

$$\left\|\int_E \phi_k - \sum_{i\le n} \mu E_i \phi_k(t_i)\right\| \le 2^{-k}\varepsilon$$

whenever $E_0, \dots, E_n$ are disjoint measurable sets with union E and $t_0, \dots, t_n \in [0,1]$ are such that $E_i \subseteq [t_i - \delta_k(t_i), t_i + \delta_k(t_i)]$ for each i; such a gauge exists by Lemma 2H. Choose δ: $[0,1] \to]0,\infty[$ such that $\delta(t) \le \min(\varepsilon, \delta_k(t))$ and $[0,1] \cap [t - \delta(t), t + \delta(t)] \subseteq G_k$ for $t \in A_k$.

(d) Let $\langle([a_i, b_i], t_i)\rangle_{i\in\mathbf{N}}$ be a McShane partition of $[0,1]$ subordinate to δ. We seek to estimate $\|\nu([0,1]) - w\|$, where $w = \sum_{i\le n}(b_i - a_i)\phi(t_i)$.

Set $I_k = \{i\colon i \le n,\ t_i \in A_k\}$ for each k; of course all but finitely many of the I_k are empty. For $i \in I_k$, set $E_i = [a_i, b_i] \cap V_k^*$. We have

$$[a_i, b_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \subseteq G_k,$$

so

$$\sum_{i\in I_k} \mu([a_i, b_i] \setminus E_i) \le 2^{-k}\varepsilon,$$

and

$$\sum_{i\in I_k} \mu([a_i, b_i] \setminus E_i)\|\phi(t_i)\| \le 2^{-k}k\varepsilon$$

because $\|\phi(t)\| \leq k$ for $t \in A_k$. Consequently, if we write

$$w_1 = \sum_{i \leq n} \mu E_i \phi(t_i),$$

we shall have $\|w - w_1\| \leq \sum_{k \in \mathbf{N}} 2^{-k} k\varepsilon = 2\varepsilon$.

For each $i \leq n$, let $k(i)$ be such that $t_i \in A_{k(i)}$. Then we have

$$\|\phi(t_i) - \phi_{k(i)}(t_i)\| \leq \varepsilon$$

for each i. So

$$\sum_{i \leq n} \mu E_i \|\phi(t_i) - \phi_{k(i)}(t_i)\| \leq \sum_{i \leq n} (b_i - a_i)\varepsilon \leq \varepsilon,$$

because $\langle [a_i, b_i] \rangle_{i \leq n}$ is non-overlapping. Accordingly, writing

$$w_2 = \sum_{i \leq n} \mu E_i \phi_{k(i)}(t_i),$$

we have $\|w - w_2\| \leq 3\varepsilon$.

Set $H_k = \bigcup\{E_i : i \in I_k\}$ for each k. Because $E_i \subseteq [t_i - \delta_k(t_i), t_i + \delta_k(t_i)]$ for each $i \in I_k$, we have

$$\left\| \sum_{i \in I_k} \mu E_i \phi_k(t_i) - \int_{H_k} \phi_k \right\| \leq 2^{-k}\varepsilon.$$

Consequently, writing

$$w_3 = \sum_{k \in \mathbf{N}} \int_{H_k} \phi_k,$$

we have $\|w - w_3\| \leq 5\varepsilon$.

Next, for any k, $H_k \subseteq V_k^*$, so we have

$$\left\| \nu H_k - \int_{H_k} \phi_k \right\| \leq \varepsilon \mu H_k,$$

by (b) above. So writing $w_4 = \sum_{k \in \mathbf{N}} \nu H_k$ we have $\|w_3 - w_4\| \leq \varepsilon$ and $\|w - w_4\| \leq 6\varepsilon$.

If we set $H'_k = \bigcup\{[a_i, b_i]: i \in I_k\}$, then $\mu(H'_k \setminus H_k) \le \eta_k$, so that $\|\nu H'_k - \nu H_k\| \le 2^{-k}\varepsilon$, for each k. Accordingly $\|w - w_5\| \le 8\varepsilon$, where

$$w_5 = \sum_{k \in \mathbf{N}} \nu H'_k = \nu\Bigl(\bigcup_{k \in \mathbf{N}} H'_k\Bigr) = \nu\Bigl(\bigcup_{i \le n} [a_i, b_i]\Bigr) = \nu([0,1]).$$

As ε is arbitrary, $\int\phi$ exists and is equal to ν ([0, 1]).

Problem. In this theorem we are supposing that $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ in the norm topology for every t. Is it enough if $\phi(t)$ is the weak limit of $\langle\phi_n(t)\rangle_{n\in\mathbf{N}}$ for every t?

2J Corollary. *Let X be a Banach space.*

(a) *Let $\langle\phi_n\rangle_{n\in\mathbf{N}}$ be a sequence of McShane integrable functions from* [0, 1] *to X such that $\phi(t) = \lim_{n\to\infty}\phi_n(t)$ exists in X for every $t \in [0,1]$. If*

$$C = \{f\phi_n : f \in X^*,\ \|f\| \le 1,\ n \in \mathbf{N}\}$$

is uniformly integrable, then ϕ is McShane integrable. In particular, if $\{\|\phi_n\|: n \in \mathbf{N}\}$ is dominated by an integrable function, then ϕ is McShane integrable.

(b)*Let ϕ: [0, 1] $\to X$ be a Pettis integrable function and $\langle E_i\rangle_{i\in\mathbf{N}}$ a cover of* [0, 1] *by measurable sets. Suppose that $\phi \times \chi(E_i)$ is McShane integrable for each i. Then ϕ is McShane integrable.*

Proof. (a) The point is that ϕ_n, ϕ satisfy the conditions of Theorem 2I. To see this, take $E \in \Sigma$ and $\varepsilon > 0$. Because C is uniformly integrable, there is an $\eta > 0$ such that $\int_H |g| \le \varepsilon$ whenever $g \in C$ and $\mu H \le \eta$; consequently $\|\int_H \phi_n\| \le \varepsilon$ for all $n \in \mathbf{N}$ whenever $H \in \Sigma$ and $\mu H \le \eta$. Now set

$$A_n = \{t : \|\phi_i(t) - \phi_j(t)\| \le \varepsilon\ \forall i, j \ge n\};$$

then $\langle A_n\rangle_{n\in\mathbf{N}}$ is an increasing sequence with union [0, 1], so there is an n such that $\mu^* A_n \ge 1 - \eta$. Let $G \in \Sigma$ be such that $A_n \subseteq G$ and $\mu G = \mu^* A_n$. Then whenever $i, j \ge n$ we have

$$\begin{aligned}\Bigl\|\int_{E\cap G}\phi_i - \int_{E\cap G}\phi_j\Bigr\| &\le \underline{\int}_{E\cap G}\|\phi_i(t) - \phi_j(t)\|\mu(dt)\\ &\le \mu G \sup_{t\in A_n}\|\phi_i(t) - \phi_j(t)\| \le \varepsilon.\end{aligned}$$

Also $\|\int_{E\setminus G}\phi_i\|$ and $\|\int_{E\setminus G}\phi_j\|$ are both less than or equal to ε, so $\|\int_E\phi_i - \int_E\phi_j\| \le 3\varepsilon$. This shows that $\langle\int_E\phi_i\rangle_{i\in\mathbf{N}}$ is a Cauchy sequence and therefore convergent, for every $E \in \Sigma$. Accordingly the conditions of 2I are satisfied and ϕ is McShane integrable.

(b) We apply 2I with $\phi_n(t) = \phi(t)$ for $t \in \bigcup_{i \leq n} E_i$, 0 elsewhere.

Remark. Part (a) is a version of Vitali's lemma. Part (b) is a generalization of [7], Theorem 15.

2K. We now give a result connecting the McShane and Talagrand integrals. Recall that if (S, Σ, μ) is a probability space, a set A of real-valued functions is *stable* (in Talagrand's terminology) if for every $E \in \Sigma$, with $\mu E > 0$, and all real numbers $\alpha < \beta$, there are $m, n \geq 1$ such that $\mu^*_{m+n} Z(A, E, m, n, \alpha, \beta) < (\mu E)^{m+n}$, where throughout the rest of paper we write $Z(A, E, m, n, \alpha, \beta)$ for

$$\{(t,u): t \in E^m, u \in E^n, \exists f \in A, f(t(i)) \leq \alpha\ \forall i < m, f(u(j)) \geq \beta\ \forall j < n\},$$

and μ^*_{m+n} is the ordinary product outer measure on $S^m \times S^n$. Now if X is a Banach space, a function ϕ: $S \to X$ is *properly measurable* if

$$\{h\phi: h \in X^*, \|h\| \leq 1\}$$

is stable. Talagrand proved [14, Theorem 8] that ϕ is Talagrand integrable iff it is properly measurable and the upper integral $\overline{\int} \|\phi(t)\| \mu(dt)$ is finite.

In particular, a Talagrand integrable function ϕ: $S \to X$ must be scalarly measurable for the completion of μ [13, 6-1-1]. So if X is separable, ϕ must be measurable for the completion of μ ([4], II.1.2 or [13], 3-1-3); now as $\int \|\phi\| \, d\mu < \infty$, ϕ is Bochner integrable.

2L Proposition. *Let X be a Banach space such that the unit ball of X^* is w^*-separable. If ϕ: $[0,1] \to X$ is a McShane integrable function then it is properly measurable.*

Proof. Let w be the McShane integral of ϕ. Set

$$A = \{h\phi: h \in X^*, \|h\| \leq 1\} \subseteq \mathbf{R}^{[0,1]};$$

we have to show that A is stable. Note that because the unit ball of X^* is separable for the w^*-topology on X^*, and the map $h \mapsto h\phi$: $X^* \to \mathbf{R}^{[0,1]}$ is continuous for the w^*-topology on X^* and the topology of pointwise convergence on $\mathbf{R}^{[0,1]}$, A has a countable dense subset A_0.

Take a non-negligible measurable $E \subseteq [0,1]$ and $\alpha < \alpha' < \beta' < \beta$ in $\mathbf{R}$. From $m, n \geq 1$ set $H_{mn} = Z(A, E, m, n, \alpha, \beta)$, $H'_{mn} = Z(A_0, E, m, n, \alpha', \beta')$; then $H_{mn} \subseteq H'_{mn}$ and H'_{mn} is measurable for the usual (completed) product measure on $E^m \times E^n$. We seek an m with $\mu_{2m}H'_{mm} < (\mu E)^{2m}$, writing μ for Lebesgue measure on $[0,1]$ and μ_{2m} for Lebesgue measure on $[0,1]^m \times [0,1]^m$.

Set $\varepsilon = \frac{1}{6}(\beta' - \alpha')\mu E$, and choose a function δ: $[a,b] \to]0,\infty[$ such that

$$\left\| w - \sum_{i \le n} (b_i - a_i)\phi(t_i) \right\| \le \varepsilon$$

for every McShane partition $\langle([a_i, b_i], t_i)\rangle_{i \le n}$ of $[a,b]$ subordinate to δ.

Take $k \ge 1$ such that $\mu^* D \ge \frac{1}{2}\mu E$, where $D = \{s\colon s \in E,\ \delta(s) \ge 1/k\}$. Let $\langle[a_i, b_i]\rangle_{i<m}$ be an enumeration of the intervals of the form $[j/k, (j+1)/k]$ which meet D in a set of positive measure; set $G = \bigcup_{i<m}]a_i, b_i[$, so that G is open and $\mu G \ge \frac{1}{2}\mu E$.

Now suppose, if possible, that $\mu_{2m}H'_{mm} = (\mu E)^{2m}$. The set

$$\left\{(t,u)\colon t,u \in \prod_{i<m} (D \cap]a_i, b_i[)\right\} \subseteq E^m \times E^m$$

has non-zero outer measure, so must meet H'_{mm}; take (t,u) in the intersection. Write

$$\Delta(s) = \{s'\colon |s' - s| < \min(\delta(s), 1/k)\}$$

for each $s \in [0,1]$. Let $\langle t(i)\rangle_{m \le i \le n}$ be a finite sequence in $[a,b] \setminus G$ such that $[a,b] \setminus G \subseteq \bigcup_{m \le i \le n} \Delta(t(i))$. Because $[a,b] \setminus G$ is itself a finite union of closed intervals, we can find a family $\langle[a_i, b_i]\rangle_{m \le i \le n}$ of non-overlapping closed intervals such that $[0,1] \setminus G = \bigcup_{m \le i \le n} [a_i, b_i]$ and $[a_i, b_i] \subseteq \Delta(t(i))$ for each i. Now if we set $u(i) = t(i)$ for $m \le i \le n$, we see that $\langle([a_i, b_i], t(i))\rangle_{i \le n}$ and $\langle([a_i, b_i], u(i))\rangle_{i \le n}$ are both McShane partitions of $[a, b]$ subordinate to δ. So we must have

$$\begin{aligned} &\left\| \sum_{i \le m} (b_i - a_i)(\phi(t(i)) - \phi(u(i))) \right\| \\ &\qquad = \left\| \sum_{i \le n} (b_i - a_i)(\phi(t(i)) - \phi(u(i))) \right\| \le 2\varepsilon. \end{aligned}$$

Now $(t,u) \in H'_{mm}$, so there is an $f \in A$ such that $f(t(i)) \le \alpha'$ and $f(u(i)) \ge \beta'$ for every $i < m$. f is of the form $h\phi$ for some h of norm at most 1, so

$$\left| \sum_{i<m} (b_i - a_i)(f(t(i)) - f(u(i))) \right| \le 2\varepsilon.$$

However, $f(t(i)) \leq \alpha'$ for each i and $\Sigma_{i<m} b_i - a_i = \mu G$, so

$$\sum_{i<m} (b_i - a_i) f(t(i)) \leq \alpha' \mu G;$$

similarly $\Sigma_{i<m}(b_i - a_i) f(u(i)) \geq \beta' \mu G$, and we get

$$2\varepsilon \geq (\beta' - \alpha')\mu G \geq (\beta' - \alpha')\tfrac{1}{2}\mu E = 3\varepsilon,$$

which is absurd.

2M Corollary. *Let X be a Banach space such that the unit ball of X^* is w^*-separable. If ϕ: $[0,1] \to X$ is a McShane integrable and $\overline{\int}\|\phi(s)\|\mu(ds) < \infty$, then ϕ is Talagrand integrable.*

3. Examples

In this section we give examples to show that the results above are more or less complete in their own terms. In particular, a McShane integrable function need not be Talagrand integrable (3A, 3E) and a Talagrand integrable function need not be McShane integrable (3C).

3A *Example.* There is a bounded McShane integrable function ϕ: $[0,1] \to l^\infty(\mathfrak{c})$ which is not properly measurable and therefore not Talagrand integrable.

Proof. Enumerate as $\langle H_\xi \rangle_{\xi < \mathfrak{c}}$ the family of all Borel subsets of any power $[0,1]^n$ of the unit interval which have positive Lebesgue measure in the appropriate dimension. Then we can choose inductively a disjoint family $\langle D_\xi \rangle_{\xi < \mathfrak{c}}$ of finite sets such that $D_\xi^{k(\xi)} \cap H_\xi \neq \emptyset$ for each ξ, taking $k(\xi)$ such that $H_\xi \subseteq [0,1]^{k(\xi)}$. Define ϕ: $[0,1] \to l^\infty(\mathfrak{c})$ by saying that $\phi(s) = e_\xi$ when $s \in D_\xi$, 0 when $s \in [0,1] \setminus \bigcup_{\xi < \mathfrak{c}} D_\xi$, writing e_ξ for the unit vector of $l^\infty(\mathfrak{c})$ with $e_\xi(\eta) = 1$ if $\eta = \xi$, 0 otherwise.

To see that ϕ is McShane integrable, with integral 0, let $\varepsilon > 0$. For each $\xi < \mathfrak{c}$ let $G_\xi \supseteq D_\xi$ be a relatively open subset of $[0,1]$ of measure at most ε; let $\delta(s)$ be the distance from s to $[0,1] \setminus G_\xi$ if $s \in D_\xi$, 1 if $s \in [0,1] \setminus \bigcup_{\xi < \mathfrak{c}} D_\xi$. If $\langle([a_i, b_i], t_i)\rangle_{i \in I}$ is any McShane partition of $[0,1]$ subordinate to δ, then

$$\left|\left(\sum_{i \in J} (b_i - a_i)\phi(t_i)\right)(\xi)\right| = \sum_{i \in J,\, t_i \in D_\xi} (b_i - a_i) \leq \mu G_\xi \leq \varepsilon$$

for any $J \subseteq I$, $\xi < \mathfrak{c}$, so that

$$\left\| \sum_{i \in J} (b_i - a_i)\phi(t_i) \right\| \leq \varepsilon$$

for every finite $J \subseteq I$. As ε is arbitrary, ϕ is McShane integrable, with integral 0.

To see that ϕ is not properly measurable, set

$$A = \{h\phi : h \in (l^\infty(\mathfrak{c}))^*, \|h\| \leq 1\},$$

and consider, for $m, n \geq 1$, the set $H_{mn} = Z(A, [0,1], m, n, 0, 1)$.

Suppose, if possible, that there are $m, n \geq 1$ such that $\mu^*_{m+n} H_{mn} < 1$. In this case there is a non-negligible measurable $H \subseteq ([0,1]^m \times [0,1]^n) \setminus H_{mn}$. Set

$$H' = \{u : u \in [0,1]^n, \mu_m\{t : (t,u) \in H\} > 0\};$$

then $\mu_n H' > 0$, so there is a $\xi < \mathfrak{c}$ such that $H_\xi \subseteq H'$, and a $u \in D^n_\xi \cap H'$. Now

$$H^{-1}[\{u\}] = \{t : (t,u) \in H\}$$

is non-negligible, and D_ξ is finite, so there is a $t \in H^{-1}[\{u\}]$ such that no coordinate of t belongs to D_ξ. In this case, taking $h_\xi(x) = x(\xi)$ for $x \in l^\infty(\mathfrak{c})$, and $f = h_\xi \phi \in A$, we see that $f(u(j)) = 1$ for each $j < n$, but that $f(t(i)) = 0$ for each $i < m$; so that $(t,u) \in H_{mn}$ and $(t,u) \notin H$, which is absurd.

Thus ϕ is not properly measurable and therefore not Talagrand integrable.

Remark. Observe that ϕ is not measurable, and either for this reason, or because it is not Talagrand integrable, cannot be Bochner integrable.

3B. We need the following long-known fact which we do not find in print.

LEMMA. *Let (S, Σ, μ) be a probability space and $\langle F_i \rangle_{i \in \mathbf{N}}$ a stochastically independent sequence of measurable sets in S such that, for some $k \in \mathbf{N}$, $\sum_{i \in \mathbf{N}} (\mu F_i)^k < \infty$. Then the family $A = \{\chi(F_i) : i \in \mathbf{N}\}$ of their characteristic functions is stable.*

Proof. Take $E \in \Sigma$ with $\mu E > 0$, and $\alpha < \beta$ in $\mathbf{R}$. If $\alpha < 0$ or $\beta > 1$ then

$$\mu_2 Z(A, E, 1, 1, \alpha, \beta) = 0 < (\mu E)^2.$$

If $0 \leq \alpha < \beta \leq 1$ take l so large that $\sum_{i \geq l}(\mu F_i)^k < (\mu E)^k$ and $\mu F_i \leq \mu E$ for every $i \geq l$; then $\sum_{i \geq l}(\mu F_i)^m < (\mu E)^m$ for every $m \geq k$. Set

$$J = \{i: i < l, \mu(E \cap F_i) > 0\},$$

and let $m \geq k$ be so large that

$$\sum_{i \in J}\left(\frac{\mu(E \setminus F_i)}{\mu E}\right)^m < 1.$$

Now

$$\sum_{i \in J}(\mu(E \setminus F_i))^m < (\mu E)^m \text{ and } \sum_{i \in \mathbf{N} \setminus J}(\mu(E \cap F_i))^m < (\mu E)^m.$$

So

$$H = \left(E^m \setminus \bigcup_{i \in J}(S \setminus F_i)^m\right) \times \left(E^m \setminus \bigcup_{i \in \mathbf{N} \setminus J} F_i^m\right)$$

has $\mu_{2m} H > 0$. But also $H \cap Z(A, E, m, m, \alpha, \beta) = \emptyset$, so

$$\mu_{2m} Z(A, E, m, m, \alpha, \beta) < (\mu E)^{2m},$$

as required.

3C *Example*. There is a bounded Talagrand integrable function ϕ: $[0, 1] \to l^\infty(\mathbf{N})$ which is not McShane integrable.

Proof. (a) Let $\langle F_m \rangle_{m \in \mathbf{N}}$ be a stochastically independent sequence of measurable subsets of $[0, 1]$ with $\mu F_m = 1/(m + 1)$ for each m, where μ is Lebesgue measure on $\mathbf{R}$. Then the set $A = \{\chi(F_m)\ m \in \mathbf{N}\}$ of their characteristic functions is stable, by 3B. So the closed absolutely convex hull of A in $\mathbf{R}^{[0,1]}$ is also stable [13, 11-2-1], and if we define ϕ: $[0, 1] \to l^\infty(\mathbf{N})$ by writing

$$\phi(t)(m) = \begin{cases} 1 & \text{if } t \in F_m, \\ 0 & \text{otherwise,} \end{cases}$$

ϕ is properly measurable, therefore Talagrand integrable. Its Talagrand integral is w, where $w(n) = \mu F_n = 1/(n + 1)$ for each n.

(b) Suppose, if possible, that ϕ is McShane integrable; then its McShane integral must be w. Let $\delta\colon [0,1] \to]0,\infty[$ be a gauge such that

$$\left\| w - \sum_{i \le n} (b_i - a_i)\phi(t_i) \right\| \le \tfrac{1}{4}$$

for every McShane partition $\langle([a_i, b_i], t_i)\rangle_{i \le n}$ of $[0,1]$ subordinate to δ.

For each $m \in \mathbf{N}$, set

$$G_m = [0,1] \cap \bigcup \{]t - \delta(t), t + \delta(t)[: t \in F_m\}.$$

For $t \in [0,1]$ set

$$\tilde{\Delta}(t) = [0,1] \cap \bigcap \{G_m : m \in \mathbf{N}, t \in F_m\};$$

then $\tilde{\Delta}(t)$ is a measurable set including $[0,1] \cap]t - \delta(t), t + \delta(t)[$. For measurable sets $H \subseteq [0,1]$ set

$$H^* = \left\{t : \mu\left(H \cap \tilde{\Delta}(t)\right) \ge \tfrac{3}{4}\mu H\right\};$$

observe that H^* is always measurable, because

$$[0,1] \setminus H^* = \bigcup \left\{ \bigcap_{i \in I} F_i : I \subseteq \mathbf{N} \text{ is finite}, \mu\left(H \cap \bigcap_{i \in I} G_i\right) < \tfrac{3}{4}\mu H \right\}.$$

Let $\mathscr{H}$ be the set of non-negligible measurable subsets H of $[0,1]$ such that $\mu H^* > 0$.

(c) If $E \subseteq [0,1]$ is measurable and not negligible, there is an $H \in \mathscr{H}$ such that $H \subseteq E$. To see this, start by taking $E' = E \setminus \bigcup\{U : U \subseteq [0,1]$ is relatively open, $\mu(U \cap E) = 0\}$; then $\mu E' = \mu E > 0$ and

$$\mu\left(E' \cap \tilde{\Delta}(t)\right) \ge \mu(E' \cap]t - \delta(t), t + \delta(t)[) > 0$$

for every $t \in E'$. Next, for each $t \in E'$, take $r(t) \in \mathbf{N}$ such that

$$\mu\left(E' \cap \tilde{\Delta}(t)\right) \ge \tfrac{3}{4}\mu\left(E' \cap \bigcap \{G_i : i \le r(t), t \in F_i\}\right);$$

set $J(t) = \{i : i \le r(t), t \in F_i\}$. Let K be such that

$$\mu^*\{t : t \in E', J(t) = K\} > 0,$$

and set $H = E' \cap \bigcap_{i \in K} G_i$. If $t \in E'$, $J(t) = K$ then

$$\mu\big(H \cap \tilde{\Delta}(t)\big) = \mu\big(E' \cap \tilde{\Delta}(t)\big) \geq \tfrac{3}{4}\mu H;$$

so $H^* \supseteq \{t\colon J(t) = K\}$ has positive outer measure.

(d) We can therefore find, by exhaustion, a finite disjoint family $H_0, \ldots, H_k$ in $\mathscr{H}$ such that $\mu(\bigcup_{i \leq k} H_i) \geq \frac{3}{4}$. Now observe that for each $i \leq r$,

$$\sum\big\{\mu F_j\colon F_j \cap H_i^* = \varnothing\big\} < \infty,$$

so there is an $n \geq 3$ such that $F_n \cap H_i^* \neq \varnothing$ for every $i \leq r$. For each $i \leq r$, therefore, we can find a $t_i \in F_n \cap H_i^*$, in which case

$$\mu(H_i \cap G_n) \geq \mu\big(H_i \cap \tilde{\Delta}(t_i)\big) \geq \tfrac{3}{4}\mu H_i,$$

and accordingly $\mu G_n \geq (\frac{3}{4})^2 > \frac{1}{2}$.

(e) Returning to the definition of G_n, we see that there will be $t_0, \ldots, t_m \in F_n$ such that

$$\mu\Big([0,1] \cap \bigcup_{i \leq m}]t - \delta(t_i), t_i + \delta(t_i)[\Big) > \tfrac{1}{2}.$$

It is now possible to find non-overlapping intervals $[a_0, b_0], \ldots, [a_m, b_m] \subseteq [0,1]$ such that $t_i - \delta(t_i) \leq a_i \leq b_i \leq t_i + \delta(t_i)$ for each i and

$$\bigcup_{i \leq m} [a_i, b_i] = [0,1] \cap \bigcup_{i \leq m} [t_i - \delta(t_i), t_i + \delta(t_i)].$$

(Induce on m.) Next, we can extend $\langle([a_i, b_i], t_i)\rangle_{i \leq m}$ to a McShane partition $\langle([a_i, b_i], t_i)\rangle_{i \leq p}$ of $[0,1]$ subordinate to δ, by covering each component of $[0,1] \setminus \bigcup_{i \leq n}[a_i, b_i]$ separately. Now consider

$$v = \sum_{i \leq p} (b_i - a_i)\phi(t_i).$$

By the choice of δ, $\|w - v\| \leq \frac{1}{4}$, so

$$|v(n)| \leq \frac{1}{4} + |w(n)| = \frac{1}{4} + \frac{1}{n+1} \leq \frac{1}{2}.$$

On the other hand,

$$v(n) = \sum\{b_i - a_i\colon i \leq p,\, t_i \in F_i\} \geq \sum_{i \leq m} b_i - a_i > \frac{1}{2},$$

which is impossible.

This contradiction shows that ϕ is not McShane integrable.

Remark. Because ϕ is Talagrand integrable, it must be Pettis integrable; thus we see that the condition 'ϕ is measurable' in Gordon's theorem (1Ca) is necessary.

3D. We conclude with notes on two well-known examples.

Example. There is a bounded McShane integrable, Talagrand integrable function ϕ: $[0, 1] \to L^\infty([0, 1])$ which is not Bochner integrable.

Proof. Let $\phi(t)$ be the equivalence class in L^∞ of the characteristic function of the interval $[0, t]$ for each t. (See [2], §22, Example 1.) As remarked in [13], exercise 3-3-2, ϕ is Talagrand integrable but not Bochner integrable; as remarked in [7], p. 567, in a slightly different context, ϕ is McShane integrable.

Remark. We observe indeed that ϕ above could be called 'Riemann integrable', as its integrability can be witnessed by constant gauge functions. It is easy to see that such a function must be both McShane integrable and Talagrand integrable.

Because the unit ball of $L^\infty([0, 1])$ is w*-separable, there is an isometric embedding of $L^\infty[0, 1]$ in $l^\infty(\mathbf{N})$ (indeed, $L^\infty([0, 1])$ is isomorphic to $l^\infty(\mathbf{N})$—see [10], p. 111), so there is a bounded McShane integrable, Talagrand integrable function from $[0, 1]$ to $l^\infty(\mathbf{N})$ which is not Bochner integrable.

3*E Example.* There is a McShane integrable function ϕ: $[0, 1] \to l^2(\mathbf{N})$ which is not Talagrand integrable.

Proof. Let c_n be the nth unit vector of $l^2(\mathbf{N})$ and set $\phi(t) = 2^n(n + 1)^{-1}e_n$ for $2^{-n-1} \leq t < 2^{-n}$. Then $\int \|\phi\| = \infty$ so ϕ is not Talagrand integrable, but by [7], Theorem 15, it is McShane integrable.

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