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LUSTERNIK-SCHNIRELMANN COCATEGORY

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1. Introduction

In the last ten years there has been a revival of interest in the (Lusternik-Schnirelmann) category of a topological space X. Recall the classical definition: a space X has category $\leq n$ if and only if it can be covered by $n + 1$ open sets, each of which is contractible in X. This revival came from rational homotopy theory. Finite category seems to be the right finiteness restriction on a rational space.

Cocategory is much less well understood than category. In fact it is not clear that we have the right definition of it yet. The first attempt to define cocategory was made by Ganea in 1960 [4] and [5]. His invariant, which we will call inductive cocategory, satisfied many of the properties for which one would hope. The most fundamental of these are the following:

1. The spaces of cocategory one are the H-spaces.
2. In a space of cocategory n, Whitehead products in the homotopy of length greater than n vanish.
3. In a fibration, the cocategory of the fiber can be no bigger than the cocategory of the total space plus one.

However, inductive cocategory has a rather inscrutable definition. Not many papers were written about it. Then in his Oxford thesis, Hopkins pointed out that there is more than one natural choice for a definition of cocategory. He introduced symmetric cocategory, which he proves satisfies the first two properties above. He also shows that symmetric cocategory is at least as big as inductive cocategory, but he is unable to determine if they are equal (see [8]).

At about the same time Sbaï investigated rational cocategory. There is an obvious choice for a rational definition of cocategory, using the Quillen model and dualizing a definition of Félix and Halperin in [2]. Sbaï was trying

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to show that inductive cocategory localized to give rational cocategory. He was only able to show that inductive cocategory was less than or equal to rational cocategory.

In this paper, we introduce a new definition which we will call simply cocategory. Cocategory has some advantages over the previous two definitions. It dualizes Whitehead's definition of category, so it is defined by a map making a suitable diagram commute. This means it fits into the general framework for numerical invariants devised by Peterson in [10]. No previous definition of cocategory has ever done that. Also, it is perfectly obvious that the spaces of cocategory one are the H-spaces. With the other definitions, too much thought is involved for such a basic property.

Cocategory satisfies the first two properties above. As with symmetric cocategory, we do not know at this point if it satisfies the fibration property. In particular, we don't know how many of the three invariants cocategory, inductive cocategory, and symmetric cocategory are distinct. However, we do know that rational cocategory does not satisfy the fibration property and so is not the rationalization of inductive cocategory. This disproves a conjecture of Sbaï. The relations that we know of between the definitions are as follows. Inductive cocategory is less than or equal to symmetric cocategory and (for a rational space) rational cocategory, and is not always equal to rational cocategory. Cocategory is also less than or equal to rational cocategory (again for a rational space).

The fact that rational cocategory does not satisfy the fibration property could be a serious drawback. After all, one certainly wants n-stage rational Postnikov towers to have rational cocategory less than or equal to n. We can not at this point prove a suitable weakening of the fibration property, but we feel we know what the right weakening should be.

The methods of proof in this paper include the results on homotopy inverse limits of Bousfield and Kan in [1], and the machinery of rational homotopy theory. A good reference for the latter is [14]. The paper is organized as follows. In the first section we develop the thin product, a functor dual to the fat wedge needed to define cocategory. We also define cocategory. The second section contains an analysis of the homotopy of the thin product which enables us to prove the Whitehead product property above. In the third section we show that, for rational spaces, cocategory is less than or equal to rational cocategory. Finally, the fourth section contains an example to show that rational cocategory does not satisfy the fibration property, thereby showing that rational cocategory is not the same as inductive cocategory for rational spaces. This disproves a conjecture of Sbaï. We also make some conjectures which would clarify the situation.

The research in this paper formed part of the author's Ph.D. thesis. I would like to thank my thesis advisor, David Anick, for many helpful discussions.

2. Cocategory

In this section we define a new functor, the thin product, and use it to define cocategory. Throughout this section we will be using the results of [1], especially Chapter 10, and we refer the reader there for definitions. In particular, we will assume familiarity with homotopy inverse limits and homotopy direct limits.

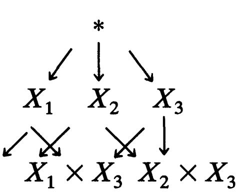
We would like to dualize the Whitehead definition of category mentioned in the preceding section. To do this we realize the fat wedge as a homotopy direct limit.

Throughout this paper we will use $\mathscr{T}$ for the category of pointed topological spaces and pointed continuous maps, and X will denote such a space. If C is a small category $\mathscr{T}^C$ will denote the diagram category. Recall that the fat wedge of $X_1, \dots, X_n$ is the subspace of the product consisting of points at least one of whose coordinates is the basepoint. Whitehead showed that, for reasonable X, the category of X is the least n so that the diagonal map factors up to homotopy through the fat wedge, minus one.

Let $\mathscr{D}^n$ denote the category of proper subsets of $\{1, \dots, n\}$ and inclusions. Define a functor $F: \mathscr{T}^n \to \mathscr{T}^{\mathscr{D}^n}$ by

$$F(X_1, \dots, X_n)(A) = \prod_{i \in A} X_i,$$

and by defining $F(X_1, \dots, X_n)(A \subseteq B)$ to be the natural inclusion.

$F(X_1, X_2, X_3) =$ 

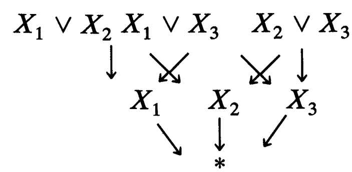
It is then easy to see that $\varinjlim F = T^n(X_1, \dots, X_n)$, the fat wedge. To see that $\underrightarrow{\operatorname{holim}} F$ is also the fat wedge requires a little more work and so we put it off for the moment. (for the definition of $\underrightarrow{\operatorname{holim}}$ see [1].)

Now it is easy to see how to dualize the fat wedge. Let $\mathscr{E}^n = (\mathscr{D}^n)^{op}$ be the category of proper subsets of $1, \dots, n$ ordered by superset. Define a functor $G: \mathscr{T}^n \to \mathscr{T}^{\mathscr{E}^n}$ by

$$G(X_1, \dots, X_n)(A) = \bigvee_{i \in A} X_i$$

and by letting $G(X_1, \dots, X_n)(A \supseteq B)$ be the map which is the identity on

$\bigvee_{i \in B} X_i$ and sends $\bigvee_{i \in (A-B)} X_i$ to $*$.

$G(X_1, X_2, X_3) =$ 

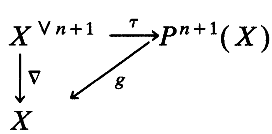
Notice that $\underleftarrow{\lim}\ G = \bigvee_i X_i$ unlike the previous case. We abbreviate $G(X, \dots, X)$ by $G(X)$.

Definition 1. The thin product of $X_1, \dots, X_n$, denoted by $P^n(X_1, \dots, X_n)$, is the basepoint component of $\underleftarrow{\text{holim}}\ G(X_1, \dots, X_n)$. We will abbreviate $P^n(X, \dots, X)$ by $P^n(X)$.

As it is the composition of two functors, P^n is a functor. If Y is a space let $\mathbf{Y}$ denote the constant $\mathscr{C}^n$ diagram. We have a natural map $\bigvee_i \mathbf{X}_i \to G(X_1, \dots, X_n)$ defined by simply mapping extra coordinates to $*$. Taking homotopy limits defines a natural map $\tau : \bigvee_i X_i \to P^n(X_1, \dots, X_n)$. Actually $\underleftarrow{\text{holim}}\ \mathbf{Y} = \hom(\mathscr{C}^n, Y)$ [1, pg. 300], but $\mathscr{C}^n$ has an initial object so as a space it is contractible and $\underleftarrow{\text{holim}}\ \mathbf{Y}$ is naturally homotopy equivalent to Y.

It is of course possible to define thinner products by taking homotopy inverse limits over subdiagrams of G. Now that we have the thin product we define cocategory in the expected way.

Definition 2. The cocategory of X, cocat X, is defined to be the least n for which there is a map g making the following diagram commute:



If there is no such n we define cocat $X = \infty$.

We then have the following basic lemma.

Lemma 1. (1) *If X is a homotopy retract of Y, then* cocat $X \leq$ cocat Y, *so cocategory is a homotopy invariant.*

(2) cocat $X = 0$ *if and only if X is contractible.*

(3) cocat $X \leq 1$ *if and only if X is an H-space.*

Proof. (1) This follows from the naturality of the thin product.

(2) This follows from $P^1(X) = \underleftarrow{\text{holim}}\ * = *$.

(3) $P^2(X) = \underleftarrow{\text{holim}}\ (X \to * \leftarrow X) = X \times X$. So cocat $X = 1$ if and only if X is not contractible and the fold map extends to the product—that is, if and only if X is a non-trivial H-space. □

We also have the lemma below, which guarantees that if the fold map does factor through the n-fold thin product, cocat $X \leq n$.

LEMMA 2. *Suppose* cocat $X \leq n$. *Then for all* $N \geq n$ *there is a factorization*

$$\begin{array}{ccc} X^{\vee N+1} & \xrightarrow{\tau} & P^{N+1}(X) \\ \downarrow{\scriptstyle\nabla} & \swarrow{\scriptstyle g} & \\ X & & \end{array}$$

Proof. It suffices to show that there is a map $f : P^{m+1}(X) \to P^m(X)$ for all m making the following diagram commute:

$$\begin{array}{ccc} X^{\vee m+1} & \xrightarrow{\tau} & P^{m+1}(X) \\ \downarrow{\scriptstyle X^{\vee m-1} \vee \nabla} & & \downarrow{\scriptstyle f} \\ X^{\vee m} & \xrightarrow{\tau} & P^m(X) \end{array}$$

Now maps into a homotopy inverse limit are determined by a universal property, as follows. For A an object of $\mathscr{C}^m$, we form the over category $\mathscr{C}^m/A$ whose objects are morphisms $B \to A$ and whose morphisms are commuting triangles. We also use the notation $\mathscr{C}^m/A$ for the classifying space of the over category, and we let $\mathscr{C}^m/-$ denote the induced diagram of spaces. Given any space Y, we can form the diagram $Y \times \mathscr{C}^m/-$ made up of the spaces $Y \times \mathscr{C}^m/A$. The universal property of $\underleftarrow{\text{holim}}$ says that

$$[Y, P^m(X)] = [Y \times \mathscr{C}^m/-, G(X)].$$

In particular there are structure maps for all A objects of $\mathscr{C}^m$:

$$P^m(X) \times \mathscr{C}^m/A \to \bigvee_{i \in A} X_i$$

where $X_i = X$ for all i.

Define $u : \mathscr{C}^m \to \mathscr{C}^{m+1}$ by $u(A) = A \cup \{m+1\}$. Then u induces

$$u : \mathscr{C}^m/A \to \mathscr{C}^{m+1}/u(A).$$

We define f_A to be the composition

$$P^{m+1}(X) \times \mathscr{C}^m/A \xrightarrow{(\mathrm{id}, u)} P^{m+1}(X)$$
$$\times \mathscr{C}^{m+1}/u(A) \longrightarrow \bigvee_{i \in A} X_i \vee X_{m+1} \xrightarrow{h_A} \bigvee_{i \in A} X_i.$$

Here the middle arrow is the structure map of $P^{m+1}(X)$. h_A is the identity on $\bigvee_{i \in (A - \{m\})} X_i$ and if $m \in A$, it folds X_m and X_{m+1} together.

One can then verify that the f_A commute with the maps of $G(X)$ so define a map $f: P^{m+1}(X) \to P^m(X)$. Another diagram chase shows that f is the map we wanted. □

3. Cocategory and Whitehead products

In this section we will calculate $\pi_* P^n(X_1, \ldots, X_n)$ and use this to show that in a space of cocategory $\leq n$ Whitehead products of length $> n$ vanish. We will also show that $\underrightarrow{\mathrm{holim}}\ F(X_1, \ldots, X_n) = T^n(X_1, \ldots, X_n)$ as promised in the previous section. Our basic tool is a spectral sequence due to Bousfield and Kan; see [1].

PROPOSITION 1 (Bousfield-Kan). *If I is a small category and $\mathbf{D} \in \mathscr{T}^I$ then*:

(1) *There is a spectral sequence $\{E_r^{s,t}\}$ with $E_2^{s,t} = \underleftarrow{\lim}^s \pi_t \mathbf{D}$ for $0 \leq s \leq t$ and 0 otherwise which is closely related to $\pi_* \underleftarrow{\mathrm{holim}}\ \mathbf{D}$*

(2) *There is a spectral sequence $\{\overline{E}_r^{s,t}\}$ with $\overline{E}_2^{s,t} = \underleftarrow{\lim}^s H^{-t} \mathbf{D}$ for $s \geq 0$ and 0 otherwise which is closely related to $H^* \underrightarrow{\mathrm{holim}}\ \mathbf{D}$*

Here "closely related" is a technical term which means that in good cases, in particular when the spectral sequence collapses, the $E_\infty^{s,s+j}$ are the quotients in a composition series for $\pi_j \underleftarrow{\mathrm{holim}}\ \mathbf{D}$. The differentials d^r in the first spectral sequences above map $E_r^{s,t}$ to $E_r^{s+r,t+r-1}$.

In order to use this spectral sequence we will need to calculate the E_2 term. This can be done using a result of Roos.

LEMMA 3 (Roos). *Let I be a partially ordered set, A a contravariant diagram of abelian groups over I with structure maps $p_\alpha^\beta: A_\beta \to A_\alpha$ for $\alpha < \beta$*

in I. Suppose A is flasque—that is, for all $\beta \in I$ *the natural map*

$$\varprojlim_\alpha (A, \alpha \leq \beta) \to \varprojlim_\alpha (A, \alpha < \beta)$$

is surjective. Then $\varprojlim^s A = 0$ *for* $s > 0$.

Roos gives only an outline of the proof in [11]. Also his hypotheses are too strong. Thus, we will fill in some of the details.

Proof. The plan is to construct a natural cochain complex $\prod A$ whose cohomology groups are the $\varprojlim^* A$.

Let $J(k)$ be the set of $k+1$-chains $\alpha_0 < \cdots < \alpha_k$ in I. For $C \in J(k)$ let $A_C = A_{\alpha_0}$. Define $\prod^k A = \prod_{C \in J(k)} A_C$. Define $d^k : \prod^k A \to \prod^{k+1} A$ by

$$(d^k)_D = p_{\alpha_0}^{\alpha_1} \operatorname{pr}_{D-\alpha_0} + \sum_i (-1)^i \operatorname{pr}_{D-\alpha_i}.$$

Then $\prod A = (\prod^k A, d^k)$ is a cochain complex. Note that $H^0 \prod A = \varprojlim A$. We will show below that for flasque A, $H^s \prod A = 0$ for $s > 0$.

On the other hand, any A will embed in a flasque diagram. Indeed, let

$$S_\beta = \prod_{\alpha \leq \beta} A_\alpha$$

with the evident projections as structure maps. It is easy to see that S is flasque, and there is an embedding of A into S defined by

$$A_\beta \xrightarrow{(p_\alpha^\beta)} S_\beta,$$

where p_α^α is the identity. This, together with the vanishing of the positive degree cohomology of $\prod A$ if A is flasque, implies that $H^s \prod A = \varprojlim^s A$ for general A. The proof is a standard homological algebra comparison argument.

Now we prove that $H^s \prod A = 0$ for flasque A and $s > 0$. Suppose $(y_{\alpha_0\alpha_1}) \in \ker d^1$. We want to define $(x_\alpha) \in \prod^0 A$ such that $d^0(x_\alpha) = (y_{\alpha_0\alpha_1})$. Define $x_\alpha = 0$ on minimal elements of I. Suppose we have defined x_α for $\alpha < \beta$ such that $p_{\alpha_0}^{\alpha_1} x_{\alpha_1} - x_{\alpha_0} = y_{\alpha_0\alpha_1}$ for $\alpha_0 < \alpha_1 < \beta$. Let $z_\alpha = x_\alpha + y_{\alpha\beta}$ for $\alpha < \beta$. Then one checks that $(z_\alpha) \in \varprojlim (A, \alpha < \beta)$. By flasqueness there is an x_β with $p_\alpha^\beta x_\beta = z_\alpha$ for $\alpha < \beta$. This extends the definition of (x_α) so by induction $H^1 \prod A = 0$. We can now finish the proof using induction on s and a similar argument. □

Bousfield and Kan define $\varprojlim^1$ for a diagram of not necessarily abelian groups. The resulting object is not a group, only a set. The above lemma can be extended to this case.

We can now calculate the E_2 term of our spectral sequences.

THEOREM 1. (1) *If $j > 1$, the diagram $\pi_j G(X_1, \ldots, X_n)$ is flasque. Hence the spectral sequence collapses and*

$$\pi_j P^n(X_1, \ldots, X_n) = \varprojlim \pi_j G(X_1, \ldots, X_n)$$

for $j > 0$.

(2) *The diagram $H^j F(X_1, \ldots, X_n)$ is flasque. Hence the spectral sequence collapses and*

$$H^j\Big(\operatorname{holim}_{\longrightarrow} F(X_1, \ldots, X_n)\Big) = \varprojlim H^j F(X_1, \ldots, X_n).$$

Proof. (1) Fix $j > 1$. Suppose $B \subseteq A \subseteq \{1, \ldots, n\}$. Then we have maps

$$i_B^A : \bigvee_{i \in B} X_i \to \bigvee_{i \in A} X_i \text{ and } r_A^B : \bigvee_{i \in A} X_i \to \bigvee_{i \in B} X_i.$$

We will use the same notation for the induced maps on homotopy. These maps satisfy $r_A^C i_B^A = i_{B \cap C}^C r_B^{B \cap C}$. Define G_A recursively by

$$G_A = \Big\{ f \in \pi_j \bigvee_{i \in A} X_i : f \notin \sum_{B \subset A} i_B^A G_B \Big\}.$$

Then by definition $\Sigma_{B \subseteq A} i_B^A G_B = \pi_j \vee_{i \in A} X_i$. This sum is actually direct. Indeed, suppose $\Sigma_{B \subseteq A} i_B^A f_B = 0$. Choose a maximal C with $f_C \neq 0$. Then

$$f_C = r_A^C i_C^A f_C = -\sum_{B \neq C} r_A^C i_B^A f_B = -\sum_{B \neq C} i_{B \cap C}^C r_B^{B \cap C} f_B.$$

But this is a contradiction since $B \cap C \neq C$ by maximality.

It is now easy to see that

$$\varprojlim \big(\pi_j G(X_1, \ldots, X_n), B \subset A\big) \cong \bigoplus_{B \subset A} G_B$$

and

$$\varprojlim \big(\pi_j G(X_1, \ldots, X_n), B \subseteq A\big) = \bigoplus_{B \subseteq A} G_B$$

with the map being the obvious surjection. Thus $\pi_j G(X_1, \ldots, X_n)$ is flasque. So we have $E_2^{s,t} = 0$ for $t > 1$, $s > 0$ and

$$E_2^{0,t} = \varprojlim \pi_t G(X_1, \ldots, X_n).$$

$E_2^{1,1} = \varprojlim{}^1 \pi_1 G(X_1, \ldots, X_n)$ can be non-zero. However $d^2 : E_2^{1,1} \to E_2^{3,2} = 0$ so the spectral sequence still collapses. This term is in the 0 stem and so only affects the connectivity of the thin product. Its presence means $\underleftarrow{\text{holim}}\ G(X_1, \ldots, X_n)$ is not connected in general, and is the reason we take the thin product to be the basepoint component.

(2) We can repeat the same argument in this case, modifying the definitions of i_B^A and r_A^B. □

COROLLARY 1. $\underrightarrow{\text{holim}}\ F(X_1, \ldots, X_n) = T^n(X_1, \ldots, X_n)$ *for simply connected* $X_1, \ldots, X_n$.

Proof. We have a map

$$\underrightarrow{\text{holim}}\ F(X_1, \ldots, X_n) \to \varprojlim F(X_1, \ldots, X_n) = T^n(X_1, \ldots, X_n).$$

The above theorem shows this map induces an isomorphism on cohomology. □

We can now prove the following:

THEOREM 2. *In a space* X *with* cocat $X \leq n$ *all Whitehead products of length* $> n$ *vanish.*

Proof. Suppose $x_1, \ldots, x_{n+1} \in \pi_* X$. We have $n + 1$ sections

$$s_i : X \to \bigvee X_i \quad (X_i = X)$$

of the fold map. Let $y_i = s_{i*} x_i$. Then for any Whitehead product $W(x_i)$, $\nabla_* W(y_i) = W(x_i)$. Consider the constant diagram $\bigvee \mathbf{X_i}$. The obvious map between this and $G(X_1, \ldots, X_{n+1})$ induces a map between spectral sequences. Now the constant diagram is obviously flasque. So we get that the composition

$$\pi_* \bigvee X_i \overset{\pi_*}{\to} \pi_* P^{n+1}(X_1, \ldots, X_{n+1}) \overset{\cong}{\to} \varprojlim \pi_* G(X_1, \ldots, X_{n+1})$$

is the map which sends $G_{\{1,\ldots,n+1\}}$ to 0 and fixes the other G_A. In particular it sends $W(y_i)$ to 0. Thus $W(x_i) = \nabla_* W(y_i) = g_* \tau_* W(y_i) = 0$. □

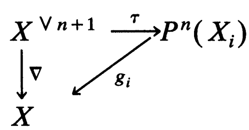
We also have the following much easier result about the cocategory of a product.

PROPOSITION 2. $\operatorname{cocat}(X_1 \times X_2) = \max(\operatorname{cocat} X_1, \operatorname{cocat} X_2)$.

Proof. Let

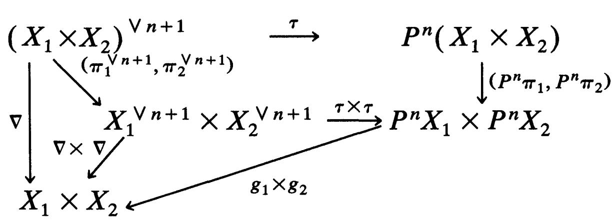
$$n = \max(\operatorname{cocat} X_1, \operatorname{cocat} X_2).$$

Since X_1 and X_2 are retracts of $X_1 \times X_2$, $\operatorname{cocat}(X_1 \times X_2) \geq n$. To show the opposite inequality, we can assume n is finite. Then by Lemma 2, there are maps g_i making



homotopy commute.

Consider the diagram



The left triangle commutes by inspection, the right triangle by assumption, and the quadrilateral by the fact that τ is a natural transformation. Thus the whole diagram commutes and $\operatorname{cocat}(X_1 \times X_2) \leq n$. □

Although we do not need it in the sequel, we point out that cocategory behaves well under localization. We only consider rational localization, but the same proof works for localization at any set of primes.

PROPOSITION 3. *Let X be a simply connected space and let X_0 be its rational localization. Then* $\operatorname{cocat} X_0 \leq \operatorname{cocat} X$.

Proof. First we claim that $(P^nX)_0 = P^nX_0$. Indeed the map $X \to X_0$ induces $P^nX \to P^nX_0$. The space P^nX_0 is rational since its homotopy groups are inverse limits of rational vector spaces, hence rational vector spaces. In fact the induced map on homotopy is just localization, so P^nX_0 is the localization of P^nX. Hence given a map $g : P^nX \to X$ we can localize to get $g_0 : P^nX_0 \to X_0$. It is easy to see that if g extends the fold map so does g_0. □

4. Rational cocategory

In this section we will explore the relation between the thin product definition of cocategory, applied to a rational space X, and the rational cocategory of X. We use the standard terminology and tools of rational homotopy theory, a careful exposition of which may be found in [14] or [6].

Throughout this section, X will be a rational space; i.e. a simply connected space whose homotopy and homology groups are $\mathbf{Q}$ vector spaces. We must also assume that X is finite type, so the above vector spaces are finite dimensional in each dimension. $(L(V), d)$ (or just $L(V)$) will denote a Quillen minimal model of X, unique up to isomorphism. $(L(V), d)$ is a free differential graded Lie algebra (DGLA).

ΛW will be a Sullivan (minimal) model, which is a free commutative graded differential algebra (CGDA). We will use $L * M$ for the coproduct of two DGLAs and occasionally $\coprod_i L_i$ for an indexed coproduct. Sometimes we will abuse notation and still use $L * M$ even when the differential is not the free product of the differentials of L and M. $L^{>n}$ will denote the differential Lie ideal consisting of linear combinations of m-fold brackets for $m > n$.

Let $\mathscr{C}$ denote either the category of CGDAs or that of DGLAs. For objects or maps of $\mathscr{C}$ there is a notion of minimal model. This is an especially good element of the homotopy class of the given object or map. See [14] or [6]. Given $f : A \to B$ in $\mathscr{C}$ with models $\phi : M_A \to A$, $\psi : M_B \to B$, and $\alpha : M_A \to M_B$, we say that A is a retract of B (via f) if there is a $g : M_B \to M_A$ with $g \circ f$ homotopic to the identity.

Let us recall one of the main results of [2]. Let ΛW denote the Sullivan model of X. As above, let $\Lambda^{>n}W$ consist of m-fold products for $m > n$.

THEOREM 3 (Félix-Halperin). *$\operatorname{cat} X \leq n$ if and only if the canonical projection $\pi : (\Lambda W, d) \to (\Lambda W/\Lambda^{>n}W, \bar{d})$ has a homotopy retraction.*

Motivated by this, Sbaï defines rational cocategory as follows, in [12].

DEFINITION 3 (Sbaï). The rational cocategory of X, $\operatorname{cocat}_0 X$, is the least n so that the canonical projection $L(V) \to L(V)/L^{>n}(V)$ has a homotopy retraction.

We would like to prove an analog of Theorem 3 for cocategory. That is, we would like to show that $\operatorname{cocat} X = \operatorname{cocat}_0 X$. Unfortunately, we are currently unable to do this. However, we can show:

THEOREM 4. *Let X be a rational space. Then* cocat $X \leq$ cocat$_0$ X.

To prove this theorem we will follow the proof of Félix and Halperin in [2] [pages 7–14] as far as we can.

The first step is to find a rational model for the thin product.

LEMMA 4. *Let $X_1, \dots, X_{n+1}$ be rational spaces with associated Quillen (DGLA) models L_i. Let $L = \coprod_i L_i$, the model of the wedge, $\bigvee_i X_i$. Let*

$$I = \bigcap_j \ker\left(L \xrightarrow{\pi_j} \coprod_{i \neq j} L_i \right).$$

Then L/I represents $P^{n+1}(X_1, \dots, X_{n+1})$. Furthermore, the natural projection $L \to L/I$ represents $\tau : \bigvee_i X_i \to P^{n+1}(X_1, \dots, X_{n+1})$.

Proof. The Quillen model is obtained through a number of adjoint functor pairs each of which is an equivalence of homotopy categories. It is easy to see that any such must preserve homotopy inverse limits (which are defined in DGLAs and CGDAs just as for topological spaces). Hence if $\overline{G}(X_i)(A) = \coprod_{i \in A} L_i$, then $\underleftarrow{\operatorname{holim}}\ \overline{G}(X_i)$ represents the thin product. Also, the map $f : \coprod(L_i, d_i) \to \coprod_{i \in A}(L_i, d_i)$ obtained by mapping (L_i, d_i) for $i \notin A$ to 0 represents τ. This map sends I to 0. So we have the following diagram.

$$\begin{array}{ccc} & L & \\ \swarrow^{\pi} & & \searrow^{f} \\ L/I & \xrightarrow{\ \bar f\ } & \underleftarrow{\operatorname{holim}}\ \overline{G}(X_i) \end{array}$$

Now τ, and therefore f, is surjective on homotopy by the results of Section 3. This requires simple connectivity as $\pi_1 \tau$ is not surjective in general. Thus $\bar f$ is surjective on homotopy as well. We will show that it is injective on homotopy and thus a quism.

Suppose there is a non-zero homotopy class $\alpha \in H_*(L/I)$ with $(H_* \bar f)\alpha = 0$. Choose a representing cycle $x + I$ with $x \in L$ having a minimal number of $\pi_i x \neq 0$. Let j be the first index with $\pi_j x \neq 0$. Since $x + I$ is a cycle, $dx \in I$. Thus $d\pi_j x = \pi_j\, dx = 0$. Since $(H_* \bar f)\alpha = 0$, $\pi_j x$ must be a boundary in $\coprod_{i \neq j} L_i$. This is because π_j factors through $\bar f$—in fact, π_j was one of the maps used to define $\bar f$. Suppose $y \in \coprod_{i \neq j} L_i$ has $dy = \pi_j x$. Let ι_j: $\coprod_{i \neq j} L_i \to L$ be the inclusion. Let $z = x - \iota_j \pi_j x$. Then $x - z = \iota_j \pi_j x = d\iota_j y$, so z is a cycle representing α. Denote by τ_A the projection $L \to \coprod_{i \notin A} L_i$ and by ι_A^B the inclusion $\coprod_{i \notin A} L_i \to \coprod_{i \notin B} L_i$ $B \subset A$. Then $\pi_j z = 0$ and $\pi_i z = \pi_i x - \pi_i \iota_j \pi_j x = \pi_i x - \iota^i_{i,j} \pi_{i,j} x$. So if $\pi_i x = 0$, $\pi_i z = 0$ as well.

This violates the minimality of x, so $\bar{f}$ is injective on homotopy and hence a quism. □

Now we want to take the pushout of the diagram below.

$$\begin{array}{ccc} L & \xrightarrow{\pi} & L/I \\ {\scriptstyle\nabla}\downarrow & & \\ L(V) & & \end{array}$$

Here $L_i = L(V)$ for all i.
Just as in [2] we find that this pushout is

$$\Gamma_n(L(V)) = \left(L/I *_L L * L(\bar{V})^{*n}, D\right) = \left(L/I * L(\bar{V})^{*n}, D\right).$$

Here $\bar{V}$ is the suspension of V. Also, letting $Q(D)$ denote the indecomposable part of D and V_i denote the ith copy of V we have $Q(D)(\overline{v_i}) = v_i - v_{i+1}$. Thus we get:

Proposition 4. *Let X be a rational space with Quillen model $L(V)$. Then* cocat $X \leq n$ *if and only if the natural map $L(V) \to \Gamma_n(L(V))$ has a homotopy retraction.*

Following [2], we should analyse the homotopy type of Γ_n. The precise analog of their Theorem 3.1 would be that $\Gamma_n(L(V))$ has the same homotopy type as the product of $L(V)/L^{>n}(V)$ with some extra Eilenberg-MacLane spaces. However, this analog is false. Using the Mayer-Vietoris sequence or the Quillen model, one can calculate $\Gamma_1(S^n)$. We find that $\Gamma_1(S^3)$ has the homotopy type of $S^3 \vee S^6$, and that $\Gamma_1(S^2)$ has the homotopy type of the complex projective plane CP^2. The corresponding $L(V)/L^{>n}(V)$ have the rational homotopy types of S^3 and CP^∞ respectively. Note in particular that the cocategory of Γ_n can be infinite, in stark contrast to the dual case.
The best we can do is the following:

Lemma 5. *The natural map*

$$L(V) \to L(V)/L^{>n}(V)$$

factors through $\Gamma_n(L(V))$.

Proof. Recall that L denotes the $(n+1)$-fold free product of $L(V)$ with itself, and I denotes the ideal consisting of linear combinations of brackets in which every factor occurs. The fold map $\nabla : L \to L(V)$ sends I into $L^{>n}(V)$. Thus we have a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\pi} & L/I \\ \downarrow & & \downarrow{\scriptstyle\nabla} \\ L(V) & \longrightarrow & L(V)/L^{>n}(V) \end{array}$$

Since $\Gamma_n(L(V))$ is the homotopy pushout of the left corner of this diagram, we get the required map $\Gamma_n(L(V)) \to L(V)/L^{>n}(V)$ factoring the projection up to homotopy. □

This lemma then implies Theorem 4.

5. A counterexample

In this section we will apply the previous work. We find a counterexample to Sbaï's conjecture that rational cocategory and inductive cocategory localized at 0 are the same. We also make some conjectures that would help clarify the situation if they are true.

We start by pointing out that the main result of Félix and Thomas in [3] carries over with no change to the DGLA situation. That is, we have:

PROPOSITION 5. *A rational space X of rational cocategory 2 is a (generalized) two-stage Postnikov tower.*

The proof is the same, with the Quillen model instead of the Sullivan model and the free product instead of the tensor product.

Now, localize all spaces rationally. Recall that for odd n, $K(\mathbf{Q}, n) \simeq S^n$ and for even n, $K(\mathbf{Q}, n) \simeq \Omega S^{n+1}$. Define $f : S^5 \times S^5 \to \Omega S^{11}$ by the cohomology class ab where a, b are the fundamental classes of the two copies of S^5. Let B be the fiber of this map. Then the homotopy of B is generated by elements which we also call a and b in dimension 5 and the Whitehead product $c = [a, b]$ in dimension 9. Hence by the fibration property, ind cocat $B = 2$.

B is what is called coformal: that is, its Quillen model depends only on its homotopy Lie algebra. Sbaï in [12] shows that the rational cocategory of a coformal space is just the length of the longest nonzero. Whitehead product. Thus $\text{cocat}_0\, B = 2$.

Define a map $\Omega S^3 \times \Omega S^3 \times S^3 \to S^5 \times S^5$ by $(\alpha\gamma, \beta\gamma)$ where α, β and γ are the fundamental classes. Then this map factors through B and we get a fibration

$$F \to \Omega S^3 \times \Omega S^3 \times S^3 \to B.$$

The homotopy groups of F are generated by $\alpha, \beta, \gamma, [\alpha, \gamma], [\beta, \gamma]$, and an element C in dimension 8. By the fibration property, ind cocat $F = 2$. On the other hand, it is easy to see that F is not a 2-stage Postnikov tower. Indeed, if $F \to X_1 \to X_2$ is a fibration with X_1 and X_2 products of Eilenberg-MacLane spaces, we can assume that α, β, and γ are all mapped into nonzero elements of $\pi_* X_1$. Then $[\alpha, \gamma]$ and $[\beta, \gamma]$ must come from elements of $\pi_* X_2$. So the only choice involved is where to put C. Using the Serre spectral sequence and the Sullivan model of F, it is easy to see that neither one works. Therefore, $\mathrm{cocat}_0 F > 2$.

We now define precisely what we mean by a generalized n-stage Postnikov tower, which we need to make a conjecture.

DEFINITION 4. Let X be a space. Define the fiber length of X, f.l. X, as follows:

1. f.l. $X = 0$ if and only if X is contractible.
2. f.l. $X \leq n$ if and only if there is a Y with f.l. $Y <$ n and a fibration $X \to Y \to Z$ where Z is an H-space.

We will mostly be interested in this definition over the rationals, where the fiber length is just the shortest length of a generalized Postnikov tower for X. Note that ind cocat $X \leq$ f.l. X, and the example above shows that the inequality can be strict. However I make the following conjecture.

CONJECTURE 1. *For rational* X, $\mathrm{cocat}_0 X =$ f.l. X.

Were this conjecture true, it would give us the following analog to the fibration property of inductive cocategory, which holds in the above example.

CONJECTURE 2. *In a fibration* $F \to E \to B$, $\mathrm{cocat}_0 F \leq \mathrm{cocat}_0 E + \mathrm{cocat}_0 B$.

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