ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 3, Fall 1992

ON PROPAGATION OF SINGULARITIES FOR FUCHSIAN QUASILINEAR DIFFERENTIAL OPERATORS

BY

JOSÉ RUIDIVAL DOS SANTOS FILHO

Introduction

A Meyer type flow, of a Tricomi gas for nozzle problem, is expressed in terms of solutions of the system

$$(0.1)\qquad \begin{pmatrix} s \\ \theta \end{pmatrix}_{\psi} = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix}\begin{pmatrix} s \\ \theta \end{pmatrix}_{\phi}$$

where s is the speed, θ is the inclination of the velocity, ψ is the stream function and ϕ is the velocity potential (see Bers, [1]). Therefore, for sufficiently smooth solutions, one could reduce the problem to the study of solutions of the equation

$$(0.2)\qquad u_{xx} - uu_{yy} - (u_y)^2 = 0.$$

A generic propagation of singularity result was proved in Guillemin–Schaeffer [3] for a linearization of (0.2), (considering Taylor expansion of u and u_y). This result was completed for the n-dimensional case by Santos Filho [6].

Based in the theory of paradifferential operators of Bony [2], see also Meyer [5], we can prove a result which, in particular, states that for sufficiently smooth solutions of (0.2) singularities can not be isolated in the set $\{(x, y); u(x, y) = 0, \nabla u(x, y) \neq 0\}$. The paper is organized as follows: In §1 we state the theorem and recall the main definitions and basic theorems of Bony's theory. In §2, we prove the main result. Finally, in §3, we state a generalization of our theorem.

Received December 7, 1990.
1991 Mathematics Subject Classification. Primary 35S05, 35M05; Secondary 76N15, 35G20.

1. Statement of the theorem

Our equation has the form

$$u_{x_1x_1} - uu_{x_2x_2} - (u_{x_2})^2 = 0. \tag{1.1}$$

where $(x_1, x_2) \in \mathbf{R}^2$ and u is real valued.

Let $s \in \mathbf{R}$ and $0 < \delta < 1$ we assume

$$s > 2 + 2\delta. \tag{H1}$$

Let $u \in H^{s-\delta}(\mathbf{R}^n)$. We assume the following:

$$H_p(x^0, \xi^0) = \theta Z(x^0, \xi^0) \quad \text{with } \theta \neq 0 \text{ and } |\xi^0| = 1, \tag{H2}$$

where

$$H_p(x^0, \xi^0) \left(= \sum \left(\frac{\partial p}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial p}{\partial x_i} \frac{\partial}{\partial \xi_i} \right) \right)$$

is the Hamiltonian vector field of

$$p(x, \xi) = \xi_1^2 - u(x)\xi_2^2$$

and $Z(x, \xi) = \Sigma \xi_i \partial_{\xi_i}$ the radial vector field, at (x^0, ξ^0).

We also assume

$$u \in H^s_{(x,\xi)} \cap H^{s-\delta}(\mathbf{R}^n) \quad \text{if } (x, t\xi) \neq (x^0, \xi^0), \forall t > 0. \tag{H3}$$

and

$$\theta(s - \tfrac{1}{2}) + 2\partial_{x_2} u(x^0) > 0 \tag{H4}$$

Here $H^t(\mathbf{R}^n) = H^t$ is the usual Sobolev space and $H^t_{(x,\xi)}$ is its microlocal version (see Hörmander [4]). From Bony's main result, see [2], we can consider that $(x^0, \xi^0) = (0, (0, 1))$ and hence we prove:

THEOREM. *Let u satisfy* (1.1). *Under the hypothesis* (H1) *to* (H4) *we have* $u \in H^s(\mathbf{R}^n)$.

We now recall some facts regarding the theory of Bony. Let $\phi \in C_0^\infty(\mathbf{R}^n)$ such that is supported in the ball centered at the origin and radius 1 and which is equal to one in the ball with same center and radius $1/2$. Taking $\psi(\xi) = \phi(\xi 2^{-1}) - \phi(\xi)$. For $f \in S(\mathbf{R}^n)$ (the Schwartz space), consider the Littlewood-Paley decomposition of f, $f = S_0(f) + \Sigma\Delta_k(f)$, where $(S_0(f))^\wedge = \phi f^\wedge$ and $(\Delta_k(f))^\wedge = \psi(\xi 2^{-k}) f^\wedge$; here $^\wedge$ means Fourier transform. For $f \in C^r$, $r > 0$, the sum converges uniformly to f. Let $f \in H^t(\mathbf{R}^n)$, $t > n/2$. Define the paraproduct

$$\Pi(f, g) = \sum S_{k-6}(f)\Delta_k(g), \qquad g \in S(\mathbf{R}^n);$$

here $S_j(f) = S_0(f) + \Sigma_{l=1}^{j}\Delta_l(f)$, $j \geq 1$. We summarize results of Bony [2] and Meyer [5], see Theorems 2.1, 2.3, 2.4 and 2.5 of [2] and Theorems 3 and 5 of [5].

THEOREM 1.1. *Let $f, g \in H^t$, $t > n/2$ and $h \in H^s$.*

(a) *$\Pi(f, \cdot\,)$ can be extended to a continuous operator from H^r into H^r.*

(b) *$E = \Pi(f, \cdot\,)\circ\Pi(g, \cdot\,) - \Pi(ab, \cdot\,)$ is a $(t - n/2)$-smoothing operator, that is, it maps continuously H^r into $H^{r+(t-n/2)}$.*

(c) *The adjoint $(\Pi(f, \cdot\,))^*$ of $\Pi(f, \cdot\,)$ applies H^r into H^r and $\Pi(f, \cdot\,) - (\Pi(f, \cdot\,))^*$ is a $(t - n/2)$-smoothing.*

(d) *If $s > n/2$ then*

$$fh = \Pi(f, h) + \Pi(h, f) + r$$

where $|r|_{s+t-n/2-\varepsilon} \leq C|f|_t|h|_s$.

If $-t + n/2 < s \leq n/2$, then

$$fg = \Pi(f, g) + r$$

where $|r|_{s+t-t/2-\varepsilon} \leq C|f|_t|h|_s$.

DEFINITION. We say $\sigma \in B_r^m$ if $\sigma \in S_{1,1}^m$,

$$\left\|\partial_\xi^\alpha \sigma(x, \xi)\right\|_{C^r} \geq C_\alpha(1 + |\xi|)^{m-|\alpha|}$$

and for each ξ, the support of $\sigma^\wedge(\cdot, \xi)$ is contained in $\{\eta; |\eta| < |\xi|/10\}$. Here $S_{\rho,\delta}^m$ is Hörmander's class of symbols; see [4].

Examples. (1) The symbol of $\Pi(a, \cdot\,)$ belongs to B_r^0, where $r = s - n/2 > 0$, if $a \in H^s(\mathbf{R}^n)$.

(2) $\sigma(\xi) \in B_r^m$ for each $\sigma \in S_{1,0}^m$ and $r > 0$.

THEOREM 1.2. (a) *Let $F(x, y)$ be a C^∞ function, where $y = (y_0, \ldots, y_\alpha, \ldots)|\alpha| \leq (m - 1)$, such that $F(x, 0) = 0$. Then*

$$F(x, U(x)) = \sum_{|\beta|=0}^{m-1} \Pi\big((\partial_{y\beta}F)(x, U(x)), u\big) + E,$$

where $U(x) = (u(x), \ldots, \partial^\beta u(x), \ldots)$ *with* $u \in H^t$, $t - (m-1) > n/2$ *and* $E \in H^{(2t-2(m-1)-n/2)}$.

(b) *If* $\sigma \in S^m_{1,1}$ *then for* $s > 0$ $\sigma(x, D)$ *can be extended to a continuous operator from* H^{s+m} *into* H^s.

(c) *Let* $r > 0$, $\sigma \in B^{m_1}_r$ *and* $\tau \in S^{m_2}_{1,1}$ *then*

$$\tau(x, D) \circ \sigma(x, D) = \omega(x, D) + \rho(x, D)$$

where

$$\omega(x, \xi) = \sum_{|\alpha| \leq [r]} \frac{1}{i^{|\alpha|}} \frac{l}{\alpha!} \partial^\alpha_\xi \tau(x, \xi) \partial^\alpha_x \sigma(x, \xi) \quad \textit{and} \quad \rho(x, \xi) \in S^{m_1+m_2-r}_{1,1}.$$

Remark. Theorem 1.2 implies the classical Schauder's Lemma, which says that $H^t(\mathbf{R}^n)$, $t > n/2$ is invariant under non-linear transformations.

2. Proof of the theorem

Consider

$$D^2_{x_1} u - u D^2_{x_2} u + (u_{x_2})^2 = 0. \tag{2.1}$$

Multiplying this equation by χ, where $\chi \in C_0^\infty$, $\chi = 1$ if $|x| \leq \varepsilon/4$, $\chi = 0$ if $|x| \geq \varepsilon$, for $\varepsilon > 0$ small, we obtain

$$D^2_{x_1}(\chi u) - D^2_{x_2}(\chi u) + \chi((\chi u)_{x_2})^2 = f$$

where $f \in H^{s-1}(\mathbf{R}^2)$ from (H1) and (H3), by Schauder's Lemma.

We write the equation above in the form

$$P(\chi u) + Q(\chi u) = f \tag{2.1$'$}$$

where $P = D^2_{x_1} - u D^2_{x_2}$ and $Qv = \chi(\partial_{x_2}(v))^2$. We can assume $r = s - \delta - 1$ is a non-integer positive real number.

Let $\mathscr{C} \subset S^{s-(m-1)/2-\lambda}_{1,0}(\mathbf{R}^{2n})$, $\lambda > \frac{1}{2} + \delta$ be a bounded subset of $S^{s-1/2}(\mathbf{R}^{2n})$, which consists of real valued symbols. For each $c \in \mathscr{C}$ we put $C = c(x, D)$ and hence by (2.1)$'$ we get

$$\begin{aligned} \Im m(Cf, Cu) = {} & \Im m(C(Pu - \Pi, Cu) + \Im m(\Pi(Cu), Cu) \\ & + \Im m([C, \Pi]u, Cu) + \Im m(C(Q(u)), Cu) \end{aligned} \tag{2.2}$$

here $\Pi = D_{x_1}^2 - \Pi(u, \cdot) \circ D_{x_2}^2$. For convenience, we write the right hand side of (2.2) as, I + II + III + IV, respectively.

We will analyse the terms I, II, III and IV of the right hand side of (2.2), keeping the non-absorbable terms (in a sense we will make precise along the lines of this proof). By a linear change of variables, we assume $(x^0, \xi^0) = (0, e_n)$, $e_n = (0, \dots, 0, 1)$, and by replacing p by $-p$ (if necessary) we assume $\theta > 0$.

Step 1. *Analysis of term I.* We have

$$(P - \Pi)(\chi u) = -(M_u - \Pi(u, \cdot))D_{x_2}^2(\chi u),$$

where M_u is the multiplication by u. From Theorem 1.1(d) we have

$$(-M_u + \Pi(u, \cdot))D_{x_2}^2(\chi u) \in H^{2r-1-\varepsilon}, \qquad \forall \varepsilon > 0.$$

From this we know that there exist $K_{1,0}$ and $K_{1,1}$ positive constants, uniformly on $\mathscr{C}$, such that for all $\mu > 0$,

$$|\mathrm{I}| \leq \mu K_{1,1} \left\| (\mathrm{I} - \Delta)^{1/4} C(\chi u) \right\|_{L_2}^2 + \frac{1}{\mu} K_{1,0} \tag{2.3}$$

where $(\mathrm{I} - \Delta)^{1/4}$ is the pseudo-differential operator whose symbol is $(1 + |\xi|)^{1/4}$.

Step 2. *Analysis of term II.* We have

$$\Pi - \Pi^* = \Pi(u, \cdot) \circ D_{x_2}^2 - D_{x_2}^2 \circ (\Pi(u, \cdot))^*,$$

and from Theorem 1.1(c), $-\Pi(\bar{u}, \cdot) + (\Pi(u, \cdot))^* = R_{2,1}$, where $R_{2,1}$ is an r-smoothing; and since u is real-valued, we have $\Pi - (\Pi)^* = [\Pi(u, \cdot), D_{x_2}^2] - D_{x_2}^2 \cdot R_{2,1}$. In the other hand, from Theorem 1.2,

$$\left[\Pi(u, \cdot), D_{x_2}^2\right] = \omega(x, D) + R_{2,2}$$

where

$$\omega(x, \xi) = -\sum_{1 \leq |\alpha| \leq [r]} (i)^{-|\alpha|} \frac{1}{\alpha!} \alpha_\xi^\alpha \xi_2^2 \partial_x^\alpha (\sigma(\pi(u, \cdot))$$

and $\sigma(R_{2,2}) \in S_{1,1}^{2-r}$. Here $[r]$ is the greatest integer less or equal to r.

From Theorem 1.1 and Theorem 1.2 it can be shown that $\operatorname{Im}(A \circ C(\chi u), C(\chi u))$ satisfies the same type of estimate in (2.3), where A is taken to be $D_{x_2}^2 \circ R_{2,1}$ or $R_{2,2}$ or

$$\sum_{[r] \geq |\alpha| > 1} \frac{1}{\alpha!} \partial_\xi^\alpha \xi_2^2 \sigma(\pi(D^\alpha u, \cdot)))(x, D)$$

or

$$\left(\sum_{|\alpha|=1} \frac{1}{\alpha!} \partial_\xi^\alpha \xi_2^2 (M_{D^\alpha u(x)} - M_{D^\alpha u(0)}) \right)(D).$$

Which finally gives this result:

There exist $K_{2,0}$ and $K_{2,1}$ positive constants, uniformly on $\mathscr{C}$, such that for all $\mu > 0$, we have

$$\begin{aligned}(2.4) \qquad & \left| \mathrm{II} - D_{x_2} u(0) (D_{x_2} \circ C(\chi u), C(\chi u)) \right| \\ & \leq K_{2,1} \mu \left\| (\mathrm{I} - \Delta)^{1/4} C(\chi u) \right\|_{L^2}^2 + K_{2,0} \frac{1}{\mu}.\end{aligned}$$

Step 3. *Analysis of the term III.* By assuming $\sigma(C) \in B_r^m$ we have

$$\begin{aligned}\sigma([C, \pi]) = \sum_{|\alpha|=1} \Bigg(& \frac{1}{\alpha!} \partial_\xi^\alpha c \sigma(\pi(D^\alpha u, \cdot)) \xi_2^2 \\ & - \frac{1}{i^{|\alpha|}} \frac{1}{\alpha!} \partial_\xi^\alpha (\xi_2^2 \sigma(\pi(u, \cdot))) \partial_x^\alpha c(x, \xi) \Bigg) + \sigma(R_{3,1})\end{aligned}$$

where $\sigma(R_{3,1}) \in S^{s+1/2-\tilde{r}+1-\lambda}$, and $\tilde{r} = \mathrm{Min}\{r, 2\}$. Hence as before:

There exist $K_{3,0}$ and $K_{3,1}$ positive constants, uniformly on $\mathscr{C}$, such that for all $\mu > 0$, we have

$$\begin{aligned}(2.5) \qquad & \big| \mathrm{Im}([C, \pi](\chi u), C(\chi u)) \\ & + \mathrm{Re}(H_c(\xi_1^2 - u\xi_2^2)(0, D)(\chi u), C(\chi u)) \big| \\ & \leq \mu K_{3,1} \left\| (\mathrm{I} - \Delta)^{1/4} (Cu) \right\|_{L^2}^2 + \frac{1}{\mu} K_{3,0}.\end{aligned}$$

Step 4. *Analysis of the term IV.* Using Taylor's formula, by Schauder's Lemma since (H1) and (H3) hold, we have

$$\begin{aligned}\chi(\partial_{x_2}(\chi u))^2 = {} & \chi(\partial_{x_2} u(0))^2 \\ & + \chi 2 (\partial_{x_2} u(0)) (\partial_{x_2}(\chi u) - (\partial_{x_2} u)(0)) + \tilde{q}(\partial_{x_2}(\chi u(x)))\end{aligned}$$

where $\tilde{q}(\partial_{x_2}(u(0))) = 0$ and $\nabla \tilde{q}(\partial_{x_2}(u(0))) = 0$. Therefore, as before:

There exist $K_{4,0}$ and $K_{4,1}$ positive constants, uniformly on $\mathscr{C}$, such that for all $\mu > 0$, we have

$$(2.6)\quad \left|\operatorname{Im}(C(Q(\chi u)), C(\chi u)) - \operatorname{Im}\left(C\left(\chi 2(\partial_{x_2}u)(0)\partial x_2(\chi u)\right), C(\chi u)\right)\right| \\ \leq \mu K_{4,1}\left\|(\mathrm{I} - \Delta)^{1/4}(Cu)\right\|_{L^2}^2 + \frac{1}{\mu}K_{4,0}.$$

Step 5. *End of the proof*. Using (2.3), (2.4), (2.5) and (2.6), and since $\operatorname{Im}(Cf, C(\chi u)$ is also absorbable (in the sense the above inequality is true for $|\operatorname{Im}(Cf, C(\chi u))|$), we have:

There exist $K_{5,0}$ and $K_{5,1}$ positive constants, uniformly on $\mathscr{C}$, such that for all $\mu > 0$,

$$(2.7)\quad \Big|\tfrac{1}{2}\operatorname{Re}\Big(\big(2\partial_{x_2}u(0)D_{x_2} + \chi 2(\partial_{x_2}u)(0)D_{x_2}\big)C(\chi u), Cu\Big) \\ + \tfrac{1}{2}\operatorname{Re} H_c\big(\xi_1^2 + u(x)\xi_2^2\big)(0, D)), Cu\big)\Big| \\ \leq \mu K_{5,1}\left\|(I - \Delta)^{1/4}(Cu)\right\|_{L^2}^2 + \frac{1}{\mu}K_{5,0}.$$

Observe that the first term of the right hand side of (2.7) can be expressed in an invariant form, namely, for the general case it is equal to $\operatorname{Re}((\sigma_{sub}(P_L)C(\chi u), Cu)$, where P_L is the linearization of P at u, $\sigma(P_L) = \Sigma\partial_{(\partial^\alpha u)}(Pu)\xi^\alpha$ and $\sigma_{sub}(P_L)$ is the subprincipal symbol of P_L (see [4]). This will give us

$$(2.8)\quad \left|\tfrac{1}{2}\operatorname{Re}\left(H_{\sigma(P)}c^2(0,D)(\chi u), \chi u\right) + \operatorname{Re}\left(\sigma_{sub}(P_L)c^2(0,D)(\chi u), \chi u\right)\right| \\ \leq \mu K_1\left\|(I - \Delta)^{1/4}(Cu)\right\|_{L^2}^2 + \frac{1}{\mu}K_0,$$

for positive constants K_0 and K_1 and for all $\mu > 0$, uniformly on $\mathscr{C}$. Here $\sigma_{sub}(P_L)(0, \xi)$ is in this case equal to $2\partial_{x_2}u(0)\xi_2$.

At this point, we take a explicit class $\mathscr{C}$ of symbols, which is taken in such a way we can apply the sharp Garding inequality for $\operatorname{Re}(Su, u)$ where $\sigma(S) = \frac{1}{2}H_{\sigma(P)}c^2 + \sigma_{sub}(P_L)c^2)(0, \xi)$. Namely we take

$$c_\gamma(x, \xi) = (\chi(x))^2(\psi(\xi))^2\xi_2^{(s-1)/2}\left(1 + \gamma\xi_2^2\right)^{-\lambda/2}$$

where $0 < \gamma < 1$ and $\psi \in S^0_{1,0}$, homogeneous of degree 0 for $|\xi| \geq \frac{1}{2}$, $\phi(\xi) = 1$ for $|\xi_1| \leq \frac{1}{2}\xi_2$ and $\psi(\xi) = 0$ for $|\xi_2| \geq \xi_2$. So

$$\tfrac{1}{2}2H_{\sigma P}\left(c_\gamma^2\right)(0, \xi_n) \geq \left(s - \tfrac{1}{2} - \gamma\lambda\right)\theta\xi_2 c_\gamma^2,$$

where θ is given in (H2). So by taking γ small the proof of the theorem is finished from (H4) and Garding inequality, see [4].

3. Remarks

A generalization of our theorem can be expressed in the following form: Let

$$\sum_{|\alpha|=m} A_\alpha(x, u(x), \dots, \partial^\beta u(x), \dots)_{|\beta| \le p_\alpha} \partial^\alpha u + q(x; u(x), \dots, \partial^\beta u(x), \dots)_{|\beta| \le p_{m-1}} = 0. \tag{3.1}$$

where $p_{m-1}, p_\alpha \le m-1$ ($p_\alpha = -\infty$ (resp. p_{m-1}) if A_α (resp. q) depends only on $x \in \mathbf{R}^n$), $A_\alpha = A_\alpha(x, y)$ and $q = q(x, \tilde{y})$ are C^∞ real-valued defined in an appropriate $\mathbf{R}^{N+m}$ and u is a real function defined in $\mathbf{R}^n$.

Let $s \in \mathbf{R}$ and $0 < \delta < 1$ and assume

(H1)

(a) $s > p_{m-1} + \frac{n}{2} + \delta$

(b) $s > \delta + \frac{\operatorname{Max} p(\alpha) + m}{2} + \frac{n}{4}$

(c) $s > \frac{n}{2} + \operatorname{Max} p(\alpha) + 1 + 2\delta$.

Let $u \in H^{s-\delta}(\mathbf{R}^n)$. We assume

(H2) $\quad H_p(x^0, \xi^0) = \theta Z(x^0, \xi^0)$ with $\theta \ne 0$ and $|\xi^0| = 1$,

where

$$H_p(x^0, \xi^0) \quad \left(= \sum \left(\frac{\partial p}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial p}{\partial x_i} \frac{\partial}{\partial \xi_i} \right) \right)$$

is the Hamiltonian vector field of $p(x, \xi) = \sum_{|\alpha|=m} A_\alpha(x, u(x), \dots)\xi^\alpha$ and $Z(x, \xi) = \sum \xi_i \partial_{\xi_i}$ the radial vector field, at (x^0, ξ^0). Observe that from (H1), $A_\alpha(x, u(x), \dots) \in C^{1+\delta}(\mathbf{R}^n)$, so H_p is a well defined Hölder continuous vector field.

Also we assume

(H3) $\quad u \in H^s_{(x,\xi)} \cap H^{s-\delta}(\mathbf{R}^n)$ if $(x, t\xi) \ne (x^0, \xi^0)$, $\forall t > 0$.

and

$$\text{(H4)} \qquad \theta\left(s - \frac{m-1}{2}\right) + 2\sigma_{sub}(x_0, \xi_0) > 0$$

where

$$\sigma_{sub}(x_0, \xi_0) = \operatorname{Re}\left(\sum_{|\beta| = m-1} (\partial_{y_\beta} q)(x_0, u(x_0), \dots)\xi_0^\beta\right) + \frac{1}{2}\sum_{|\beta|=1} (\partial_x^\beta A_\alpha)(x_0, u(x_0), \dots)(\partial_\xi^\beta \xi^\alpha)(\xi_0).$$

Using the same method one can prove:

THEOREM. *Let u satisfy* (1.1). *Under the hypothesis* (H1) *to* (H4) *we have* $u \in H^s(\mathbf{R}^n)$.

It should be said that even for the linear case this theorem says something new. In particular, it says that the solutions with prescribed singularities in a ray constructed in [3] and [6] cannot be arbitrarily smooth. In fact, it says that the solutions constructed therein are sharp regarding the regularity aspect.

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