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ANALYTIC REPRODUCING KERNELS AND MULTIPLICATION OPERATORS

BY

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Introduction

Recall that if E is a set and K is a function from $E \times E$ into the complex plane $\mathbf{C}$, then K is *positive definite* (denoted $K \gg 0$) provided

$$\sum_{j,k=1}^{n} \bar{a}_j a_k K(w_j, w_k) \geq 0$$

for any finite set of complex numbers $a_1, \dots, a_n$ and any finite subset $w_1, \dots, w_n$ of E. It is well known that if $K \gg 0$ on E, then

$$\left\{ \sum_{j=1}^{n} a_j K(\cdot, w_j) \colon a_1, \dots, a_n \in \mathbf{C} \text{ and } w_1, \dots, w_n \in E \right\}$$

has dense span in a Hilbert space $H(K)$ of functions on E with

$$\left\| \sum_{j=0}^{n} a_j K(\cdot, w_j) \right\|^2 = \sum_{j,k}^{n} \overline{a_j} a_k K(w_j, w_k).$$

A fundamental property of $H(K)$ is the *Reproducing Property* which says that

$$f(w) = \langle f(\cdot), K(\cdot, w) \rangle$$

for every w in E and f in $H(K)$. Thus evaluation at w is a bounded linear functional for each w in E. Conversely, it is well known that if F is a Hilbert space of functions defined on E such that evaluation at w is a bounded

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linear functional for each w in E, then there is a unique K defined on $E \times E$ such that $F = H(K)$. It follows from the reproducing property that for every z, w in E

$$K(z,w) = \langle K(\cdot,w), K(\cdot,z)\rangle = \overline{\langle K(\cdot,z), K(\cdot,w)\rangle} = \overline{K(w,z)}.$$

If K is analytic in the first variable, then it follows that K is coanalytic in the second variable. In this case K is an *analytic* kernel. Throughout this paper, E will always be a subset of the complex plane $\mathbf{C}$ and K will be an analytic kernel.

If ϕ is an analytic function on E and ϕf is in $H(K)$ for all f in $H(K)$, then ϕ is a *multiplier* of $H(K)$. In this case, the Closed Graph Theorem implies that the *multiplication operator* M_ϕ defined by $M_\phi f = \phi f$ is a bounded linear operator on $H(K)$. If $H(K)$ contains the constants, then ϕ is necessarily in $H(K)$.

The purpose of this paper is to provide a framework in which a general investigation of the multiplication operators on $H(K)$ can begin. Our attention will be focused on analytic kernels defined on $E \times E$ where E is a region in the plane containing the origin. In this case, $K(z, \overline{w})$ is an analytic function on $E \times E$ in the two variables z and $\overline{w}$. Hence there is an open disk $B(0, R)$ about the origin such that $K(z, w)$ is represented by the double power series $\sum_{j,k=0}^{\infty} a_{j,k} z^j \overline{w}^k$. Moreover, the series converges absolutely and uniformly on compact subsets of $B(0, R) \times B(0, R)$. If A denotes the matrix $[a_{j,k}]$, then such a K can be written more compactly in the form $K(z, w) = \overline{\mathbf{z}}^* A \overline{\mathbf{w}} = \langle A\overline{\mathbf{w}}, \overline{\mathbf{z}}\rangle_{l^2_+}$ where $\mathbf{z}$ denotes the column vector whose transpose is $(1, z, z^2, \dots)$. (Here l^2_+ denotes the usual sequence space $l^2(\mathbf{Z}_0^+)$.) This product makes sense even for the case of A being unbounded as a matrix on l^2_+ provided that w and z are both in the disk of absolute convergence for the double power series representation for K. Recall that if A is a formal matrix on l^2_+, then A is *positive* ($A > 0$) if $A_n = [a_{j,k}]_{j,k=0}^n$ is positive for each $n = 0, 1, 2, \dots$.

It is well known that $K \gg 0$ if and only if $A > 0$. Henceforth, for positive matrices A, $H(A)$ will denote the space $H(K)$ where $K = \overline{\mathbf{z}}^* A \overline{\mathbf{w}}$.

The main goal of this paper is to provide a model for the multiplication operators on $H(A)$ and to relate properties of these operators to properties of A.

Section 1 recalls some basic properties of reproducing kernel Hilbert spaces and several examples of such spaces are presented.

As the study of the multiplication operators on $H(A)$ is somewhat awkward when A is unbounded, Section 2 is devoted to a dilation result which shows that one can reduce to the case where A is bounded.

The principal result of Section 3 is Theorem 3.1 in which it is shown how to produce bases for $H(A)$ via factorizations of the form $A = B^*B$. Additionally, it is shown that a particularly nice basis for $H(A)$ is obtained when the Cholesky algorithm is used to factor A into the form U^*U, where U is upper triangular.

Section 4 contains applications of the results of Section 3 to the multiplication operators on $H(A)$. Using Theorem 3.1 it is shown that the multiplication operator M_z on $H(A)$ is bounded if and only if the unilateral shift S on l^2_+ leaves the range of B^* invariant. Moreover, if M_z is bounded, it is shown that M_z is unitarily equivalent to the restriction of the unilateral shift S to the range space of B^*. (The definition of the range space of an operator is recalled in Section 4). Additionally a characterization of the multipliers of $H(A)$ is given, as well as a characterization of when M_z is polynomially bounded. The authors would like to thank John Froelich for pointing out a simplification of the proof of Theorem 3.1.

Section 1

The following are basic properties of $H(K)$ for which the reader is referred to Aronszajn [2].

PROPERTY 1.1 [2, pp 353–354]. *If $K_1 \gg 0$ and $K_2 \gg 0$ on $E \times E$, then $K_1 + K_2 \gg 0$ on $E \times E$. Moreover $H = H(K_1 + K_2) = H(K_1) + H(K_2)$ with*

$$\|f\|_H^2 = \inf\left\{\|f_1\|_{H_1}^2 + \|f_2\|_{H_2}^2 \colon f_i \in H(K_i),\ f_1 + f_2 = f\right\}.$$

In particular, if $H(K_1) \cap H(K_2) = \{0\}$, then $H(K) = H(K_1) \oplus H(K_2)$.

PROPERTY 1.2 [2, pp 351–352]. *If K_1 is the restriction of K to a subset $E_1 \times E_1$ of $E \times E$, then $K_1 \gg 0$ and $H(K_1) = \{f|E_1 \colon f \in H(K)\}$ with*

$$\|f|E_1\|_{K_1} = \inf\left\{\|g\|_K \colon g \in H(K),\ g(z) = f(z) \text{ for all } z \in E_1\right\}.$$

PROPERTY 1.3 [2, pp 354–355]. *The following are equivalent:*
(a) *As classes of functions, $H(K_1)$ is the same as $H(K_2)$;*
(b) *There are positive constants m and M such that*

$$mK_1 \ll K_2 \ll MK_1;$$

(c) *$H(K_1)$ and $H(K_2)$ are equivalently normed Hilbert spaces.*

PROPOSITION 1.4. *If E is a region in the plane containing $B(0, R)$, K is an analytic kernel on E, and K_1 is the restriction of K to $B(0, R)$, then the restriction map $V\colon H(K) \to H(K_1)$ is a unitary operator. Moreover if M_ϕ is a multiplication operator on $H(K)$, then $VM_\phi f = M_{\tilde{\phi}} f$ where $\tilde{\phi}$ denotes the restriction of ϕ to $B(0, R)$.*

Proof. Since the functions in $H(K)$ are uniquely determined by their values on $B(0, R)$, the result follows from Property 1.2. ■

The content of Proposition 1.4 is that the study of the multiplication operators on $H(K)$, where K is an analytic kernel and E is a region containing the origin, can be reduced to the study of multiplication operators on $H(A)$ where $K = \bar{\mathbf{z}}^* A \bar{\mathbf{w}}$ on a disk $B(0, R)$.

PROPOSITION 1.5. *If $H(K_1)$ and $H(K_2)$ consist of the same class of functions, then M_z on $H(K_1)$ is similar to M_z on $H(K_2)$.*

Proof. By Property 1.3 (c), the identity map is an invertible map from $H(K_1)$ to $H(K_2)$ that intertwines M_z. ■

Before proceeding it is useful to keep some examples in mind. For general diagonal matrices, the determination of these spaces can be accomplished by well known techniques, but the determination will also be seen to be a consequence of Theorem 3.1 of Section 3.

Example 1.1. If A is the diagonal matrix with positive entries $\{a_0, a_1, a_2, \dots\}$ and $(\limsup a_n^{1/n})^{-1} = R < \infty$, then

$$H(A) = \left\{ \sum_{n=0}^{\infty} b_n z^n : \left(\sum_{n=0}^{\infty} |b_n|^2 / a_n \right) < \infty \right\}.$$

In this case M_z is a (bounded) unilateral weighted shift if and only if its weight sequence $\{\sqrt{a_n / a_{n+1}} : n = 0, 1, \dots\}$ is uniformly bounded. The literature on weighted shifts is extensive and the reader is referred to Shields [5] for basic properties. The following four examples are specific cases of this example.

Example 1.2. If A is the identity, then $K(z, w) = (1 - \bar{w}z)^{-1}$ and $H(A)$ is the Hardy space H^2 of the unit disk $\mathbf{D}$, M_z is the unilateral shift, and the multipliers of $H(A)$ are the bounded analytic functions on $\mathbf{D}$.

Example 1.3. If $a_n = n + 1$, then $H(A)$ can be identified with the Bergman space of analytic functions on $\mathbf{D}$ which are square integrable with respect to area measure on $\mathbf{D}$. In this case, M_z is a cyclic operator whose spectrum is $\sigma(M_z) = \mathbf{D}^-$ and whose essential spectrum is $\sigma_e(M_z) = \partial \mathbf{D}$. Moreover, the Fredholm index of $M_{z-\lambda}$ is $\mathit{index}(M_{z-\lambda}) = -1$ for all λ in $\mathbf{D}$. As in the Hardy space case, the multipliers of M_z are the bounded analytic functions.

Example 1.4. If $a_n = (n+1)^{-1}$, then $H(A)$ can be identified with the Dirichlet space of analytic functions on $\mathbf{D}$ whose derivatives are in the Bergman space. In this case, $H(A)$ is not naturally identified as a subspace of an L^2 space. Again, the operator M_z is bounded, $\sigma(M_z) = D^-$, $\sigma_e(M_z) = \partial\mathbf{D}$, and $index(M_{z-\lambda}) = -1$ for all λ in $\mathbf{D}$. The multipliers of $H(A)$ are not as easily described as in the Hardy space case. Moreover it is shown in Stegenga [6] that there is a function continuous on the closed unit disk and analytic on the open disk which is not a multiplier of $H(A)$.

Example 1.5. If $a_n = (n!\, 2^n)^{-1}$, then $K(z,w)$ converges on the whole complex plane and $H(A)$ is a Hilbert space of functions known as the *Fock* space. In this case, M_z is an unbounded subnormal operator.

The following proposition enlarges the class of examples beyond the diagonal case. Recall first, that if $\phi \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ is a formal trigonometric series, then the formal *Laurent matrix* T_ϕ is the matrix $[a_{j-k}]_{j,k=0}^{\infty}$ and $T_\phi^* = T_{\bar{\phi}}$. If $a_n = 0$ for each negative n, then T_ϕ is a lower triangular matrix. In this case, T_ϕ is a formal *analytic Toeplitz matrix*, and one readily verifies that the formal matrix product $T_\phi A T_{\bar{\phi}}$ is well defined for any matrix A with complex entries.

PROPOSITION 1.6. *If $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $B(0, R_1)$, T_ϕ is the formal Toeplitz matrix with symbol ϕ, $K(z,w) = \bar{\mathbf{z}}^* A \bar{\mathbf{w}}$ defines an analytic kernel on $B(0,R)$, and $R_2 = \min(R, R_1)$, then the map W from $H(A)$ into $H(T_\phi A T_{\bar{\phi}})$ defined by $(Wf)(z) = \phi(z) f(z)$ for z in $B(0, R_2)$ is a unitary map that intertwines M_z on the respective spaces.*

Proof. First note that

$$K_\phi(z,w) = \bar{\mathbf{z}}^* T_\phi A T_{\bar{\phi}}\, \bar{\mathbf{w}} = \phi(z)\overline{\phi(w)} K(z,w)$$

where $K(z,w) = \bar{\mathbf{z}}^* A \bar{\mathbf{w}}$. Hence K_ϕ is an analytic kernel on $B(0,R_2)$, $K_\phi(\cdot, w) = \phi(\cdot) K(\cdot, w)\overline{\phi(w)}$, and

$$\begin{aligned}
&\big\langle W(K(\cdot, w_1)), W(K(\cdot, w_2))\big\rangle_\phi \\
&= \big\langle \phi(\cdot)K(\cdot,w_1), \phi(\cdot)K(\cdot,w_2)\big\rangle_\phi \\
&= \big\langle \phi(\cdot)K(\cdot,w_1), \phi(\cdot)K(\cdot,w_2)\overline{\phi(w_2)}\big\rangle_\phi (1/\phi(w_2)) \\
&= \big\langle \phi(\cdot)K(\cdot,w_1), K_\phi(\cdot,w_2)\big\rangle_\phi (1/\phi(w_2)) \\
&= \phi(w_2)K(w_2,w_1)/\phi(w_2) \\
&= \langle K(\cdot,w_1), K(\cdot,w_2)\rangle.
\end{aligned}$$

As the span of $\{K(\cdot, w): w \in B(0, R_2)\}$ is a dense linear manifold in $H(K)$ and W is an isomorphism of this manifold onto a dense manifold in $H(K_\phi)$, W extends to an isomorphism. Clearly $WM_z = M_z W$. ∎

Example 1.6. If $\{z_n\}_{n=0}^{\infty}$ is a non-Blaschke sequence in $\mathbf{D}$, and ϕ is an analytic function on $\mathbf{D}$ with zero set $\{z_n\}$, then by Proposition 1.5, M_z on $H(T_\phi T_{\bar{\phi}})$ is unitarily equivalent to the unilateral shift S on the Hardy space $H(I) = H^2$ via the map $W: H(I) \to H(T_\phi T_{\bar{\phi}})$ defined by $(Wf)(z) = \phi(z)f(z)$. Note the nonzero functions in $H^2 = H(I)$ cannot vanish on $\{z_n\}$, while every function in $H(T_\phi T_{\bar{\phi}})$ must vanish on $\{z_n\}$. Hence $H(T_\phi T_{\bar{\phi}}) \cap H(I) = \{0\}$ and by, Property 1.1,

$$H = H(I + T_\phi T_{\bar{\phi}}) \cong H(I) \oplus H(T_\phi T_{\bar{\phi}}).$$

Moreover, by Proposition 1.5, M_z on $H(I + T_\phi T_{\bar{\phi}})$ is unitarily equivalent to $S \oplus S$. Hence, $\operatorname{index}(M_z - \lambda I) = -2$ for λ in $\mathbf{D}$.

Note also that there is no factorization property for H in the sense of Richter [4], as there is a function f in H with $f(0) = 0$ but no g in H such that $f(z) = zg(z)$. To see this, note that $H_\lambda = \{f: f(\lambda) = 0\}$ is the kernel of a bounded linear functional and hence has codimension 1 in H. Thus, since the range of $M_z - \lambda I$ is contained in H_λ, and $\operatorname{index}(M_z - \lambda I) = -2$, the range of $M_z - \lambda I$ has codimension 1 in H_λ.

It is also worth remarking that this example shows that M_z on $H(A_1)$ may be unitarily equivalent to M_z on $H(A_2)$, while $H(A_1)$ and $H(A_2)$ can consist of quite different functions. This contrasts sharply with the diagonal case in which it is known that if A_1 and A_2 are diagonal matrices such that M_z on $H(A_1)$ is similar to M_z on $H(A_2)$, then $H(A_1)$ and $H(A_2)$ consist of the same class of functions.

We close this section with a result that asserts that any subtleties involved in characterizing the multipliers of $H(A)$ or the commutants of the multiplication operators on $H(A)$ have nothing to do with the continuity of the spectral decomposition of the matrix A. No assumption as to the boundedness of A is made in the result and the reader is referred to the literature for the spectral decomposition of an unbounded positive operator.

THEOREM 1.7. *If $K_1(z, w) = \bar{\mathbf{z}}^* A_1 \bar{\mathbf{w}}$ and A_1 has the spectral decomposition $A_1 = \int_0^\infty \lambda \, dE(\lambda)$, then there is a diagonalizable matrix A_2 such that $H(A_1)$ and $H(A_2)$ consist of the same class of functions.*

Proof. For each integer n, let $\lambda_n = 2^n$, $\Delta_n = [\lambda_n, \lambda_{n+1}]$, $P_n = E(\Delta_n)$, and

$$V_n = \int_{\Delta_n} \sqrt{\frac{\lambda}{\lambda_n}} \, dE(\lambda).$$

Note that $V_n^* = V_n$, $\|V_n\|^2 < 2$, and $V = E(0) + \sum_{n=-\infty}^{\infty} V_n$ is invertible with $\|V\|^2 \leq 2$ and $\|V^{-1}\|^2 \leq 1/2$. If $A_2 = \sum_{n=-\infty}^{\infty} \lambda_n P_N$, then $A_1 = V^* A_2 V$. Since V is bounded and invertible, Property 1.3 implies that $H(A_1) = H(V^* A_2 V)$ and $H(A_2)$ consist of the same class of functions. ∎

Section 2

The purpose of this section is to show that the study of the multiplication operators on $H(A)$ can be reduced to a setting where A is bounded by introducing a dilation of the kernel.

DEFINITION 2.1. If A is a matrix and $0 < r < 1$, then the *r-dilation* A_r of A is given by $A_r = D_r A D_r$ where D_r is the diagonal matrix with diagonal entries $\{1, r, r^2, \ldots\}$.

DEFINITION 2.2. If K is an analytic kernel on $E \times E$ and $0 < r < 1$, then the *r-dilation* K_r of K is defined by $K_r(z,w) = K(rz, rw)$ for (z,w) in $E_r \times E_r$ where $E_r = \{z\colon rz \in E\}$.

For a function f defined on E, the function f_r is defined on E_r by $f_r(z) = f(rz)$.

Note that $K_r \gg 0$ on E_r if and only if $K \gg 0$ on E. Moreover, if $K(z,w) = \bar{\mathbf{z}}^* A \bar{\mathbf{w}}$, then $K_r(z,w) = \bar{\mathbf{z}}^* A_r \bar{\mathbf{w}} = \overline{\mathbf{rz}}^* A \overline{\mathbf{rw}}$.

THEOREM 2.3. *The operator V from $H(K)$ into $H(K_r)$ defined by $Vf = f_r$ is a unitary operator.*

Proof. Note

$$V\left(\sum_{j=1}^{n} a_j K(z, w_j)\right)$$
$$= \sum_{j=1}^{n} a_j K(rz, w_j) = \sum_{j=1}^{n} a_j K\left(rz, r\frac{w_j}{r}\right) = \sum_{j=1}^{n} a_j K_r\left(z, \frac{w_j}{r}\right),$$

and

$$\left\langle K_r\left(z, \frac{w_j}{r}\right), K_r\left(z, \frac{w_k}{r}\right)\right\rangle$$
$$= K_r\left(\frac{w_k}{r}, \frac{w_j}{r}\right) = K(w_k, w_j) = \left\langle K(z, w_j), K(z, w_k)\right\rangle.$$

Since V is an isomorphism from a dense linear manifold of $H(K)$ onto a dense linear manifold of $H(K_r)$, V is unitary. ■

COROLLARY 2.4. *The function ϕ is a multiplier of $H(K)$ if and only if ϕ_r is a multiplier of $H(K_r)$. Moreover, M_ϕ on $H(K)$ is unitarily equivalent to M_{ϕ_r} on $H(K_r)$.*

Proof. Note $V(\phi f) = (\phi f)_r = \phi_r f_r = \phi_r V(f)$. ■

Corollary 2.4 asserts that it suffices to study the dilated kernel K_r as the multipliers of $H(K_r)$ are the dilations of the multipliers of $H(K)$. In particular, if M_z is bounded on $H(K)$, then M_z is unitarily equivalent to $M_{rz} = rM_z$ on $H(K_r)$. Proposition 2.5 below points out that the dilated matrix A_r is considerably better behaved than the original matrix A.

PROPOSITION 2.5. *If $K(z,w) = \sum_{j,k=0}^{\infty} a_{j,k} z^j \overline{w}^k = \overline{\mathbf{z}}^* A \overline{\mathbf{w}}$ converges for $|z|, |w| < R$ and $0 < r < R$, then A_r is a trace class operator on l_+^2.*

Proof. Since $A_r = [a_{j,k} r^{j+k}]$ and $K(r,r) = \sum_{j,k=0}^{\infty} a_{j,k} r^{j+k}$ is absolutely convergent, the entries of A_r are absolutely summable. ■

Section 3

By 2.4 and 2.5, it is sufficient to study multiplication operators on spaces $H(A)$ where A is bounded, so we shall assume throughout this section A is bounded. Theorem 3.1 below is the main theorem of this section and it provides a natural model for $H(A)$ as a linear manifold inside l_+^2. The theorem assumes that A has been decomposed into the form $A = B^*B$ for some bounded operator B on l_+^2. Since $A > 0$, a natural choice to keep in mind for B is the positive square root of A, but we shall see other factorizations are important.

Recall that the *range space* of a bounded operator B on l_+^2 is the Hilbert space

$$R(B) = \{Bf : f \in l_+^2\} = \{Bf : f \in (\ker B)^\perp\}.$$

If $f, g \in (\ker B)^\perp$, then $\langle Bf, Bg\rangle_{R(B)} = \langle f, g\rangle_{l_+^2}$ defines the inner product. Note that convergence of vectors in $R(B)$ implies componentwise convergence of the vectors of sequences in l_+^2. The operator B is a unitary operator onto $R(B)$ when restricted to $(\ker B)^\perp$. Throughout this section the notation B^{-1} will be used to denote the inverse of this restriction of B.

For the choice of $B = A^{1/2}$, Theorem 3.1 can be deduced from Theorem IV, p. 357 of Aronszajn [2]. The significance of Theorem 3.1 lies in the other factorizations of A.

THEOREM 3.1. *If $A = B^*B$ for some bounded operator B on l^2_+, then the operator V from* $(\ker B^*)^\perp$ *into $H(A)$ defined by $(Vf)(z) = \langle B^*f, \bar{\mathbf{z}}\rangle_{l^2_+}$ is a unitary operator.*

Before giving the proof of Theorem 3.1 it is worth pointing out some of its consequences.

COROLLARY 3.2. *If $A = B^*B$ and $\{f_n\}$ is an orthonormal basis for* $(\ker B^*)^\perp$, *then $\{\langle B^*f_n, \bar{\mathbf{z}}\rangle_{l^2_+}\}$ is an orthonormal basis for $H(A)$.*

COROLLARY 3.3. *If $A = B^*B$, then the range space $R(B^*)$ is unitarily equivalent to $H(A)$ via the map W defined by*

$$W(B^*f) = \langle B^*f, \bar{\mathbf{z}}\rangle_{l^2_+}$$

Proof. Note B^* is a unitary operator from $(\ker B^*)^\perp$ onto $R(B^*)$ and apply Theorem 3.1. ∎

Note that Corollary 3.3 makes clear that the range space is independent of the factorization of A.

COROLLARY 3.4. *If $A = B^*B$ and $\{e_n\}$ is the canonical basis for l^2_+, then the power series $\sum_{n=0}^{\infty} a_n z^n$ is in $H(A)$ if and only if $\sum_{n=0}^{\infty} a_n e_n$ is in $R(B^*)$. In particular, z^n is in $H(A)$ if and only if e_n is in $R(B^*)$.*

Proof. By Corollary 3.3, z^n is in $H(A)$ if and only if $z^n = \langle B^*f, \bar{\mathbf{z}}\rangle_{l^2_+}$ for some B^*f in $R(B^*)$. The result now follows by observing that $\langle B^*f, \bar{\mathbf{z}}\rangle_{l^2_+} = z^n$ if and only if $B^*f = e_n$. ∎

Note Corollary 3.4 indicates z is a multiplier of $H(K)$ if and only if the unilateral shift S leaves $R(B^*)$ invariant. Thus V establishes a unitary equivalence between M_z on $H(K)$ and S on $R(B^*)$. This relationship is explored in more depth in Section 4.

Proof of Theorem 3.1. Let H be the set of all functions on $\mathbf{D}$ of the form $g(z) = \langle B^*f, \bar{\mathbf{z}}\rangle_{l^2}$ where $f \in (\ker B^*)^\perp$. When endowed with the norm $\|g\|_H = \|f\|_2$, H is easily identifiable with either $(\ker B^*)^\perp$ or the range space $R(B^*)$. If $z \in \mathbf{D}$, then evaluation at z is a bounded linear functional as $|g(z)| = |\langle B^*f, \bar{\mathbf{z}}\rangle| \leq \|B^*f\|_2\|\bar{\mathbf{z}}\|_2 \leq \|B^*\|\,\|g\|_H\|\bar{\mathbf{z}}\|_2$. Also since $\{\bar{\mathbf{z}}\colon z \in \mathbf{D}\}$ is dense in l^2_+, if $g(z) = \langle B^*f, \bar{\mathbf{z}}\rangle = 0$ for all z in $\mathbf{D}$, then $f = 0$. Hence $g = 0$ in H and H is a reproducing kernel Hilbert space on $\mathbf{D}$.

Note that if $\{f_n\}$ is an orthonormal basis for $(\ker B^*)^\perp$, then $g_n = \langle B^* f_n, \bar{\mathbf{z}}\rangle$ is an orthonormal basis for H. Hence the kernel function for H is

$$\sum_n \langle B^* f_n, \bar{\mathbf{z}}\rangle \overline{\langle B^* f_n, \bar{\mathbf{w}}\rangle} = \sum_n \langle f_n, B\bar{\mathbf{z}}\rangle \langle B\bar{\mathbf{w}}, f_n\rangle = \langle B\bar{\mathbf{w}}, B\bar{\mathbf{z}}\rangle$$
$$= \langle A\bar{\mathbf{w}}, \bar{\mathbf{z}}\rangle = K(z, w).$$

The uniqueness of the kernel function implies $H = H(A)$ from which the result follows. ■

Example 3.1. If A is the diagonal matrix with diagonal entries $\{a_0, a_1, a_2, \dots\}$ and $B^* = B = A^{1/2}$, then Theorem 3.1 implies that $\{\sqrt{a_n} z^n : a_n \neq 0\}$ is an orthonormal basis for $H(A)$. If $a_n \neq 0$ for each integer n, then $R(B^*)$ contains the canonical basis $\{e_n\}$ for l_+^2 and $H(A)$ contains the polynomials in z.

The next example illustrates how to construct a basis for $H(A)$ by using the Cholesky algorithm to factor the nonnegative matrix A into the product U^*U where U is upper triangular. Recall that the Cholesky algorithm is based on the following well known fact about 2×2 matrices.

LEMMA 3.5. *If*

$$T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_3 \end{pmatrix}$$

is a 2×2 block matrix such that T_1 is invertible on the range of T_2, then $T \geq 0$ if and only if $T_1 \geq 0$ and $T_3 - T_2^* T_1^{-1} T_2 \geq 0$.

To perform the Cholesky algorithm one applies Theorem 3.5 to a positive matrix $A = [a_{j,k}]_{j,k=0}^\infty$ by setting $T_1 = [a_{0,0}]$, $T_2 = [a_{0,k}]_{k=1}^\infty$, and $T_3 = [a_{j,k}]_{j,k=1}^\infty$. Then $A = R_0^* R_0 + P_1$ where

$$R_0 = \frac{1}{\sqrt{a_{0,0}}} \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0}^* & A_1 \end{pmatrix}$$

($\mathbf{0}$ denotes the matrix $[0]_{j=1}^\infty$), and $A_1 = T_3 - a_{0,0}^{-1} T_2^* T_2$.
Applying Lemma 3.5 iteratively to $A_1, A_2, \dots$ one obtains

$$A = \sum_{n=0}^\infty R_n^* R_n = \left(\sum_{n=0}^\infty R_n\right)^* \left(\sum_{m=0}^\infty R_m\right) = U^*U$$

where the second equality follows from the fact that R_n is upper triangular with non zero entries only in the n-th row. The only point where the algorithm can break down is if the upper left most entry of one of the matrices A_n should happen to be zero. It then follows that the corresponding matrix T_2 is also zero, and in this case R_n is set equal to zero and the iteration continues. The factorization $A = U^*U$ obtained is known as the *Cholesky decomposition* of A. For more details on the Cholesky decomposition in Hilbert space the reader is referred to [3]. Note that even if A is only a formal (possibly unbounded) positive matrix, then this algorithm still yields a factorization of A.

Example 3.2. If $A > 0$ has been factored into its Cholesky decomposition, then Corollary 3.2 implies that $R_n(z) = \{\langle R_n^*, \bar{\mathbf{z}}\rangle\colon R_n \neq 0\}$ is an orthonormal basis for $H(A)$. This applies even if A is unbounded. To see this first form the bounded operator A_r and observe that if $A = U^*U$, $U = [u_{i,j}]$, then $A_r = U_r^*U_r$ with $U_r = [u_{i,j}r^j]$.

This basis is noteworthy for two reasons. First, the lowest power of z occurring in the power series of $R_n(z)$ is z^n. Second, the basis is computable via the Cholesky algorithm!

Section 4

This section is devoted to a discussion of the multiplication operators on $H(A)$. Before proceeding it is desirable to give a criterion for the operator M_ϕ to be bounded on $H(A)$. As was pointed out in the introduction, this is equivalent to describing the multipliers of $H(A)$.

THEOREM 4.1. *If $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic on $\mathbf{D}$ and $A = B^*B$ where B is bounded, then ϕ is a multiplier of $H(A)$ if and only if the corresponding formal Toeplitz matrix T_ϕ leaves the range of B^* invariant.*

Proof. By Corollary 3.3 the operator W from $R(B^*)$ to $H(A)$ defined by $W(B^*f) = \langle B^*f, \bar{\mathbf{z}}\rangle_{l^2_+}$ is unitary. Let $\{e_n\}_{n=0}^{\infty}$ denote the standard basis for l^2_+ and write $B^*f = \sum_{n=0}^{\infty} b_n e_n$. If ϕ is a multiplier of $H(K)$, then

$$\begin{aligned}\phi(z)W(B^*f) &= \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{m=0}^{\infty} b_m z^m\right)\\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} a_{j-k}b_k\right)z^j = \sum_{j=0}^{\infty}\left[T_\phi(B^*f)\right]_j z^j\end{aligned}$$

is in $H(A)$. Thus the j-th component of the vector $T_\phi B^*f$ is the j-th component of the power series for $\phi(z)W(B^*f)$. Since $\phi(z)W(B^*f)$ is in

$H(A)$, it follows that the components of $T_\phi B^*f$ define a vector in the range of B^*.

For the converse, assume the range of B^* is invariant under T_ϕ. Since convergence in $R(B^*)$ implies componentwise convergence, and since T_ϕ is lower triangular, the Closed Graph Theorem implies that T_ϕ is a bounded operator on $R(B^*)$. The reverse argument of the first part of the proof shows that $M_\phi = WT_\phi W^*$ is bounded. ■

Corollary 4.2 now provides a model for M_z on $H(A)$.

COROLLARY 4.2. *The following are equivalent for an analytic function ϕ on* **D** *and a bounded matrix $A = B^*B$.*

(1) *The operator T_ϕ is bounded on $R(B^*)$.*

(2) *The operator M_ϕ is bounded on $H(A)$.*

(3) *The Toeplitz matrix T_ϕ leaves the range of B^* invariant.*

Moreover, if M_ϕ on $H(A)$ is bounded, then it is unitarily equivalent to T_ϕ on $R(B^*)$ which is in turn equivalent to $B^{-1*}T_\phi B^*$ on $(\ker B^*)^\perp$.

Proof. Apply Theorems 4.1 and 3.1. ■

COROLLARY 4.3. *The operator M_z is bounded on $H(A)$ if and only if the range of B^* is invariant under the unilateral shift S on l^2_+.*

Theorem 4.4 below provides another necessary condition for M_z on $H(A)$ to be bounded.

THEOREM 4.4. *If $A = [a_{j,k}]_{j,k=0}^\infty$, $a_{0,0} > 0$, and if M_z is bounded on $H(A)$, then $A_n = [a_{j,k}]_{j,k=0}^n$ is nonsingular for each $n = 0, 1, \ldots$.*

Proof. By dilating we may assume that A is bounded. Applying the Cholesky algorithm, A may be written in the form $A = U^*U$ where U is upper triangular and where $u_{j,j} = 0$ implies that the j-th row of U is zero. Since U is upper triangular, $A_n = U_n^*U_n$ where $U_n = [u_{j,k}]_{j,k=0}^n$. Hence, if A_n is singular for some integer n, then $u_{j,j} = 0$ for some smallest integer j. Since $a_{0,0} > 0$, $u_{0,0} > 0$ and $j > 0$. By Corollary 4.3, M_z is bounded if and only if $R(U^*)$ is invariant for S. Hence, $S^j(U^*e_0) = U^*(\sum_{n=0}^\infty \alpha_n e_n)$ for some vector $\sum_{n=0}^\infty \alpha_n e_n$ in l^2_+. But

$$S^j(U^*e_0) = \sum_{n=j}^\infty \overline{u_{0,n-j}}\, e_n.$$

Let n_0 be the smallest integer for which $\alpha_n \neq 0$. If $n_0 < j$, then

$$\left\langle U^*\left(\sum_{n=0}^{\infty} \alpha_n e_n\right), e_{n_0}\right\rangle = \alpha_{n_0}\overline{u_{n_0, n_0}} \neq 0.$$

But this is impossible since $\langle S^j(U^*e_0), e_j\rangle = 0$ for all $n_0 < j$. Hence n_0 must exceed j since $u_{j,j} = 0$. But, $\langle S^j(U^*e_0), e_j\rangle \neq 0$, whereas $\langle U^*(\sum_{n=j+1}^{\infty}\alpha_n e_n), e_j\rangle = 0$. Since this is impossible, S cannot leave $R(U^*)$ invariant and M_z is not bounded. ■

THEOREM 4.5. *Let the functions in $H(A_1)$ be analytic on $B(0, R)$. If ϕ_1 is a cyclic vector for $M_1 = M_z$ on $H(A_1)$, then M_1 is unitarily equivalent to $M_2 = M_z$ on $H(A_2)$ if and only if there is an analytic function ϕ_2 on $B(0, R)$ such that*

$$T_{\phi_1}A_2T_{\overline{\phi_1}} = T_{\phi_2}A_1T_{\overline{\phi_2}}.$$

Proof. Suppose V: $H(A_1) \to H(A_2)$ is a unitary operator such that $VM_1 = M_2V$. If $\phi_2 = V\phi_1$, then ϕ_2 is a cyclic vector for M_2 since ϕ_1 is a cyclic vector for M_1. By Proposition 1.6, the operators

$$W_1\colon H(A_1) \to H(T_{\phi_2}A_1T_{\overline{\phi_2}}) \quad \text{and} \quad W_2\colon H(A_2) \to H(T_{\phi_1}A_2T_{\overline{\phi_1}})$$

defined by

$$W_1(f) = \phi_2 f \quad \text{and} \quad W_2(f) = \phi_1 f$$

are unitary. Thus $\{p\phi_2\phi_1\colon p \text{ is a polynomial}\}$ is dense in both $H(T_{\phi_2}A_1T_{\overline{\phi_2}})$ and in $H(T_{\phi_1}A_2T_{\overline{\phi_1}})$. Since

$$\|p\phi_1\|_{H(A_1)} = \|p(M_2)V(\phi_1)\|_{H(A_2)} = \|p(M_1)V(\phi_2)\|_{H(A_1)} = \|p\phi_2\|_{H(A_2)},$$

it follows that

$$\|p\theta_2\phi_1\|_{H(T_{\phi_2}A_1T_{\overline{\phi_2}})} = \|p\phi_1\theta_2\|_{H(T_{\phi_1}A_2T_{\overline{\phi_1}})}.$$

Hence $H(T_{\phi_1}A_2T_{\overline{\phi_1}}) = H(T_{\phi_2}A_1T_{\overline{\phi_2}})$ as functional Hilbert spaces with the same norm. Since reproducing kernels are unique, $T_{\phi_1}A_2T_{\overline{\phi_1}} = T_{\phi_2}A_1T_{\overline{\phi_2}}$.

Conversely, if $T_{\phi_1}A_2T_{\overline{\phi_1}} = T_{\phi_2}A_1T_{\overline{\phi_2}}$, then $H(T_{\phi_1}A_2T_{\overline{\phi_1}}) = H(T_{\phi_2}A_1T_{\overline{\phi_2}})$, and the operator V: $H(A_1) \to H(A_2)$ defined by $V = W_2^*W_1$ is a unitary operator such that $VM_1 = M_2V$. ■

COROLLARY 4.6. *The operator M_z on $H(A)$ is unitarily equivalent to the unilateral shift S on l_+^2 if and only if $A = T_\phi T_{\bar{\phi}}$ for some function ϕ analytic in a neighborhood of zero.*

Proof. Apply Theorem 4.5 with $A_1 = I$ and $\phi_1 = 1$. ■

In Theorem 4.5 it is assumed that M_z on $H(A_1)$ is cyclic. Theorems 4.7 and 4.8 below replace the cyclic assumption with the condition that $H(A_2)$ contains the constants. It is easy to see that M_z can be cyclic on $H(A)$ with $H(A)$ not containing the constants. The following example illustrates more clearly the difference between Theorems 4.5 and 4.8.

Example 4.1. Let $G = \mathbf{D} \setminus (-1, 1/2]$ and let $L_a^2(G)$ denote the space of analytic functions on G which are square integrable with respect to area measure on G. It is well known that $L_a^2(G)$ is a reproducing kernel Hilbert space with kernel defined on $G \times G$. Additionally M_z is bounded on $L_a^2(G)$, index$(M_z - \lambda I) = -1$ for $\lambda \in G$, and $L_a^2(G)$ contains the constants. Moreover it is shown in [1] that M_z is not a cyclic operator.

THEOREM 4.7. *If A_1, A_2 are bounded, $M_1 = M_z$ is bounded on $H(A_1)$, $M_2 = M_z$ is bounded on $H(A_2)$, and $H(A_2)$ contains the constants, then M_1 is similar to M_2 if and only if there is an analytic function ϕ on $\mathbf{D}$ such that the range of $A_1^{1/2}$ and the range of $T_\phi A_2^{1/2}$ are identical.*

Proof. By Corollary 4.2, M_i is unitarily equivalent to $A_i^{-1/2} S A_i^{1/2}$ for $i = 1, 2$. Since M_2 is assumed bounded, Corollary 3.4 implies that $H(A_2)$ contains the constants if and only if the range of $A_2^{1/2}$ contains the canonical basis $\{e_n\}_{n=0}^\infty$ for l_+^2. If an invertible R: $(\ker A_1^{1/2})^\perp \to (\ker A_2^{1/2})^\perp$ exists such that

$$RA_1^{-1/2} S A_1^{1/2} = A_2^{-1/2} S A_2^{1/2} R,$$

then on the range of $A_2^{1/2}$,

$$SA_1^{1/2} R^{-1} A_2^{-1/2} = A_1^{1/2} R^{-1} A_2^{-1/2} S.$$

Since $\{e_n\}$ is contained in the range of $A_2^{1/2}$, $A_1^{1/2} R^{-1} A_2^{-1/2} = T_\phi$ for some formal Toeplitz matrix. Since R^{-1} maps onto $(\ker A_1^{1/2})^\perp$ and $A_1^{1/2} R^{-1} = T_\phi A_2^{1/2}$, the range of $A_1^{1/2}$ is equal to the range of $T_\phi A_2^{1/2}$.

For the converse, suppose the range of $A_1^{1/2}$ and the range of $T_\phi A_2^{1/2}$ are the same. By Proposition 1.5, M_z on $H(T_\phi A_2^{1/2} A_2^{1/2} T_{\bar{\phi}})$ and $M_2 = M_z$ on $H(A_2)$ are unitarily equivalent. Since range$(A_1^{1/2}) =$ range$(T_\phi A_2^{1/2})$, Proposition 1.2 implies that M_1 is similar to M_2. ■

THEOREM 4.8. *If A_1, A_2 are bounded, $H(A_2)$ contains the constants, and $M_i = M_z$ is bounded on $H(A_i)$, $i = 1, 2$, then M_1 is unitarily equivalent to M_2 if and only if $A_1 = T_\phi A_2 T_{\bar{\phi}}$ for some analytic ϕ on* **D**.

Proof. If M_1 is unitarily equivalent to M_2, then Corollary 4.2 implies there is a unitary operator W such that $WA_1^{-1/2}SA_1^{1/2} = A_2^{-1/2}SA_2^{1/2}W$. Hence

$$SA_1^{1/2}W^*A_2^{-1/2} = A_1^{1/2}W^*A_2^{-1/2}S$$

on the range of $A_2^{1/2}$. Since $H(A_2)$ contains the constants, Corollary 3.4 implies the canonical basis $\{e_n\}$ for l_+^2 is contained in the range of $A_2^{1/2}$. Thus $A_1^{1/2}W^*A_2^{-1/2} = T_\phi$ for some formal Toeplitz matrix T_ϕ. Hence $A_1^{1/2}W^* = T_\phi A_2^{1/2}$ and $T_\phi A_2^{1/2}T_{\bar{\phi}} = T_\phi A_2^{1/2}A_2^{1/2}T_{\bar{\phi}} = A_1^{1/2}W^*WA_1^{1/2} = A_1$. The converse follows from Proposition 1.5. ∎

We close this section with a characterization of when M_z is *polynomially bounded*. Recall an operator T on a Hilbert space H is polynomially bounded provided there is a constant C such that $\|p(T)\| \leq C\|p\|_\infty$ for all polynomials p. Here $\|p\|_\infty$ denotes the *uniform (or supremum) norm* of p on the disk **D**. Also recall the *disk algebra* A(**D**) is the uniform algebra of functions continuous on the closed disk **D**, analytic on the open disk **D**, and endowed with the uniform norm. The following is well known.

PROPOSITION 4.9. *Let $H(A)$ consist of functions analytic on* **D**. *The operator M_z on $H(A)$ is polynomially bounded if and only if the disk algebra A(**D**) is contained in the set of multipliers of $H(A)$.*

Proof. If M_z is polynomially bounded, then there is a constant C such that $\|p(M_z)\| \leq C\|p\|_\infty$ for all polynomials p. Let $\{p_n\}_{n=0}^\infty$ be a sequence of polynomials in A(**D**) converging to ϕ. Since $\{p_n\}$ is Cauchy, $\{p_n(M_z)\}$ is Cauchy. Thus $p_n(M_z)$ converges to T where T is a bounded operator on $H(A)$. Since $Tf = \lim p_n f$ and $p_n f$ converges pointwise to ϕf on **D**, $Tf = \phi f$ for f in $H(A)$. Thus $\phi \in A$(**D**).

For the converse, assume the disk algebra is contained in the multipliers of $H(A)$. Let W: A(**D**) $\to B(H(A))$ be given by $W(\phi) = M_\phi$. To show M_z is polynomially bounded it suffices to show W is continuous. For this we use the Closed Graph Theorem. Suppose g_n converges to g in A(**D**) and $W(g_n) = M_{g_n}$ converges to T in $B(H(A))$. Then for each f in $H(A)$, $Tf = \lim M_{g_n}f = \lim g_n f$. Thus $\{g_n f\}$ is convergent in $H(A)$ and hence $(g_n f)(z)$ converges pointwise for z in **D**. Since $(g_n f)(z)$ converges pointwise to $(gf)(z)$, Tf is an analytic function that agrees with gf on **D**. Hence $Tf = gf$ and $T = M_g$. By the Closed Graph Theorem there is a constant C such that $\|M_\phi\| \leq C\|\phi\|_\infty$ for all ϕ in A(**D**). ∎

COROLLARY 4.10. *If A is bounded, then M_z on $H(A)$ is polynomially bounded if and only if $R(A^{1/2})$ is invariant under T_ϕ for all $\phi \in A(\mathbf{D})$.*

REFERENCES

1. J. AKEROYD, D. KHAVINSON and H.S. SHAPIRO, *Remarks concerning cyclic vectors in Hardy and Bergman spaces*, preprint.
2. N. ARONSZAJN, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., vol. 68 (1950), pp. 337–404.
3. S. POWER, *The Cholesky decomposition in Hilbert space*, Inst. Math. Appl. Conf. Ser., vol. 22 (1986), pp. 186–187.
4. S. RICHTER, *Invariant subspaces in Banach spaces of analytic functions*, Trans. Amer. Math. Soc., vol. 304 (1987) pp. 585–616.
5. A.L. SHIELDS, "Weighted shift operators and analytic function theory," in *Topics in operator theory* (C. Pearcy, Ed.), Math Surveys, vol. 13, Amer. Math. Soc., Providence, R.I., 1974, pp. 49–128.
6. D.A. STEGENGA, *Multipliers of the Dirichlet space*, Illinois J. Math., vol. 24 (1980), pp. 113–139.

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