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A CONVERSE OF THE JORDAN-BROUWER THEOREM

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The well known Jordan-Brouwer separation theorem asserts that the image of any embedding $f\colon S^{n-1} \to S^n$ separates S^n into exactly two connected components. There is also a more general version due to Brouwer concerning embeddings from $(n-1)$-dimensional to n-dimensional closed connected manifolds. A natural question arising in this context is the following: *Can we weaken the hypothesis of the theorem and ask f to be an immersion*? This problem was initially aborded by M. Vaccaro (1957, [5]) who gave, as a counter-example in the PL-category, an immersed S^2 ("house-with-two-rooms"; see [6, p. 2]) whose complement in $\mathbf{R}^3$ is connected. Nevertheless, the answer to the above question can be made positive by adding appropriate conditions on the immersion. In this sense, M.E. Feighn (1988, [1]) proved that given any proper C^2-immersion, $f\colon M^{n-1} \to N^n$, with $H_1(N, \mathbf{Z}_2) = 0$, then $N - f(M)$ is disconnected. More recently J.J. Nuño Ballesteros and M.C. Romero Fuster [4] have shown that the differentiability condition may be released provided that the self-intersection set of f is not dense in any connected component of M^{n-1}. In particular, all the quasi-regular topological immersions satisfy this condition. Now, it is not difficult to observe that for most of these immersions the number of connected components of the complement is bigger than two. So it also seems quite natural to ask: *Is it possible, under some conditions on f, M and N, to assert that f is an embedding if and only if the number of connected components of $N - f(M)$ is exactly* 2? We give here a positive answer for the smooth immersions with normal crossings between manifolds M^{n-1} and N^n such that $H_1(M, \mathbf{Z}_2) = 0$ (if $n > 2$) and $H_1(N, \mathbf{Z}_2) = 0$. This is obtained as an immediate consequence of our main result:

THEOREM 1. *Let $f\colon M^{n-1} \to N^n$ be a smooth immersion with normal crossings and suppose that $H_1(M, \mathbf{Z}_2) = 0$ (if $n > 2$) and $H_1(N, \mathbf{Z}_2) = 0$. If the self-intersection set A of f is not empty, then $H_0(N - f(M)) \geq 3$.*

Here, by a smooth immersion with normal crossings we understand a smooth immersion $f\colon M^{n-1} \to N^n$ such that, $\forall q \in f(M)$, the images under

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the tangent map, TF: $TM \to TN$, at the points $P_j \in f^{-1}(q)$, define a set of linear subspaces in general position in T_qN. (These immersions are also called completely regular by some authors [3] and form a particular case of the quasi-regular immersions.) It can be seen [2] that the self-intersection set (or multiple point locus), A, of such an immersion is stratified by submanifolds S_m, $2 \leq m \leq n$, of codimension $m - 1$, made of the points of M whose images through f in N have multiplicity m exactly. The set, $B = f(A)$, of multiple values of f is also stratified by submanifolds $B_m = f(S_m)$ of codimension $m - 2$ in N. Moreover, $f(M)$, A and B are ANR, the restriction f: $M - A \to f(M) - B$ is a homeomorphism and the restrictions f/S_m: $S_m \to B_m$ are m-fold coverings.

Before proving the above theorem we would like to make some remarks on the conditions imposed on f, M and N:

1. A 2-sphere with two points identified illustrates the fact that the normal crossings conditions is essential.

2. The immersion (with normal crossings) of the Klein bottle in $\mathbf{R}^3$ shows that the condition $H_1(M, \mathbf{Z}_2) = 0$ is essential too (for $n \neq 2$).

3. Finally, we can immerse a 2-sphere in $\mathbf{R}^3$ with normal crossings as in Figure 1 below. Now, we consider as ambient manifold N, the product $S^2 \times S^1$ obtained by taking a small enough neighbourhood of $f(S^2)$ and conveniently identifying the internal (V_0) with the external (V_1) boundary. It is not difficult to see that $N - f(S^2)$ has exactly two connected components which proves that the condition $H_1(N, \mathbf{Z}_2) = 0$ is also essential.

The proof of the theorem is based on the following lemmas:

LEMMA 1 (Herbert [3, p. 27]). *Each nonempty m-fold multiple point locus, A_m, of a smooth immersion f: $M^{n-1} \to N^n$ with normal crossings carries a* mod 2 *fundamental class, uniquely determined by how it restricts at the points of the embedded submanifold $S_m = A_m - A_{m+1}$.*

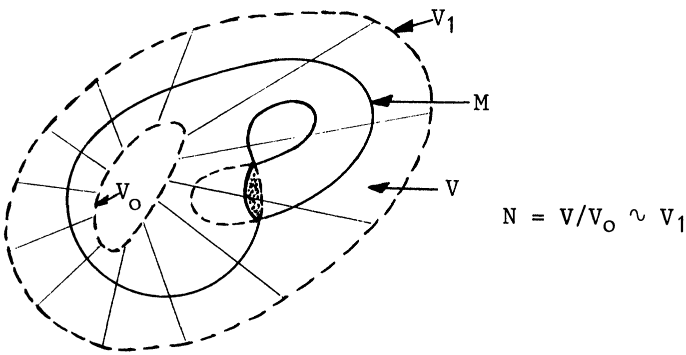

FIG. 1 Example of 2-sphere immersed in $S^2 \times S^1$

In what follows, all the homology groups considered have coefficients in $\mathbf{Z}_2$.

LEMMA 2. *With the conditions on f as above, if $H_1(N) = 0$ then*

$$H_0(N - f(M)) \simeq \mathbf{Z}_2 \oplus H_{n-1}(f(M)).$$

Proof. From the exact sequence of $(N, f(M))$,

$$\begin{array}{ccccccccc} 0 \to H_n(f(M)) & \to & H_n(N) & \to & H_n(N_1 f(M)) & \to & H_{n-1}(f(M)) & \to & H_{n-1}(N) \to \cdots \\ \| & & & & & & & & \wr\| \\ 0 & & & & & & & & H_1(N) = 0 \end{array}$$

we get $H_n(N, f(M)) = H_n(N) \oplus H_{n-1}(f(M))$, and now from the duality theorem, it follows that

$$H_0(N - f(M)) = H_n(N) \oplus H_{n-1}(f(M)),$$

as required. □

Note. The Jordan-Brouwer separation theorem in the smooth category follows as an immediate consequence of this lemma.

LEMMA 3. *Let $f\colon M^{n-1} \to N^n$ be a smooth immersion with normal crossings, and suppose that $H_1(M) = 0$ if $n > 2$ and $H_1(N) = 0$. Then, if we denote by β_{n-1} the $(n-1)$th Betti number, we have that $A \neq \varnothing \Rightarrow \beta_{n-1}(f(M)) \geq 2$.*

Proof. We have the exact sequence diagram

$$\begin{array}{ccccccccccc} \to H_{n-1}(A) & \to & H_{n-1}(M) & \to & H_{n-1}(M, A) & \to & H_{n-2}(A) & \to & H_{n-2}(M) & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow \simeq & & \downarrow & & \downarrow & & \\ \to H_{n-1}(B) & \to & H_{n-1}(f(M)) & \to & H_{n-1}(f(M), B) & \to & H_{n-2}(B) & \to & H_{n-2}(f(M)) & \to & \cdots \end{array}$$

in which the third vertical arrow is an isomorphism. This follows from the excision theorem applied to $f\colon (M, A) \to (f(M), B)$, for, as observed before, $f\colon M - A \to f(M) - B$ is a homeomorphism.

From this we can write the exact sequence

$$\cdots \to H_{n-1}(A) \to H_{n-1}(B) \oplus H_{n-1}(M) \xrightarrow{f_*} H_{n-1}(f(M)) \to$$
$$\to H_{n-2}(A) \xrightarrow{\alpha} H_{n-2}(B) \oplus H_{n-2}(M) \to H_{n-2}(f(M)) \to \cdots.$$

We first treat the case $n = 2$. In this dimension the manifold M must be a union of circles and this exact sequence becomes

$$\begin{aligned} 0 &\to H_1(M) \to H_1(f(M) \\ &\to H_0(A) \xrightarrow{\alpha} H_0(B) \oplus H_0(M) \xrightarrow{\delta} H_0(f(M)) \to \cdots; \end{aligned}$$

here $\operatorname{Im} \alpha = \ker \delta \simeq H_0(B)$, so $\dim \ker \alpha \geq 1$ for $\beta_0(A) = 2\beta_0(B)$.

But $H_1(f(M)) = H_1(M) \oplus \ker \alpha$, and hence $\beta_1(f(M)) \geq 1 \oplus \dim \ker \alpha \geq 2$.

We now prove the result for $n > 2$. In this case we are assuming that $H_{n-2}(M) = H_1(M) = 0$ and we get the exact sequence

$$0 \to H_{n-1}(M) \xrightarrow{f_*} H_{n-1}(f(M)) \to H_{n-2}(A) \xrightarrow{\alpha} H_{n-2}(B) \to \cdots$$

so

$$\beta_{n-1}(f(M)) = \dim \ker \alpha + \beta_{n-1}(M) = \dim \ker \alpha + 1.$$

So it is enough to see, as above, that $\dim \ker \alpha \geq 1$. But this follows in this case from Lemma 1. □

The theorem is now an easy consequence of the combination of Lemmas 2 and 3.

COROLLARY 1 (CONVERSE OF THE JORDAN-BROUWER SEPARATION THEOREM). *Let*

$$f\colon M^{n-1} \to N^n$$

be a smooth immersion with normal crossings between manifolds M^{n-1} such that $H_1(M^{n-1}) = 0$ if $n > 2$ and N^n with $H_1(N) = 0$.

Then f is an embedding if and only if $\beta_0(N - f(M)) = 2$.

Finally, we would like to observe that under appropriate assumptions on f, some relations between the homology of the self-intersection set, A, of f and the number of connected components of $N - f(M)$ may be found. In this sense we have the following:

COROLLARY 2. *Given a smooth immersion with normal crossings, f: $M^{n-1} \to N^n$, $n > 2$, whose self-intersection set reduces to the set of double points S_2, if $H_1(M) = H_1(N) = 0$, we have*

$$\beta_0(N - f(M)) = 2 + \dim \ker(f/A)_*,$$

where $(f/A)_*\colon H_{n-2}(A) \to H_{n-2}(B)$ *is the morphism induced by* f/A *in the homology category.*

Proof. The proof of this follows from the proof of Lemma 3 by taking $A = S_2$. □

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