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ON BERNSTEIN'S INEQUALITY

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1. Introduction

Let E^σ denote the class of entire functions of exponential type $\leq \sigma$. We consider generalizations of the classical inequality:

BERNSTEIN'S INEQUALITY [9]. *For $f \in E^\sigma \cap L^\infty(R)$, we have*

$$\|f'\|_\infty \leq \sigma \|f\|_\infty. \tag{1}$$

Akhiezer in [1], and Boas in [2], gave a generalization of (1) involving the Hilbert transform. Akhiezer proposed the following definition of the Hilbert transform for $f \in E^\sigma \cap L^\infty(R)$:

$$\tilde{H}f(x) = xH\left(\frac{f(t) - f(0)}{t}\right)(x) \tag{2}$$

where H is the classical Hilbert transform. This is justified in [1] by proving that for $f \in L^2(R)$, if also $(f(x) - \alpha)/x \in L^2(R)$, then

$$xH\left(\frac{f(t) - \alpha}{t}\right)(x) = Hf(x) - C(\alpha, f) \tag{3}$$

where $C(\alpha, f)$ is a constant depending only on α and f. In particular, this implies that $(\tilde{H}f)' = (Hf)'(x)$ for $f \in E^\sigma \cap L^2(R)$.

The following inequality is proved in [1]. For the periodic case, see [8].

THEOREM. *For $f \in E^\sigma \cap L^\infty(R)$ and $\alpha \in R$,*

$$\left\|\sin \pi\alpha f' + \cos \pi\alpha \left(\tilde{H}f\right)'\right\|_\infty \leq \sigma \|f\|_\infty. \tag{4}$$

Akhiezer's proof depends on the use of a method of Boas involving the Fourier transform, see [2].

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We extend (4) to entire functions $f \in E^{\sigma} \cap BMO(R)$. To this end we note that (3) is valid also if $f \in (L^1 + BMO)(R)$ and $(f(x) - \alpha)/x \in L^1_{\mathrm{loc}}(R)$, and therefore if $f \in E^{\sigma} \cap BMO(R)$ we have $(\tilde{H}f)' = (Hf)'$. For a proof, see [4]. We also derive the periodic case from the inequality on R. Our proof does not make use of Boas' method.

Bernstein's inequality, (1), was extended to $L^{\phi}(R)$, see [9]. We extend (4) to $L^{\phi}(R)$.

The topic considered in this note is classical, and most theorems have several proofs. We chose to present a unified exposition, repeating some known results. Some of the proofs of those results may, however, be new.

Since if $f(z) \in E^{\sigma}$ then $g(z) = f(\pi z/\sigma) \in E^{\pi}$, it is enough to consider $\sigma = \pi$.

2. Bernstein's inequalities

Let $f \in L^1_{\mathrm{loc}}(R)$. For any interval I, define

$$f_I = \frac{1}{|I|} \int_I f(y)\, dy.$$

Then $f \in BMO(R)$ if and only if

$$\sup_I \frac{1}{|I|} \int_I |f(y) - f_I|\, dy = \|f\|_{BMO} < \infty.$$

Define $k(t) = 1/t$ for $|t| > 1$ and $k(t) = 0$ for $|t| \leq 1$.
If $f \in L^1_{\mathrm{loc}}(R)$ and if

$$\lim_{N \to \infty} \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon < |x-t| < N} \frac{f(x)}{t - x}\, dx$$

exists, then this limit is called the Hilbert transform of f and is denoted Hf. In particular, this limit exists a.e. for $f \in (L^1 + L^p)(R)$, $1 \leq p < \infty$ and $f \in L^1(T)$, see [9].

If the above limit does not exist, but

$$\lim_{N \to \infty} \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon < |t-x| < N} f(x) \left(\frac{1}{t - x} + k(x) \right) dx$$

exists, then this limit is defined to be the Hilbert transform of f up to an additive constant and is denoted Hf. The definition up to an additive constant is necessary to ensure that the Hilbert transform commutes with translations and dilations. This definition is valid for $f \in BMO(R)$.

We will need several lemmas.

LEMMA 1. *For $x \in R$,*

$$H\left(\frac{\sin t}{t}\right)(x) = \frac{1 - \cos x}{x}.$$

Proof. In [4] it is shown that for $f \in L^1(T)$ if $f(x)/x \in L^1_{\text{loc}}(R)$ then

$$H\left(\frac{f(t)}{t}\right)(x) = \frac{Hf(x) - Hf(0)}{x}.$$

Applying the theorem to $f(x) = \sin x$ gives

$$H\left(\frac{\sin t}{t}\right)(x) = \frac{H(\sin t)(x) - H(\sin t)(0)}{x} = \frac{-\cos x + 1}{x}.$$

LEMMA 2. *For $f \in E^\pi \cap L^2(R)$ and $x \in R$, we have*

$$Hf(x) = \sum_{-\infty}^{\infty} f(n) \frac{1 - \cos \pi(x - n)}{\pi(x - n)}. \tag{5}$$

Proof. For $f \in E^\pi \cap L^2(R)$,

$$f(x) = \sum_{-\infty}^{\infty} f(n) \frac{\sin \pi(x - n)}{\pi(x - n)} \tag{6}$$

where $\{f(n)\}_{n=-\infty}^{n=\infty} \in l^2$, see [9].

The $L^2(R)$ convergence of (6) implies the $L^2(R)$ convergence of (5). Furthermore, since $\{f(n)\} \in l^2$, the series (5) converges absolutely and almost uniformly in x and is a continuous function as is $Hf(x)$. Therefore, (5) converges pointwise for all $x \in R$.

LEMMA 3. *For $x \in R$,*

$$\frac{\sin \pi x - \pi x}{\pi x^2} = -\sum_{n \neq 0} \frac{\sin \pi(x - n)}{\pi n(x - n)}.$$

Proof. Since

$$\frac{\sin \pi z - \pi z}{\pi z^2} \in E^\pi \cap L^2(R),$$

from (6) we have

$$\frac{\sin \pi x - \pi x}{\pi x^2} = \sum_{-\infty}^{\infty} \left(\frac{\sin \pi n - \pi n}{\pi n^2} \right) \frac{\sin \pi (x-n)}{\pi (x-n)}$$
$$= - \sum_{n \neq 0} \frac{\sin \pi (x-n)}{\pi n (x-n)}.$$

LEMMA 4. *For* $x \in R$,

$$\frac{1 - \cos \pi x}{\pi x^2} = - \sum_{n \neq 0} \frac{1 - \cos \pi (x-n)}{\pi n (x-n)}.$$

Proof. In [4] it is shown that for $f \in L^1(T)$ if $(f(x) - \alpha_0 - \alpha_1 x)/x^2 \in L^1_{\mathrm{loc}}$ then

$$H\left(\frac{f(t) - \alpha_0 - \alpha_1 t}{t^2} \right)(x) = \frac{Hf(x) - Hf(0) - xH\left(\frac{f(t) - \alpha_0}{t} \right)(0)}{x^2}.$$

Applying this to $f(x) = \sin \pi x$ gives

$$H\left(\frac{\sin \pi t - \pi t}{\pi t^2} \right)(x) = \frac{H(\sin \pi t)(x) - H(\sin \pi t)(0) - xH(\sin \pi t/t)(0)}{\pi x^2}$$
$$= \frac{-\cos \pi x + 1}{\pi x^2}.$$

Therefore by Lemma 2,

$$\frac{1 - \cos \pi x}{\pi x^2} = \sum_{-\infty}^{\infty} \left(\frac{\sin \pi n - \pi n}{\pi n^2} \right) \frac{1 - \cos \pi (x-n)}{\pi (x-n)}$$
$$= - \sum_{n \neq 0} \frac{1 - \cos \pi (x-n)}{n \pi (x-n)}.$$

The interpolation formula below was proved by Akhiezer in the case $f \in E^{\pi} \cap L^{\infty}(R)$ using Boas' technique.

THEOREM 5. *For* $f \in E^{\pi} \cap BMO(R)$ *and* $\alpha \in R$,

$$\sin \pi \alpha f'(x) + \cos \pi \alpha (Hf)'(x) = \sum_{-\infty}^{+\infty} f(n + \alpha + x) \frac{(-1)^n - \cos \pi \alpha}{\pi (\alpha + n)^2}.$$

Proof. Since

$$\frac{f(z)-f(0)}{z} \in E^{\pi} \cap L^2(R),$$

we have

$$\frac{f(x)-f(0)}{x} = \sum_{n\neq 0} \frac{\sin \pi(x-n)}{\pi(x-n)} \cdot \frac{f(n)-f(0)}{n} + \frac{\sin \pi x}{\pi x} f'(0).$$

Therefore,

$$\begin{aligned} f(x) &= x \sum_{n\neq 0} \frac{\sin \pi(x-n)}{\pi(x-n)} \cdot \frac{f(n)-f(0)}{n} + \frac{\sin \pi x}{\pi} f'(0) + f(0) \\ &= x \sum_{n\neq 0} \frac{\sin \pi(x-n)}{\pi(x-n)} \cdot \frac{f(n)}{n} - xf(0) \sum_{n\neq 0} \frac{\sin \pi(x-n)}{\pi n(x-n)} \\ &\quad + \frac{\sin \pi x}{\pi} f'(0) + f(0) \\ &= x \sum_{n\neq 0} \frac{\sin \pi(x-n)}{\pi n(x-n)} f(n) + \frac{\sin \pi x}{\pi x} f(0) + \frac{\sin \pi x}{\pi} f'(0) \quad \text{(see [9])}. \end{aligned}$$

We also have

$$\begin{aligned} &H\left(\frac{f(t)-f(0)}{t}\right)(x) \\ &\quad = \sum_{n\neq 0} \frac{1-\cos \pi(x-n)}{\pi(x-n)} \frac{f(n)-f(0)}{n} + \frac{1-\cos \pi x}{\pi x} f'(0). \end{aligned}$$

Therefore

$$\begin{aligned} \tilde{H}f(x) &= xH\left(\frac{f(t)-f(0)}{t}\right)(x) \\ &= x \sum_{n\neq 0} \frac{1-\cos \pi(x-n)}{\pi(x-n)} \cdot \frac{f(n)-f(0)}{n} + \frac{1-\cos \pi x}{\pi} f'(0) \\ &= x \sum_{n\neq 0} \frac{1-\cos \pi(x-n)}{\pi(x-n)} \cdot \frac{f(n)}{n} - xf(0) \sum_{n\neq 0} \frac{1-\cos \pi(x-n)}{\pi n(x-n)} \\ &\quad + \frac{1-\cos \pi x}{\pi} f'(0) \\ &= x \sum_{n\neq 0} \frac{1-\cos \pi(x-n)}{\pi n(x-n)} f(n) + \frac{1-\cos \pi x}{\pi x} f(0) + \frac{1-\cos \pi x}{\pi} f'(0). \end{aligned}$$

Therefore

$$
\begin{aligned}
&\sin \pi\alpha f(x) + \cos \pi\alpha \tilde{H}f(x) \\
&\quad = x \sum_{n\neq 0} f(n) \frac{\sin\pi\alpha \sin\pi(x-n) - \cos\pi\alpha\cos\pi(x-n) + \cos\pi\alpha}{\pi n(x-n)} \\
&\qquad + f(0)\frac{\sin\pi\alpha\sin\pi x - \cos\pi\alpha\cos\pi x + \cos\pi\alpha}{\pi x} \\
&\qquad + f'(0)\frac{\sin\pi\alpha\sin\pi x - \cos\pi\alpha\cos\pi x + \cos\pi\alpha}{\pi} \\
&\quad = x\sum_{n\neq 0} f(n)\frac{\cos\pi\alpha - \cos\pi(x+\alpha-n)}{\pi n(x-n)} \\
&\qquad + f(0)\frac{\cos\pi\alpha - \cos\pi(x+\alpha)}{\pi x} \\
&\qquad + f'(0)\frac{\cos\pi\alpha - \cos\pi(x+\alpha)}{\pi}.
\end{aligned}
$$

Taking derivatives and letting $x = -\alpha$ we obtain

$$
\begin{aligned}
&\sin\pi\alpha f'(-\alpha) + \cos\pi\alpha (Hf)'(-\alpha) \\
&\quad = \sum_{n\neq 0} f(n)\frac{(-1)^n - \cos\pi\alpha}{\pi n(\alpha+n)} - \alpha\sum_{n\neq 0} f(n)\frac{(-1)^n - \cos\pi\alpha}{\pi n(\alpha+n)^2} \\
&\qquad + f(0)\frac{1-\cos\pi\alpha}{\pi\alpha^2} \\
&\quad = \sum_{-\infty}^{+\infty} f(n)\frac{(-1)^n - \cos\pi\alpha}{\pi(\alpha+n)^2}.
\end{aligned}
$$

Given $f \in E^\pi \cap BMO(R)$ and $x \in R$, let $g(z) = f(x+\alpha+z)$. Then

$$g(z) \in E^\pi \cap BMO(R),\ g(n) = f(x+\alpha+n)$$

and

$$\sin\pi\alpha g'(-\alpha) + \cos\pi\alpha(Hg)'(-\alpha) = \sin\pi\alpha f'(x) + \cos\pi\alpha(Hf)'(x).$$

Therefore

$$\sin\pi\alpha f'(x) + \cos\pi\alpha(Hf)'(x) = \sum_{-\infty}^{+\infty} f(n+\alpha+x)\frac{(-1)^n - \cos\pi\alpha}{\pi(\alpha+n)^2},$$

and the theorem is proved.

Note that for $\alpha = \frac{1}{2}$ we obtain

$$f'(x) = \frac{4}{\pi}\sum_{-\infty}^{+\infty}\frac{(-1)^n f(n+\frac{1}{2}+x)}{(2n+1)^2}.$$

For $\alpha = 0$ we obtain

$$(Hf)'(x) = \frac{\pi}{2}f(x) - \frac{2}{\pi}\sum_{-\infty}^{+\infty}\frac{f(2n+1+x)}{(2n+1)^2}.$$

THEOREM 6. *For $f \in E^{\pi} \cap L^{\infty}(R)$ and $\alpha \in R$,*

$$\|\sin\pi\alpha f' + \cos\pi\alpha(Hf)'\|_{\infty} \le \pi\|f\|_{\infty}.$$

Proof. From the proof of Theorem 5, we have

$$\sin\pi\alpha f'(-\alpha) + \cos\pi\alpha(Hf)'(-\alpha) = \sum_{-\infty}^{+\infty} f(n)\frac{(-1)^n - \cos\pi\alpha}{\pi(\alpha+n)^2}.$$

Let $f(z) = \cos\pi z$. Since $f'(-\alpha) = \pi\sin\pi\alpha$ and $(Hf)'(-\alpha) = \pi\cos\pi\alpha$, we have

$$\sum_{-\infty}^{+\infty}\frac{1-(-1)^n\cos\pi\alpha}{\pi(\alpha+n)^2} = \pi.$$

By Theorem 5 again, we have

$$\begin{aligned}|\sin\pi\alpha f'(x) + \cos\pi\alpha(Hf)'(x)| &\le \sum_{-\infty}^{+\infty}|f(n+\alpha+x)|\frac{1-(-1)^n\cos\pi\alpha}{\pi(\alpha+n)^2}\\ &\le \pi\|f\|_{\infty},\end{aligned}$$

and the theorem is proved.

THEOREM 7. *For $f \in E^{\pi} \cap BMO(R)$ and $\alpha \in R$,*

$$\|\sin\pi\alpha f' + \cos\pi\alpha(Hf)'\|_{BMO} \le \pi\|f\|_{BMO}.$$

Proof. Fix α and define

$$F(x) = \sin\pi\alpha f'(x) + \cos\pi\alpha(Hf)'(x).$$

Then, with

$$c_{n,\alpha} = \frac{(-1)^n - \cos \pi\alpha}{\pi(\alpha + n)^2}$$

and $f_n(x) = f(n + \alpha + x)$, we have

$$\begin{aligned} F_I &= \frac{1}{|I|} \int_I \sum_{-\infty}^{\infty} c_{n,\alpha} f_n(x)\, dx \\ &= \sum_{-\infty}^{\infty} c_{n,\alpha} \frac{1}{|I|} \int_I f_n(x)\, dx \\ &= \sum_{-\infty}^{\infty} c_{n,\alpha} (f_n)_I, \end{aligned}$$

provided that the interchange of summation and integration is justified. Since *BMO* is translation and dilation invariant, we may assume $I = [0, 1]$. We have

$$\begin{aligned} \sum_{-\infty}^{\infty} |c_{n,\alpha}| \int_I |f_n(x)|\, dx &= \sum_{-\infty}^{\infty} |c_{n,\alpha}| \int_{n+\alpha}^{n+\alpha+1} |f(x)|\, dx \\ &\le C(\alpha) \sum_{-\infty}^{\infty} \int_{n+\alpha}^{n+\alpha+1} \frac{|f(x)|}{1 + x^2}\, dx \\ &= C(\alpha) \int_R \frac{|f(x)|}{1 + x^2}\, dx < \infty, \end{aligned}$$

(See [3].) Thus, for any interval I,

$$\begin{aligned} \frac{1}{|I|} \int_I |F(x) - F_I|\, dx &\le \sum_{-\infty}^{\infty} |c_{n,\alpha}| \frac{1}{|I|} \int_I |f_n(x) - (f_n)_I|\, dx \\ &\le \sum_{-\infty}^{\infty} |c_{n,\alpha}| \cdot \|f\|_{BMO} = \pi \|f\|_{BMO}. \end{aligned}$$

The proof is complete.

Zygmund, [9], proved that if ϕ is non-negative, non-decreasing and convex, and

$$f \in E^{\pi} \cap L^{\phi}(R) \cap L^{\infty}(R),$$

then

$$\int_R \phi\left(\left|\frac{f'(x)}{\pi}\right|\right) dx \le \int_R \phi(|f(x)|)\, dx.$$

This is the case $\alpha = \frac{1}{2}$ in

$$\int_R \phi\left(\frac{|\sin \pi\alpha f'(x) + \cos \pi\alpha (Hf)'(x)|}{\pi}\right) dx \leq \int_R \phi(|f(x)|)\, dx, \quad (7)$$

which we prove below. The requirement that $f \in L^\infty(R)$ was made to justify the application of the interpolation formula in Theorem 5. Using Lemma 8 below we show that $E^\sigma \cap L^\phi(R) \subset E^\sigma \cap L^\infty(R)$.

The periodic case of (7) was proved in [9].

LEMMA 8. *Let $\phi(x) \geq 0$ be defined on R^+ and assume*

$$\liminf_{x\to\infty} \frac{\phi(x)}{x} = \rho > 0.$$

Then

$$E^\sigma \cap L^\phi(R) \subset E^\sigma \cap L^\infty(R).$$

Proof. Let $f \in E^\sigma \cap L^\phi(R)$. For $|f(x)|$ large, we have $\phi(|f(x)|) > \rho|f(x)|/2$. Hence, there exist $g \in L^1(R)$ and $h \in L^\infty(R)$ such that $f = g + h$. For $\delta > 0$, define

$$f_\delta(x) = f(x)\frac{\sin \delta x}{\delta x}.$$

Then

$$f_\delta(x) = g_\delta(x) + h_\delta(x) \in E^{\sigma+\delta} \cap (L^1 + L^2)(R).$$

Let

$$\mathscr{G}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Since $g_\delta * \mathscr{G} \in L^2(R)$ and $h_\delta * \mathscr{G} \in L^2(R)$, we have

$$f_\delta * \mathscr{G} = g_\delta * \mathscr{G} + h_\delta * \mathscr{G} \in L^2(R).$$

Furthermore, $f_\delta * \mathscr{G} \in E^{\sigma+\delta}$:

$$\left|\int_R f_\delta(z-t)\mathscr{G}(t)\, dt\right| \leq C_\eta e^{(\sigma+\delta+\eta)|z|}\int_R e^{(\sigma+\delta+\eta)|t|}\mathscr{G}(t)\, dt = C'_\eta e^{(\sigma+\delta+\eta)|z|}.$$

Thus, by the Paley-Wiener Theorem, $\hat{f}_\delta(\xi)\hat{\mathscr{G}}(\xi) = 0$ for all $|\xi| > \sigma + \delta$. Since $\hat{\mathscr{G}} \neq 0$, we have $\hat{f}_\delta(\xi) = 0$ for all $|\xi| > \sigma + \delta$. Since

$$\hat{f}_\delta = \hat{g}_\delta + \hat{h}_\delta \in (L^2 + L^\infty)(R)$$

with compact support, we have $\hat{f}_\delta \in L^2(R)$ and so $f_\delta \in E^{\sigma+\delta} \cap L^2(R)$.

Applying a lemma of Stein, [7], for $\varepsilon > \delta$ we get

$$f_\delta(x) = \int_{-\infty}^{+\infty} f_\delta(x-y)\psi_{\sigma+\varepsilon}(y)\, dy$$

where

$$\psi_\sigma(y) = \frac{1}{\pi}\frac{\cos\sigma y - \cos 2\sigma y}{\sigma y^2}.$$

Since $|f_\delta(x)| \le |f(x)|$ and $f_\delta(x) \to f(x)$ as $\delta \to 0$, we have

$$f(x) = \int_{-\infty}^{+\infty} f(x-y)\psi_{\sigma+\varepsilon}(y)\, dy.$$

Since $f \in (L^1 + L^\infty)(R)$ and $\psi_{\sigma+\varepsilon} \in L^1 \cap L^\infty(R)$, we have $f \in L^\infty(R)$. This completes the proof.

Using deep results of Duffin and Schaeffer, a related result is proved in [5]. If $\phi(t) = \psi(\log t)$, where $\psi(u) \ge 0$ and ψ is non-decreasing and convex, then $E^\sigma \cap L^\phi(R) \subset E^\sigma \cap L^\infty(R)$.

THEOREM 9. *If $f \in E^\pi \cap L^\phi(R)$, $\phi(t)$ is non-negative, non-decreasing and convex, and $\alpha \in R$, then we have*

$$\int_R \phi\left(\frac{|\sin\pi\alpha f'(x) + \cos\pi\alpha (Hf)'(x)|}{\pi}\right) dx \le \int_R \phi(|f(x)|)\, dx.$$

Proof. Let $f \in E^\pi \cap L^\phi(R)$. By Lemma 8 and Theorem 5, we have

$$\sin\pi\alpha f'(x) + \cos\pi\alpha (Hf)'(x) = \sum_{-\infty}^{+\infty} f(n+\alpha+x)\frac{(-1)^n - \cos\pi\alpha}{\pi(\alpha+n)^2}.$$

Since ϕ is non-decreasing and convex,

$$\begin{aligned}
&\phi\left(\left|\frac{\sin\pi\alpha f'(x) + \cos\pi\alpha (Hf)'(x)}{\pi}\right|\right) \\
&\quad \le \phi\left[\frac{1}{\pi}\sum_{-\infty}^{+\infty} |f(n+\alpha+x)|\frac{1-(-1)^n\cos\pi\alpha}{\pi(\alpha+n)^2}\right] \\
&\quad \le \frac{1}{\pi}\sum_{-\infty}^{+\infty}\frac{1-(-1)^n\cos\pi\alpha}{\pi(\alpha+n)^2}\phi(|f(n+\alpha+x)|).
\end{aligned}$$

Integrating we obtain:

$$\int_R \phi\left(\left|\frac{\sin \pi\alpha f'(x) + \cos \pi\alpha (Hf)'(x)}{\pi}\right|\right) dx$$

$$\leq \frac{1}{\pi}\int_R \left[\sum_{-\infty}^{+\infty} \frac{1 - (-1)^n \cos \pi\alpha}{\pi(\alpha + n)^2} \phi(|f(n + \alpha + x)|)\right] dx$$

$$= \frac{1}{\pi}\sum_{-\infty}^{+\infty} \frac{1 - (-1)^n \cos \pi\alpha}{\pi(\alpha + n)^2} \int_R \phi(|f(n + \alpha + x)|)\, dx$$

$$= \int_R \phi(|f(x)|)\, dx.$$

Bernstein's inequalities for $E^\sigma \cap L^\infty(R)$ and for $E^\sigma \cap BMO(R)$ apply naturally to periodic functions. The periodic versions of the theorems above follow easily from an interpolation formula for trigonometric polynomials:

THEOREM 10. *Let T_n be a trigonometric polynomial of order n and let $\tilde{T}_n$ be its conjugate. Then for $\alpha, \theta \in R$,*

$$\sin \pi\alpha T_n'(\theta) + \cos \pi\alpha \tilde{T}_n'(\theta) = \sum_{j=0}^{2n-1} \left\{(-1)^j - \cos \pi\alpha\right\}\lambda_{j,\alpha} T_n(\theta + \theta_{j,\alpha})$$

where

$$\lambda_{j,\alpha} = \frac{1}{n}\cdot\frac{1}{4\sin^2(\theta_{j,\alpha}/2)} \quad \textit{and} \quad \theta_{j,\alpha} = \frac{j+\alpha}{n}\pi.$$

This formula for the case $\alpha = 1/2$ was proved by M. Riesz and the full formula is proved in [9] as a special case of trigonometric interpolation. It is perhaps worthwhile to observe that the interpolation formula can also be deduced from the interpolation formula for $f \in E^\pi \cap L^\infty(R)$.

In the proof we use the well-known identity

$$\frac{\pi^2}{\sin^2 \pi x} = \sum_{-\infty}^{\infty} \frac{1}{(x-n)^2}.$$

We can easily derive this identity from the interpolation formula for $f \in E^\pi \cap L^\infty(R)$:

$$f(x) = x\sum_{n\neq 0} \frac{\sin \pi(x-n)}{\pi n(x-n)} f(n) + \frac{\sin \pi x}{\pi x} f(0) + \frac{\sin \pi x}{\pi} f'(0).$$

Take $f(x) = \cos \pi x$ to get

$$\pi \cot \pi x = \sum_{n \neq 0} \left(\frac{1}{n} + \frac{1}{x-n} \right) + \frac{1}{x}.$$

Differentiating we obtain the result.

Proof of Theorem 10. Let $f(x) = T_n(\pi x/n)$. Since $T_n \in E^n \cap L^\infty(R)$ we have $f \in E^\pi \cap L^\infty(R)$. Thus from Theorem 5, with $\theta = \pi x/n$,

$$\begin{aligned}
&\sin \pi\alpha T_n'(\theta) + \cos \pi\alpha \tilde{T}_n'(\theta) \\
&\quad = \frac{n}{\pi}\{\sin \pi\alpha f'(x) + \cos \pi\alpha (Hf)'(x)\} \\
&\quad = \frac{n}{\pi} \sum_{-\infty}^{\infty} f(k+\alpha+x) \frac{(-1)^k - \cos \pi\alpha}{\pi(\alpha+k)^2} \\
&\quad = \frac{n}{\pi^2} \sum_{-\infty}^{\infty} T_n\left(\frac{\pi k}{n} + \frac{\pi\alpha}{n} + \theta \right) \frac{(-1)^k - \cos \pi\alpha}{(\alpha+k)^2} \\
&\quad = \frac{n}{\pi^2} \sum_{j=0}^{2n-1} \sum_{q=-\infty}^{\infty} T_n\left(\frac{\pi(2nq+j)}{n} + \frac{\pi\alpha}{n} + \theta \right) \frac{(-1)^{2nq+j} - \cos \pi\alpha}{(\alpha+2nq+j)^2} \\
&\quad = \frac{n}{\pi^2} \sum_{j=0}^{2n-1} \{(-1)^j - \cos \pi\alpha\} T_n(\theta + \theta_{j,\alpha}) \sum_{q=-\infty}^{\infty} \frac{1}{(\alpha+2nq+j)^2} \\
&\quad = \frac{n}{\pi^2} \sum_{j=0}^{2n-1} \{(-1)^j - \cos \pi\alpha\} T_n(\theta + \theta_{j,\alpha}) \frac{1}{(2n)^2} \sum_{q=-\infty}^{\infty} \frac{1}{(q+(\alpha+j)/2n)^2} \\
&\quad = \frac{n}{\pi^2} \sum_{j=0}^{2n-1} \{(-1)^j - \cos \pi\alpha\} T_n(\theta + \theta_{j,\alpha}) \frac{1}{(2n)^2} \frac{\pi^2}{\sin^2(\theta_{j,\alpha}/2)} \\
&\quad = \sum_{j=0}^{2n-1} \{(-1)^j - \cos \pi\alpha\} \lambda_{j,\alpha} T_n(\theta + \theta_{j,\alpha}).
\end{aligned}$$

This proves the theorem.

The interpolation formulas above were proved for $x \in R$. They extend however to $z \in C$. We give an example of this extension.

THEOREM 11. *Assume $f(z)$ is analytic in the upper half plane (UHP) and that $f(\cdot + iy) \in L^p(R)$ for all $y > 0$. Then the Hilbert transform $H(f(\cdot + iy))(x) = Hf(z)$ is analytic in the UHP.*

Proof. Let C be any closed curve in the UHP. We have

$$\begin{aligned}\int_C Hf(z)\,dz &= \int_C p.v.\int_R \frac{f(x-t+iy)}{t}\,dt\,dz\\ &= \int_C\int_{|t|<1} \frac{f(x-t+iy)-f(x+iy)}{t}\,dt\,dz\\ &\quad+\int_C\int_{|t|\geq 1} \frac{f(x-t+iy)}{t}\,dt\,dz\\ &= \int_{|t|<1}\int_C \frac{f(x-t+iy)-f(x+iy)}{t}\,dz\,dt\\ &\quad+\int_{|t|\geq 1}\int_C \frac{f(x-t+iy)}{t}\,dz\,dt=0,\end{aligned}$$

since f is analytic. This proves the theorem.

COROLLARY 12. *If $f(z) \in E^{\pi} \cap L^2(R)$ then for all $z \in C$,*

$$Hf(z) = \sum_{-\infty}^{\infty} f(n)\frac{1-\cos \pi(z-n)}{\pi(z-n)}.$$

Proof. For $z = x \in R$, this is Lemma 2. Since $Hf(z)$ and

$$\sum_{-\infty}^{\infty} f(n)\frac{1-\cos \pi(z-n)}{\pi(z-n)}$$

are entire functions which coincide on the real axis, we get the result for all $z \in C$.

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