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LATTICE STRUCTURES OF ORDERED BANACH ALGEBRAS

BY

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Dedicated to Professor George Maltese on his 60th birthday

Introduction

The central purpose of the present paper is to investigate conditions which assure that a Banach algebra A possessing a bounded approximate identity and ordered by a closed multiplicative cone A_+ is isomorphic to a sublattice and subalgebra of the set of all continuous real-valued functions on a compact Hausdorff space. For example, for the existence of such a representation it suffices that A be an almost f-algebra, or that every element can be written as a difference of two positive elements with product zero. We emphasize that for the second criterion no lattice property has to be assumed. In this context it seems to be interesting to consider the following analogous property:

(DDP) For all $z \in A$ there exists $x, y \in A_+$ with $z = x - y$ and $[0, x] \cap [0, y] = \{0\}$.

Positive elements x, y with the property $[0, x] \cap [0, y] = \{0\}$ are called *disjoint* and a partially ordered vector space possesses the *Disjoint Decomposition Property* iff condition DDP is fulfilled. We show in the first section that a partially ordered vector space is a vector lattice if and only if it possesses the DDP and the well known *Riesz Decomposition Property*, briefly RDP, which is defined by the validity of the equation $[0, x] + [0, y] = [0, x + y]$ for all positive $x, y \in A$. The main results about the lattice properties of ordered Banach algebras are presented in Section 2. Our main tool is a representation theorem for Banach algebras possessing a bounded approximate identity which are ordered by a multiplicative cone containing all squares. The proof of the Representation Theorem is given in Section 3. As an interesting consequence we obtain that an almost f-Banach lattice algebra A (in the

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sense of [14]) is isometrically isomorphic to the set $C_0(X, \mathbf{R})$ of all continuous real-valued functions vanishing at infinity on a locally compact space if and only if A possesses a bounded approximate identity with norm bound 1. In the fourth section we show that the positive cones of certain almost f-Banach lattice algebras are uniquely determined by their Banach algebra multiplications.

1. A characterization of vector lattices

Let E be a vector space over the real numbers. A subset E_+ of E is called a *wedge* if $x + y \in E_+$ and $\lambda x \in E_+$ for all $x, y \in E_+$ and for all real non-negative λ. The set E_+ endows E with a partial ordering $\leq$ if we define $x \leq y$ to mean that $y - x \in E_+$. A *partially ordered vector space* is a vector space with a fixed wedge E_+. The wedge E_+ is *antisymmetric* if $0 \leq x \leq 0$ implies $x = 0$, or equivalently, if $E_+ \cap - E_+ = \{0\}$, i.e., E_+ is a *cone*. E_+ is *generating* if the linear span of E_+ is the whole space E. The *order interval* $[x, y]$ is the set of all $z \in E$ with $x \leq z \leq y$. An element $u \in E_+$ is an *order unit* if E is the linear span of the order interval $[0, u]$.

1.1 PROPOSITION. *Let E be a partially ordered real vector space with DDP. Then E_+ is generating and antisymmetric.*

The *proof* is obvious and therefore omitted.

1.2 THEOREM. *Let E be a partially ordered real vector space. Then E is a vector lattice if and only if E possesses the RDP and the DDP. In that case, the disjoint positive elements x, y occurring in the representation $z = x - y$ of an arbitrary $z \in E$ are uniquely determined.*

Proof. It suffices to show that for all $z \in E$ the supremum of z and 0 exists. The DDP yields a disjoint representation of z, i.e., $z = x - y$ with $x, y \in E_+$ and $[0, x] \cap [0, y] = \{0\}$. We show that x is the supremum of z and 0. Moreover, this yields the uniqueness of the disjoint representation. At first we observe that x is an upper bound since we have $z \leq x$ and $0 \leq x$. Let w be another upper bound. Since $x - y = z \leq w$ we have $0 \leq x \leq w + y$. By the RDP there exists $z_1 \in [0, w]$ and $z_2 \in [0, y]$ with $x = z_1 + z_2$. In particular, we have $z_2 \in [0, x]$. Since x and y are disjoint we conclude that z_2 is equal to zero. Then $x = z_1 \in [0, w]$, i.e., we have $x \leq w$.

For a vector lattice E two elements $x, y \in E$ are called *disjoint* if $\inf\{|x|, |y|\} = 0$. Our definition of disjointness only applies to positive elements $x, y \in E_+$, but it easy to see that the two definitions coincide for positive elements in the case of a vector lattice.

Example 1. Let X be a compact Hausdorff space and $C(X, \mathbf{R})$ the set of all continuous real-valued functions on X. A function $f \in C(X, \mathbf{R})$ is called strictly positive if $f(t) > 0$ for all $t \in X$. It is not very difficult to see that every subalgebra of $C(X, \mathbf{R})$ containing the constants possesses the RDP but not the DDP relative to the strict ordering. Moreover this example shows that the Riesz Decomposition is not unique in general.

An order theoretic characterization of the strict ordering can be found in [19]. Further examples of partially ordered vector spaces with the RDP can be found in [7] and [16, p. 122].

Example 2. Let S be the set of all selfadjoint bounded operators on a complex Hilbert space, or (more generally) the selfadjoint part of a C*-algebra. Then S possesses the DDP, since for every $a \in S$ there exist positive $u, v \in S$ with $a = u - v$ and $uv = vu = 0$. Lemma 3 in [18] shows that this representation is disjoint. Thus S possesses the RDP if and only if S is a lattice which in turn is equivalent to the commutativity of S, see [18]. Moreover, this example shows that the disjoint representation is in general not unique: let a, b be positive, non-commuting operators which induce extreme rays of the cone of all positive operators. Then $x = a - b$ is a disjoint representation which is different from the above-mentioned since ab or ba is not zero.

2. Banach algebras ordered by a multiplicative cone

Let A be an associative algebra over the real or complex numbers, and let A_+ be a wedge in A. We call the wedge A_+ *multiplicative* if $a, b \geq 0$ implies $ab \geq 0$. Sometimes, a multiplicative wedge is also called a *semi-algebra*; see [4, p. 256].

A *unital algebra* is an algebra with a unit element 1; if A is unital let $A_e := A$. If A does not possess a unit element then A_e denotes the algebra $A \oplus K$, where K is the real or complex field. An element a in A or A_e is *invertible* if there exists $b \in A_e$ with $ab = ba = 1$. Note that the product $a^{-1}b$ with $a \in A_e$, $b \in A$ is contained in A if a is invertible. Thus the following property is well defined, even for non-unital algebras:

(I) For every $a \geq 0$ there exists $\lambda \geq 1$ such that $\lambda + a$ is invertible and for all $b \geq 0$ the element $(\lambda + a)^{-2} b (\lambda + a)^{-2}$ is positive.

2.1 Proposition. *Let A be an algebra endowed with a multiplicative cone A_+ satisfying condition (I). Then any product of two positive disjoint elements is zero.*

Proof. Let a_1, a_2 be positive and disjoint. Since $a_1 + a_2$ is positive there exists $\lambda \geq 1$ such that $(\lambda + a_1 + a_2)^{-1}$ exists in A_e. It is easy to see that the following inequality holds for $j = 1, 2$

$$(1) \qquad 0 \leq a_1 a_2 \leq (\lambda + a_1 + a_2)^2 a_j (\lambda + a_1 + a_2)^2.$$

Now the condition (I) allows us to multiply (1) by $(\lambda + a_1 + a_2)^{-2}$ on both sides proving the nice formula $0 \leq (\lambda + a_1 + a_2)^{-2} a_1 a_2 (\lambda + a_1 + a_2)^{-2} \leq a_j$. Since a_1, a_2 are disjoint we obtain $(\lambda + a_1 + a_2)^{-2} a_1 a_2 (\lambda + a_1 + a_2)^{-2} = 0$. The proof is complete.

Example 2 shows that the Proposition fails if the cone is not multiplicative. The next example shows that the condition (I) is essential. Theorem 2.2 shows that condition (I) is necessary for a large class of algebras.

Example 3. Let A be the algebra of all polynomials with real coefficients under the pointwise ordering. It is easy to see that the constant function 1 and the polynomial x^2 are disjoint positive elements. But their product is not zero.

2.2 THEOREM. *Let A be a unital real Banach algebra with the DDP and with a multiplicative cone A_+. Then the following assertions are equivalent*:

(a) *A_+ possesses property (I).*
(b) *$ab = 0$ if a, b are positive and disjoint.*
(c) *For all a in A there exist $a_1, a_2 \geq 0$ with $a = a_1 - a_2$ and $a_1 a_2 = a_2 a_1 = 0$.*
(d) *a^2 is positive for all a in A.*
(e) *The unit element is an interior point of A_+, i.e., 1 is an order unit.*

Proof. Proposition 2.1 shows (a) $\Rightarrow$ (b). For the implication (b) $\Rightarrow$ (c) note that any disjoint decomposition of $a \in A$ satisfies (c). The next implication is clear in view of $a^2 = a_1^2 + a_2^2$. Now we prove (d) $\Rightarrow$ (a). Let $a \in A_+$. Since A is a Banach algebra there exists $\lambda \geq 1$ such that $(\lambda + a)^{-1}$ exists in A_e. As A is unital, assumption (d) yields the positivity of $(\lambda + a)^{-2}$. Since A_+ is multiplicative, property (I) holds. Let us prove (d) $\Rightarrow$ (e). Let $a \in A$ with $\|a\| < 1$. Consider the square root $b := \sqrt{1 - a}$ which is defined by the Taylor expansion of the square root function; cf. [4]. Then $1 - a = b^2$ is positive by assumption (d), i.e., $a \leq 1$. Replacing a by $-a$ we obtain $-1 \leq a \leq 1$. For the last implication (e) $\Rightarrow$ (a) let $a \in A_+$. Choose $\lambda_0 \geq 1$ such that $(\lambda + a)^{-1}$ exists for all real $\lambda \geq \lambda_0$. Using the continuity of inversion we obtain that $(1 + a/\lambda)^{-1}$ converges to the interior point 1 when λ converges to infinity. Thus there exists $\lambda \geq 1$ such that $(\lambda + a)^{-1}$ is positive. Since A_+ is multiplicative, property (I) is verified.

We note that the implication (d) $\Rightarrow$ (e) is true for any unital Banach algebra even without the DDP. Example 1 shows that the converse is not valid unless A_+ possesses the DDP or is closed, see Theorem 2.7. Moreover Theorem 2.6 (d) $\Rightarrow$ (a), (b), (c) is not valid for Banach algebras without a unit element as the example in [17, p. 364] shows.

2.3 *Remark*. The equivalences of the first four statements of Theorem 2.2 are still valid for more general algebras than Banach algebras. In fact, we only need to show that for every *positive* $a \in A$ there exists $\lambda \geq 1$ such that $\lambda + a$ is invertible. Concrete examples of non-Banach algebras having this property are the algebra of all measurable functions on the unit interval or the Arens algebra.

There exists a vast literature about the structure of *lattice ordered algebras*, i.e., (associative) algebras which are lattice ordered by a multiplicative cone, see [3], [8], [17]. These mathematical objects can be rather pathological as pointed out in [3, p. 48]. Therefore some additional properties are needed in order to obtain a reasonable theory. The most common are the following which are meaningful for any lattice ordered algebra:

(f*) $\inf\{|a|, |b|\} = 0 \Leftrightarrow ab = 0$ for all $a, b \in A$.
(f) $\inf\{a, b\} = 0$ implies $\inf\{ca, b\} = \inf\{ac, b\} = 0$ for all $a, b \in A$ and $c \geq 0$.
(d) Every square is positive and $|ab| = |a| \cdot |b|$ for all $a, b \in A$.
(af) $\inf\{a, b\} = 0$ implies $ab = 0$ for all $a, b \in A$.

It is known that each of these conditions implies the subsequent condition: (f*) $\Rightarrow$ (f) is trivial since A is associative, for (f) $\Rightarrow$ (d) see [2, p. 404] and for (d) $\Rightarrow$ (af) see [3, p. 58 (Lemma 1) and p. 60 (Lemma 3)]. As pointed out in [14] these inclusions are in general proper. A lattice ordered algebra satisfying condition (f*), (f), (af) resp. is called an f*-*algebra*, f-*algebra*, *almost* f-*algebra* respectively. The following result is an easy consequence of known results:

2.4 Proposition. *An associative almost* f-*algebra is an* f*-*algebra if and only if* $a^2 \neq 0$ *for all* $a \neq 0$.

Proof. Let A be an f*-algebra and $a \neq 0$. Since $0 \neq |a| = \inf\{|a|, |a|\}$ the condition (f*) implies $0 \neq |a| \cdot |a| = |a^2|$. Thus $a^2 \neq 0$. Conversely, suppose that A is an almost f-algebra. Then the proof of Lemma 1 in [2, p. 406] shows that A is an f-algebra. Now let us prove the condition (f*). If $ab = 0$ then $0 \leq (\inf\{|a|, |b|\})^2 \leq |a| \cdot |b| = |ab| = 0$. Hence $(\inf\{|a|, |b|\})^2 = 0$ and our assumption yields $\inf\{|a|, |b|\} = 0$. If we have $\inf\{|a|, |b|\} = 0$ then $|a| \cdot |b| = 0$ by condition (af). It follows that $ab = 0$ since A is an f-algebra.

The following theorem is essentially due to Steinberg [17] and yields a sufficient criterion for the equivalence of the above conditions. The example 16 in [3] shows that the statement is not true in general for a non-archimedean cone.

2.5 THEOREM. *An archimedean unital lattice ordered algebra such that every square is positive is an* f*-*algebra.*

Proof. Corollary 1 in [17] shows that A is an f-algebra. By Proposition 2.4 it suffices to show that $a^2 = 0$ implies $a = 0$. Since $(1 \pm na)^2$ is positive we obtain $\pm 2na \leq 1$. Thus $a = 0$.

An *almost* f*-Banach lattice algebra* (in [14] called FF-Banachverbandsalgebra) is a Banach lattice and a Banach algebra ordered by a multiplicative cone satisfying the condition (af). Using the Corollary 2 in [12] and Proposition 2.4 one obtains a quite elementary proof of the following result in [14, Korollar 1.5, Satz 2.5].

2.6 THEOREM. *Let A be an almost f-Banach lattice algebra. Then the following assertions are equivalent*:

(a) *A is an f^*-algebra.*

(b) *$a^2 \neq 0$ for all $a \neq 0$.*

(c) *A is semi-simple.*

Theorem 2.4 in [14] shows that an almost f-Banach lattice algebra is semi-simple if and only if A is isomorphic to a separating subalgebra and sublattice of the set $C_0(X, \mathbf{R})$ of all continuous functions vanishing at infinity on a locally compact space X.

It is a matter of fact that many results valid for unital Banach algebras can be generalized to Banach algebras possessing a bounded approximate identity. More important, this class of algebras is more appropriate for applications. A *bounded approximate identity* is a net $(e_\lambda)_{\lambda \in I} \subset A$ such that ae_λ and $e_\lambda a$ converge to a when λ converges to infinity and $\|e_\lambda\| \leq C$ for some $C > 0$.

2.7 THEOREM. *Let A be a Banach algebra possessing a bounded approximate identity and endowed with a closed multiplicative cone. Then the following assertions are equivalent*:

(a) *A is an f^*-algebra.*

(b) *A is an f-algebra.*

(c) *A is an almost f-algebra.*

(d) *A is a vector lattice with property* (I).

(e) *A possesses the DDP and any product of disjoint positive elements is zero.*

(f) *A possesses the DDP and every square is positive.*
(g) *For all a in A there exist* $a_1, a_2 \geq 0$ *with* $a = a_1 - a_2$ *and* $a_1 a_2 = a_2 a_1 = 0$.
(h) *A is isomorphic to a subalgebra and sublattice of* $C(X, \mathbf{R})$ *for a suitable compact Hausdorff space X.*

Proof. Note that (a) ⇒ (b) ⇒ (c) ⇒ (f) is valid for any associative algebra. (h) ⇒ (a) is trivial. Hence it suffices to show that (d)–(h) are equivalent. The implications (d) ⇒ (e) ⇒ (f) are clear. Now we show that (f) implies (g) and (g) implies (h). First observe that both conditions imply that every square is positive. Let X be the set of all positive multiplicative functionals. Corollary 3.5 shows that A is isomorphic to a subalgebra of $C(X, \mathbf{R})$. The monomorphism $G_X: A \to C(X, \mathbf{R})$ is defined by $G_X(a) = \hat{a}$ and $\hat{a}(x) := x(a)$. It remains to show that the image $G_X(A)$ is a sublattice of $C(X, \mathbf{R})$. In the case (g) ⇒ (h) the representation $a = a_1 - a_2$ with $a_1 a_2 = 0$ shows that $\hat{a}_1(x) = \sup(\hat{a}(x), 0)$ for every $x \in X$ since x is multiplicative. In the case (f) ⇒ (g) it is enough to prove condition (I) since A possesses the DDP. Let $a \geq 0$. Since A is a Banach algebra there exists $\lambda > 0$ such that $(\lambda + a)^{-1}$ exists in A_e. The positivity of $(\lambda + a)^{-2} b (\lambda + a)^{-2}$ with $b \in A_+$ follows from the positivity of $x((\lambda + a)^{-2} b (\lambda + a)^{-2})$ for every positive multiplicative functional x. Moreover this argument yields (h) ⇒ (d), thus (d)–(h) are equivalent. The proof is complete.

3. A representation theorem

An application of Stone's representation theorem (see [9, p. 173]) shows that a unital Banach algebra with a closed multiplicative cone containing the algebraic unit as an order unit is isomorphic to a subalgebra of $C(X, \mathbf{R})$ for a suitable compact space. This space X can be defined as the set of all (continuous) positive multiplicative functionals. Note that this fact implies that the closed cone A_+ contains the closure of the wedge of all finite sums of squares, which will be denoted by $A_\times$ in our further discussion. But it may happen that the cone A_+ is strictly larger than $A_\times$ as the following example shows:

Example 4. Let B be the disc algebra with the involution * defined for $f \in B$ by

$$f^\star(z) = \bar{f}(\bar{z}) \qquad (z \in D)$$

where D denotes the closed unit disc in the complex plane. Then the selfadjoint part A is a real Banach algebra and the cone $A_\times$ consists of all functions $f: A \to \mathbf{C}$ which are pointwise non-negative for each $z \in [-1, 1]$.

Define A_+ as the multiplicative cone consisting of all functions $f: A \to \mathbf{C}$ which are real-valued on $[-1, 1]$ and non-negative on $[0, 1]$. Then the function $f: A \to \mathbf{C}$ defined by $f(z) = z$ is in A_+ but not $A_\times$. Moreover, the evaluation at a point $t \in [-1, 0)$ is a real-valued multiplicative functional on A which is not positive.

On the other hand there exists a representation theorem for *commutative complex* Banach *-algebras with a bounded approximate identity endowed with the wedge $A_\times$; cf. [13, Theorem 4.6.12 and Theorem 4.5.14]. Our representation theorem generalizes this result to Banach algebras with a closed multiplicative cone containing all squares. Moreover, our proof avoids the techniques of the representation theory of Banach *-algebras and is influenced by the geometrical methods used in [5] for unital Banach *-algebras.

3.1 Definition. Let A be a real algebra with a wedge A_+. We call a functional $f: A \to \mathbf{R}$ a *Schwarz map* if it satisfies the *Schwarz inequality* $f(a)^2 \leq f(a^2)$ for all $a \in A$. The set of all Schwarz maps is denoted by S_A, and S_A^+ is defined to be the subset of all functionals with $f(A_+) \subset [0, \infty)$.

It is easy to see that S_A and S_A^+ are convex sets. If A is non-unital then S_A is affinely bijective to the set of all unital Schwarz maps on A_e; the bijection is given by $f \mapsto \bar{f}$ where

$$\bar{f}(a + \lambda) := f(a) + \lambda \qquad (a \in A, \lambda \in \mathbf{R}).$$

Moreover f is non-negative on squares of A_e because of the Schwarz inequality; conversely, the restriction of a (positive) functional g with $g(1) \leq 1$ which is non-negative on squares yields a (positive) Schwarz map on A, cf. the *weak Cauchy-Schwarz inequality* in [13, p. 231]. Note that positivity means $g(A_+) \subset [0, \infty)$.

Let A be a Banach algebra. The *spectral radius* $|\cdot|_\sigma$ can be defined by the formula

$$|a|_\sigma = \lim_{n \in \mathbf{N}} \sqrt[n]{\|a^n\|} \qquad (a \in A).$$

It is well known that for every $a \in A_e$ with $|a|_\sigma < 1$ there exists $b \in A_e$ with $(1 - a) = b^2$. This yields $0 \leq \bar{f}(b^2) = 1 - f(a)$. Replacing a by $a^2/|a|_\sigma^2 + \varepsilon$ with $\varepsilon > 0$ we obtain in connection with the Schwarz inequality the following:

3.2 Lemma. *Let A be a real Banach algebra. Then every Schwarz map f is continuous and satisfies $|f(a)| \leq |a|_\sigma$ for all $a \in A$. In particular, S_A is convex and w^*-compact.*

For the proof of our representation theorem we need two conditions which seem to be rather technical:

(II) $\bar{f}(x^2a^2) \geq 0$ holds for every $f \in S_A^+$ and for all $x, a \in A_e$.
(III) $x^2a \in A_+$ holds for all $x \in A_e$ and all $a \in A_+$.

Observe that condition (II) is always satisfied if A is commutative. Moreover (III) is valid for a commutative algebra if every positive element is a sum of squares, i.e., that (II) and (III) are fulfilled for the wedge generated by the squares in a commutative algebra. Thus the following theorem can be applied to the selfadjoint part of a commutative Banach *-algebra in order to prove Theorem 4.5.15 in [13].

3.3 THEOREM. *Let A be a real Banach algebra with a wedge satisfying the conditions* (II) *and* (III). *Then the extreme points of S_A^+ are exactly the positive multiplicative functionals.*

Proof. Let $f \in S_A^+$ be extreme. If A possesses a unit element put $\bar{f} := f$, otherwise consider $\bar{f}: A_e \to \mathbf{R}$. For $x \in A_e$ with $|x|_\sigma < 1$ define $S_{x^2}: A_e \to \mathbf{R}$ by

$$S_{x^2}(a) := \bar{f}(x^2a) - \bar{f}(x^2)\bar{f}(a).$$

We show that $f_\pm := \bar{f} \pm S_{x^2}$ is non-negative on squares and positive. Note that

$$f_+(a) = \left(1 - \bar{f}(x^2)\right)\bar{f}(a) + \bar{f}(x^2a).$$

Since $1 - x^2 = b^2$ for some $b \in A_e$ we have $1 - \bar{f}(x^2) \geq 0$. Thus f_+ is positive by condition (III) and non-negative on squares of A_e by condition (II). Similarly it follows that

$$f_-(a) = \bar{f}(a - x^2a) + \bar{f}(x^2)\bar{f}(a) = \bar{f}(b^2a) + \bar{f}(x^2)\bar{f}(a)$$

is positive and non-negative on squares of A_e. Therefore the restrictions of $f_\pm$ on A are contained in S_A^+. The extremality yields $S_{x^2}(a) = 0$ for all $a \in A$, i.e., that $f(x^2a) = \bar{f}(x^2)f(a)$ for all $a \in A$ and for all $x \in A_e$ by the linearity of f. Substituting x by $1 + b$ with $b \in A$ one easily obtains $f(ba) = f(b)f(a)$ for all $a, b \in A$. A multiplicative positive Schwarz map is always extreme; cf. Proposition 5.2.2 in [9].

3.4 LEMMA. *Let A be a Banach algebra possessing a bounded approximate identity. Then a closed multiplicative wedge A_+ containing all squares satisfies condition* (II) *and* (III). *For every* (*continuous*) *positive non-trivial functional f there exists $\lambda > 0$ such that λf is a Schwarz map.*

Proof. Let $(e_\lambda)_{\lambda \in I}$ be a bounded approximate identity and let $x, a \in A_e$. Then we have $(xe_\lambda)^2 \in A_+$ and $(ae_\lambda)^2 \in A_+$. Thus $(xe_\lambda)^2(ae_\lambda)^2$ is positive and converges to x^2a^2. It follows by the continuity of $\bar{f}$ that $\bar{f}(x^2a^2) \geq 0$. The second statement follows similarly. Let $f\colon A \to \mathbf{R}$ be a continuous positive functional. Then

$$f\left(e_\lambda^2\right) \leq \|f\| \sup \|e_\lambda^2\| =: C.$$

Furthermore $2f(a) = \lim_{\lambda \in I} f(ae_\lambda + e_\lambda a)$ and the weak Cauchy-Schwarz inequality yields $4f(a)^2 \leq Cf(a^2)$ for all $a \in A$. Hence λf is a Schwarz map for $\lambda = 4/C$.

3.5 COROLLARY. *Let A be a Banach algebra with a bounded approximate identity and with a closed multiplicative cone containing all squares. Then A is isomorphic to a subalgebra of $C(X, \mathbf{R})$ where X is the w^*-compact space of all positive multiplicative functionals.*

Proof. X is w^*-compact by Lemma 3.2. Lemma 3.4, Theorem 3.3 and the Krein-Milman Theorem show that S_A^+ is the w^*-closed convex hull of X. The map $G_X\colon A \to C(X, \mathbf{R})$ defined by $G_X(a) := \hat{a}$ and $\hat{a}(x) := x(a)$ is obviously a continuous algebra homomorphism. It is as well an order isomorphism: A_+ is a closed cone and S_A^+ generates the topological dual cone, i.e., that for every positive continuous non-trivial functional $f\colon A \to \mathbf{R}$ there exists $\lambda > 0$ with $\lambda f \in S_A^+$; cf. Lemma 3.4 and Lemma 1.2 in [10].

3.6 *Remark.* Observe that the subalgebra $G_X(A) \subset C(X, \mathbf{R})$ is generally not a separating subalgebra. But it easy to see that $G_X(A)$ is isometrically isomorphic (with respect to the supremum norm) to a separating subalgebra of $C_0(Y, \mathbf{R})$ where

$$Y := X \setminus \{x \in X\colon \hat{a}(x) = 0 \text{ for all } a \in A\}$$

is a locally compact Hausdorff space.

3.7 COROLLARY. *An almost f-Banach lattice algebra A is isometrically isomorphic to $C_0(X, \mathbf{R})$ (X locally compact) if and only if A possesses a bounded approximate identity with norm bound at most one.*

Proof. For the non-trivial implication it is enough to show that the map G_X is an isometry. Let $a \in A$ and consider $b := |a| \geq 0$. It is an immediate consequence of Corollary 3.5 that $0 \leq be_\lambda^2 \leq |b|_{S_A^+} e_\lambda^2$, where $|b|_{S_A^+} := \sup_{f \in S_A^+} |f(b)|$. The norm is monotone, thus we obtain $0 \leq \|be_\lambda^2\| \leq |b|_{S_A^+} \|e_\lambda^2\|$ and therefore $0 \leq \|b\| \leq |b|_{S_A^+}$. But we have $|a|_{S_A^+} = |b|_{S_A^+}$ since every extreme functional in S_A^+ is a lattice homomorphism. This yields

$\|a\| = \|b\| \leq |a|_{S_A^+} \leq |a|_\sigma$ by Lemma 3.2. Thus the non-trivial implication is proved since the reverse inequality $|a|_\sigma \leq \|a\|$ is always true.

4. Uniqueness of the positive cone and of the multiplication

Example 4 shows that a multiplicative real-valued functional need not to be positive. But observe that the algebra is not lattice ordered. The main result of this paragraph says that this can not happen if the algebra satisfies one of the equivalent conditions listed in Theorem 2.7. The *proof* of this result is essentially due to E. Scheffold who introduced in [15] the following condition for a lattice ordered Banach algebra A:

$(\mathcal{O})$ $\inf\{|a|, |b|\} = 0$ implies $|ab|_\sigma = 0$ for all $a, b \in A$.

Since we are dealing with partially ordered algebras we weaken the condition $(\mathcal{O})$ to the following condition:

$(w\mathcal{O})$ $[0, a] \cap [0, b] = \{0\}$ implies $|ab|_\sigma = 0$ for all $a, b \in A_+$.

Note that example 4 satisfies the condition $(w\mathcal{O})$. It is easy to see that the proof of Theorem 2.2 in [14] can be used to establish the following result.

4.1 THEOREM. *Let A be a real algebra and a vector lattice. If the positive cone contains all squares and satisfies $(w\mathcal{O})$ then every multiplicative order bounded functional $f\colon A \to \mathbf{R}$ is a positive lattice homomorphism.*

4.2 COROLLARY. *Let A be a real Banach algebra possessing a bounded approximate identity. If A_+ is a closed multiplicative cone satisfying one of the equivalent conditions listed in Theorem* 2.7, *then A_+ is the closure of the cone generated by the squares.*

Proof. If A_+ satisfies one of the equivalent conditions of Theorem 2.7 then A_+ contains all squares and satisfies condition $(w\mathcal{O})$. It is easy to see that every multiplicative functional is order bounded. Let $a \in A_+$. Theorem 4.1 shows that $f(a) \geq 0$ for every multiplicative functional. But this property characterizes the elements of the closure of the cone generated by the squares; cf. Corollary 3.5 applied to the cone $A_\times$.

4.3 COROLLARY. *Let A be a Banach algebra possessing a bounded approximate identity. Then there exists at most one cone which makes A into an almost f-algebra.*

Observe that the assumption of an approximate identity in Corollary 4.2 is essential, even if A is a semi-simple Banach algebra; cf. Example M4 in [14].

Corollary 4.3 shows that the multiplication determines uniquely the positive cone of a unital almost f-Banach lattice algebra. This result should be seen as a converse to the following result (see [1, Theorem 8.23]): If A is an archimedean vector lattice and e is positive then there exists at most one product on A that makes A an f-algebra having e as its unit element. Now, this means that the order structure determines the multiplication uniquely. This result can be strengthened if one considers only Banach algebra multiplications:

4.4 THEOREM. *Let A be a Banach space with a closed cone A_+ and let e be a positive element. Then there exists at most one product on A that makes A a Banach algebra having e as its unit element and such that A_+ is a multiplicative cone containing all squares.*

Proof. Assume that the two products $\cdot$ and $*$ have the properties required in Theorem 4.4. By Corollary 3.5 it suffices to show that $f(x \cdot y - x * y) = 0$ for every non-trivial positive functional which is multiplicative relative to the first multiplication. But a multiplicative positive functional induces an extreme ray of the cone of all positive continuous functionals. Furthermore every extreme ray induces an extreme point of the set of all positive Schwarz maps with respect to the second multiplication, i.e., that there exists $\lambda > 0$ such that $f = \lambda g$ and g is multiplicative relative to the product $*$. But e is a unit element with respect to both products and therefore $\lambda = 1$. The proof is complete.

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