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EXTREME POSITIVE OPERATORS ON l^p

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1. Introduction

The problem of the characterization of the extreme operators was first investigated by A. Ionescu Tulcea and C. Ionescu Tulcea [13]. They considered extreme positive contractions on the space of continuous functions. Next many authors extended this result, and now we have a quite good knowledge about extreme operators on $C(K)$ (see for example [4], [5]). Thus it is natural to consider the possible extension of this problem to other classical Banach spaces. Using the results for $C(K)$ we can get characterizations of extreme l^∞-operators and l^1-operators (see [18], [14]). Note that for a Hilbert space case the set of extreme contractions coincides with the set of all isometries and coisometries (see [15], [8]). The other cases of l^p-spaces are more complicated. Some partial results on extreme l^p-contractions for $1 < p < \infty$, $p \neq 2$, are given in [6], [7], [16], [17], [12].

The purpose of this paper is to characterize the extreme points of the positive part of the unit ball of the space of operators acting on infinite dimensional l^p-spaces $1 < p < \infty$. This result extends an earlier one for the finite dimensional case [9]. Generally speaking the structure of extreme positive contractions is similar to the structure of extreme infinite doubly stochastic matrices with respect to arbitrary positive sequences (not necessarily elements of l^1). This description turns out to be more complicated compared with the finite dimensional case.

Let $1 < p < \infty$ and $q = p/(p-1)$. As usual we denote by l^p the Banach lattice of all p-summable real sequences with the norm

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}, \quad \mathbf{x} = (x_1) \in l^p$$

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and standard order ($\mathbf{x} \le \mathbf{y}$ if and only if $x_i \le y_i$ for all $i \in \mathbf{N}$). We put $\mathbf{e}_{i_0} = (\delta_{ii_0})$ (δ_{ij} denotes the Kronecker's delta). Obviously $\{\mathbf{e}_i\}$ forms the canonical basis of l^p. The adjoint space $(l^p)'$ is identified with the space l^q. For $0 \le \mathbf{x} = (x_i) \in l^p$ we let $\mathbf{x}^{p-1} = (x_i^{p-1}) \in l^q$. Note that $\mathbf{x}^{p-1}$ as a functional attains its norm at $\mathbf{x}$ and is the unique functional with this property. Moreover we have $\|\mathbf{x}^{p-1}\|_q^q = \|\mathbf{x}\|_p^p$.

We denote by $\mathscr{L}(l^p)$ the Banach space of all linear bounded operators from l^p into l^p. An operator T is said to be positive $T \ge 0$ if $T\mathbf{x} \ge 0$ whenever $\mathbf{x} \ge 0$. The positive part of the unit ball of $\mathscr{L}(l^p)$ (the set of positive contraction on l^p) is denoted by $\mathscr{P}$.

To every operator $T \in \mathscr{L}(l^p)$ corresponds a unique matrix (t_{ji}) with real entries, such that $(T\mathbf{x})_j = \sum_{i=1}^{\infty} t_{ji} x_i$. We have $T \ge 0$ if and only if $t_{ji} \ge 0$ for all $i, j \in \mathbf{N}$. The operators on l^p will be identified with their corresponding matrices. Thus for instance $(\delta_{jj_0}\delta_{ii_0})$ denotes the one dimensional operator in $\mathscr{L}(l^p)$ which maps $\mathbf{e}_{i_0}$ onto $\mathbf{e}_{j_0}$. Clearly the adjoint operator $T^* \in \mathscr{L}(l^q)$ is determined in the same manner by the transposed matrix.

Let $0 \le T \in \mathscr{P}$. We say that entries of $T = (t_{ji})$ are maximal if

$$\left\|\left(t_{ji} + \gamma\delta_{jj_0}\delta_{ii_0}\right)\right\| > 1$$

for every $\gamma > 0$ and all $i_0, j_0 \in \mathbf{N}$ such that $t_{j_0 i_0} > 0$. Obviously, if some entry of the operator T is maximal then $\|T\| = 1$ and if T is an extreme positive contraction then all entries of T are maximal. Note that there exists $T \in \mathscr{P}$ such that $\|T\| = 1$ and the entries of T are not maximal (a suitable example is given in the paper).

We define the support of an operator $T = (t_{ji}) \in \mathscr{L}(l^p)$ by

$$\operatorname{supp} T = \left\{i\colon \text{there exists } j_0 \text{ such that } t_{j_0 i} \ne 0\right\}.$$

For a positive operator $T = (t_{ji}) \in \mathscr{L}(l^p)$ we denote by $\mathscr{M}(T)$ the set of all non-negative sequences (x_i) such that

$$0 \le \sum_{k=1}^{\infty} t_{jk} x_k = y_j < \infty, \tag{1}$$

$$\sum_{k=1}^{\infty} t_{ki} y_k^{p-1} = x_i^{p-1} \tag{2}$$

for all $i, j \in \mathbf{N}$, and

(3) $x_i > 0$ if and only if $i \in \operatorname{supp} T$. That is

$$\mathscr{M}(T) = \left\{ (x_i) \geq 0\colon \operatorname{supp}(x_i) = \operatorname{supp} T \text{ and for every } i \in \mathbf{N}, \right.$$

$$\left. \sum_{j=1}^{\infty} t_{ji} \left(\sum_{k=1}^{\infty} t_{jk} x_k \right)^{p-1} = x_i^{p-1} \right\}.$$

Let $\mathbf{a} = (a_i)$, $\mathbf{b} = (b_j)$ be non-negative sequences. A matrix $P = (p_{ji})$, $i, j \in \mathbf{N}$, is said to be doubly stochastic with respect to $((a_i), (b_j))$ if $p_{ji} \geq 0$, $\sum_{j=1}^{\infty} p_{ji} = a_i$, $\sum_{i=1}^{\infty} p_{ji} = b_j$. The set of all doubly stochastic matrices with respect to $\mathbf{a}, \mathbf{b}$ will be denoted by $\mathscr{D}(\mathbf{a}, \mathbf{b})$.

To complete a characterization of extreme positive l^p-contractions we need a description of extreme points of $\mathscr{D}(\mathbf{a}, \mathbf{b})$ for arbitrary non-negative sequences $\mathbf{a}, \mathbf{b}$. This problem was investigated under various assumption on $\mathbf{a}, \mathbf{b}$ by many authors (see [20], [21], [3]). Note that the first result of this kind was given by G.D. Birkhoff [1] (see also [22], I, §5). The characterization of $\operatorname{ext} \mathscr{D}(\mathbf{a}, \mathbf{b})$ for arbitrary non-negative sequences $\mathbf{a}, \mathbf{b}$ is given in [10].

The main aim of this paper is to prove the following characterization of extreme positive l^p-contraction.

THEOREM. *Let* $1 < p < \infty$, *and let* $0 \neq T = (t_{ji}) \in \mathscr{P}$. *Then* T *is an extreme positive contraction if and only if the following conditions hold*:

(i) *the entries of* T *are maximal*;

(ii) *the matrix* $P = (t_{ji} x_i y_j^{p-1})$ *is extreme in* $\mathscr{D}((x_i^p), (y_j^p))$, *where* $(x_i) \in \mathscr{M}(T)$ *and* $y_j = \sum_{i=1}^{\infty} t_{ji} x_i$.

2. Proof of the theorem

We will use the following fact, which is a generalized version of the Schur's test [23] (see [11], §5,Th. 5.2.).

PROPOSITION 1. *For a positive operator* $T = (t_{ji}) \in \mathscr{L}(l^p)$ *let there exist positive sequences* $(x_i), (y_j)$ *such that*

$$y_j = \sum_{i=1}^{\infty} t_{ji} x_i$$

and

$$\sum_{j=1}^{\infty} t_{ji} y_j^{p-1} \leq x_i^{p-1}$$

for all $i, j \in \mathbf{N}$. *Then* $\|T\| \leq 1$.

Proof. Using the convexity of $f(t) = t^p$ for an arbitrary non-negative vector $\mathbf{u} = (u_i) \in l^p$ we have

$$\begin{aligned} \|T\mathbf{u}\|_p^p &= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} t_{ji} u_i \right)^p \\ &= \sum_{j=1}^{\infty} y_j^p \left(\sum_{i=1}^{\infty} \frac{t_{ji} x_i}{y_j} \frac{u_i}{x_i} \right)^p \\ &\leq \sum_{j=1}^{\infty} y_j^p \sum_{i=1}^{\infty} \frac{t_{ji} x_i}{y_i} \frac{u_i^p}{x_i^p} \\ &= \sum_{i=1}^{\infty} \frac{u_i^p}{x_i^{p-1}} \sum_{j=1}^{\infty} t_{ji} y_j^{p-1} \\ &\leq \sum_{i=1}^{\infty} u_i^p = \|\mathbf{u}\|_p^p. \end{aligned}$$

COROLLARY 1. *If for a positive operator $T \in \mathscr{L}(l^p)$ the set $\mathscr{M}(T)$ is non-empty then* $\|T\| \leq 1$.

For every matrix (t_{ji}) define a graph $G((t_{ji}))$ by the following formula. To the j-th row there corresponds a (row) node j, $j \in \mathbf{N}$, and to i-th column there corresponds a (column) node i, $i \in \mathbf{N}$. There is an edge joining a node i and a node j if and only if $t_{ji} \neq 0$. There are no other edges.

We say that an operator $T \in \mathscr{L}(l^p)$ is elementary provided there are no non-zero operators $T = T_1 + T_2$ and

$$\operatorname{supp} T_1 \cap \operatorname{supp} T_2 = \operatorname{supp} T_1^* \cap \operatorname{supp} T_2^* = \varnothing.$$

Note that T is elementary if and only if the graph $G(T)$ is connected. Each operator $T \in \mathscr{L}(l^p)$ can be represented as a countable sum of elementary operators T_k, $T = \Sigma T_k$ with $\operatorname{supp} T_k$ disjoint and $\operatorname{supp} T_k^*$ disjoint. Then $\|T\| = \sup_k \|T_k\|$ and $T \geq 0$ if and only if $T_k \geq 0$ for all k. Therefore T is an extreme positive contraction if and only if the T_k's are extreme positive contractions. The above decomposition shows us that for our purpose it is enough to consider elementary operators. Therefore without any loss of generality all operators in $\mathscr{L}(l^p)$ considered in the remainder of the paper will be assumed to be elementary operators.

PROPOSITION 2. *Let $T \in \operatorname{ext} \mathscr{P}$. Then the graph $G(T)$ has no cycle.*

Proof. Suppose, to get a contradiction, that the graph $G(T)$ has a simple cycle C. Let $F_n \in \mathscr{L}(l^p)$ denote the projection defined by

$$F_n \mathbf{e}_i = \begin{cases} \mathbf{e}_i & \text{if } i \le n \\ 0 & \text{otherwise} \end{cases}$$

For n sufficiently large the graph of $T_n = TF_n$ contains the cycle C. Note that the T_n's are finite dimensional operators and they are not extreme. Recall here that the finite dimensional case if the graph of a positive contraction has the cycle then it is not extreme (see [9, Th.3]), so for each T_n there exists $R_n = (r_{ji}^{(n)}) \ne 0$ such that $\|T_n \pm R_n\| \le \|T_n\| \le 1$ and $T_n \pm R_n \ge 0$, the graph $G(R_n) = C$ and $t_{j_0 i_0} = |r_{j_0 i_0}^{(n)}|$ for some $(i_0, j_0) \in C$ (not necessarily the same for all n). Choose a subsequence n_k of $\mathbf{N}$ such that $\lim_{k \to \infty} r_{ji}^{(n_k)} = r'_{ji}$ exists for all (i, j). Note that $r'_{ji} \ne 0$ for some (i, j), i.e., $R' = (r'_{ji}) \ne 0$. Obviously $T \pm R \ge 0$ and $\|T \pm R\| \le 1$. This contradiction ends the proof.

LEMMA 1. *Let the graph $G(T)$ of $T \in \mathscr{P}$ be a tree. If all entries of T are maximal then $\mathscr{M}(T)$ is non-empty.*

LEMMA 2. *Let all the entries of $T \in \mathscr{P}$ be maximal. Let $(x_i) \in \mathscr{M}(T)$ and $j_1 \in \operatorname{supp} T^*$. Then for every $\varepsilon > 0$ there exists N_0 such that for all $N > N_0$ there exists $\mathbf{u}^{(N)} \in l^p$ such that*

$$\|\mathbf{u}^{(N)}\|^p - \|T\mathbf{u}^{(N)}\|^p < \varepsilon,$$

and

$$u_i^{(N)} = x_i \quad \text{for } i \in \{k \le N \colon t_{j_1 k} \ne 0\},$$
$$u_i^{(N)} = 0 \quad \text{for } i \in \{k > N \colon t_{j_1 k} \ne 0\}.$$

The proofs of Lemmas 1 and 2 will be presented in Section 4.

Let the graph $G(T)$ of $T \in \mathscr{L}(l^p)$ be a tree (i.e., $G(T)$ has no cycles). Let $i_1 \in \operatorname{supp} T$. Note that $G(T)$ is a connected tree since T is elementary. We define inductively two families $\{I_n\}$ and $\{J_n\}$ of disjoint subsets of $\mathbf{N}$ and a family $\{E_n\}$ of disjoint subsets of $\mathbf{N} \times \mathbf{N}$. Put

$$I_1 = \{i_1\}, \quad J_1 = \{j \colon t_{j_1 i} \ne 0\}$$

and

$$\begin{aligned} I_{n+1} &= \{i \notin I_n : t_{ji} \neq 0 \quad \text{for some } j \in J_n\} \\ J_{n+1} &= \{j \notin J_n : t_{ji} \neq 0 \quad \text{for some } i \in I_{n+1}\} \\ E_{2n-1} &= \{(i,j) : i \in I_n,\, j \in J_n\} \\ E_{2n} &= \{(i,j) : i \in I_{n+1},\, j \in J_n\}, \quad n \in \mathbf{N}. \end{aligned}$$

LEMMA 3. *Let all the entries of $T \in \mathscr{P}$ be maximal. Let $(x_i) \in \mathscr{M}(T)$ and $y_j = \sum_{j=1}^{\infty} t_{ji} x_i$. If $T \pm R \in \mathscr{P}$ for some $R = (r_{ji})$ then*

$$\sum_{j=1}^{\infty} r_{ji} x_i = 0 \quad \text{and} \quad \sum_{j=1}^{\infty} r_{ji} y_j^{p-1}.$$

Proof. The graph $G(R)$ is included in the graph $G(T)$ and $|r_{ji}| \leq t_{ji}$, since $T \pm R \geq 0$. Fix $j_1 \in \operatorname{supp} T^*$. Because in the construction of the sets $I_1, J_1, I_2, \ldots$ the index i_1 is arbitrary we may and do assume that $j_1 \in J_1$.

Fix $\varepsilon > 0$. We need to show that there exists N_0 such that

$$\left| \sum_{i=1}^{N} r_{j_1 i} x_i \right| < \varepsilon \quad \text{for all } N > N_0.$$

By Lemma 2 we can find $N_0 \in \mathbf{N}$ such that for every $N > N_0$ there exists $\mathbf{u}^{(N)} \in l^p$ such that

$$\|\mathbf{u}^{(N)}\| - \|T\mathbf{u}^{(N)}\| < \varepsilon$$

and

$$(R\mathbf{u}^{(N)})_{j_1} = \sum_{i=1}^{N} r_{j_1} x_i.$$

First consider the case when $p \geq 2$. Using the Clarkson inequality [2] (see also [19], Corollary 2.1) we have

$$2\|R\mathbf{u}^{(N)}\|_p^p + 2\|T\mathbf{u}^{(N)}\|_p^p \leq \|(T+R)\mathbf{u}^{(N)}\|_p^p + \|(T-R)\mathbf{u}^{(N)}\|_p^p \leq 2\|\mathbf{u}^{(N)}\|_p^p$$

Hence we have

$$\left| \sum_{i=1}^{N} r_{j_1 i} x_i \right| = \left| (R\mathbf{u}^{(N)})_{j_1} \right| \leq \|R\mathbf{u}^{(N)}\|_p \neq \left(\|\mathbf{u}^{(N)}\|_p^p - \|T\mathbf{u}^{(N)}\|_p^p \right)^{1/p} < \varepsilon^{1/p}.$$

Therefore $\sum_{i=1}^{\infty} r_{ji} x_i = 0$ for all $j \in \mathbf{N}$ and $p \geq 2$.

Now assume that $1 < p < 2$. As an immediate consequence of differential calculus we obtain

$$(t+\tau)^p + (t-\tau)^p \geq 2t^p + p(p-1)\tau^2 t^{p-1} \geq 2t^p + p(p-1)\tau^2$$

where $|\tau| < t < 1$. By this, putting $T\mathbf{u}^{(N)} = (f_j)$ and $R\mathbf{u}^{(N)} = (g_j)$ we obtain

$$\begin{aligned} 2\sum_{j=1}^{\infty} |f_j|^p + p(p-1)g_{j_1}^2 &\leq \sum_{j=1}^{\infty} |f_j + g_j|^p + \sum_{j=1}^{\infty} |f_j - g_j|^p \\ &= \|(T+R)\mathbf{u}^{(N)}\|_p^p + \|(T-R)\mathbf{u}^{(N)}\|_p^p \\ &\leq 2\|\mathbf{u}^{(N)}\|_p^p. \end{aligned}$$

Hence

$$p(p-1)g_{j_1}^2 \leq 2\left(\|\mathbf{u}^{(N)}\|_p^p - \|T\mathbf{u}^{(N)}\|_p^p\right) < 2\varepsilon.$$

Thus we prove that $\sum_{i=1}^{\infty} r_{ji} x_i = 0$ for all $p \in (1, \infty)$. To prove that $\sum_{j=1}^{\infty} r_{ji} y_j^{p-1} = 0$ we apply the same arguments for the adjoint operators T^* and R^*.

Proof of the theorem. Suppose that $T \in \operatorname{ext} \mathscr{P}$. Then obviously the condition (i) holds. From Lemma 1 there exists $(x_i) \in \mathscr{M}(T)$. Put $y_j = \sum_{i=1}^{\infty} t_{ji} x_i$.

Suppose that $P = (t_{ji} x_i y_j^{p-1}) \notin \operatorname{ext} \mathscr{D}((x_i^p), (y_j^p))$. Then there exist $P' = (p'_{ji})$ and $P'' = (p''_{ji})$ in $\mathscr{D}((x_i^p), (y_j^p))$ such that $P' \neq P''$ and $P = (P' + P'')/2$. In view of Proposition 1, $T' = (t'_{ji})$ and $T'' = (t''_{ji})$ are positive contractions, where $t'_{ji} = p'_{ji}/x_i y_j^{p-1}$ and $t''_{ji} = p''_{ji}/x_i y_j^{p-1}$ (we admit $0/0 = 0$). We have $(T' + T'')/2 = T$, so T is not extreme. Thus the condition (ii) also holds.

Now suppose that the conditions (i) and (ii) hold. Let $R = (r_{ji})$ be such that $T \pm R \in \mathscr{P}$. Obviously the graph $G(R)$ is a subgraph of $G(T)$. By Lemma 3, $\sum_{i=1}^{\infty} r_{ji} x_i = 0$ and $\sum_{j=1}^{\infty} r_{ji} y_j^{p-1} = 0$. Thus

$$\left(t_{ji} x_i y_j^{p-1}\right) \pm \left(r_{ji} x_i y_j^{p-1}\right) \in \mathscr{D}\left((x_i^p), (y_j^p)\right).$$

Because $(t_{ji} x_i y_j^{p-1}) \in \operatorname{ext} \mathscr{D}((x_i^p), (y_j^p))$ we get $r_{ji} x_i y_j^{p-1} = 0$. Hence $r_{ji} = 0$, i.e., $T \in \operatorname{ext} \mathscr{P}$.

3. Operators with a graph of finite height

Let the graph of $T \in \mathscr{L}(l^p)$ be a tree. The family I_n is a partition of $\operatorname{supp} T$ and the family J_n is a partition of $\operatorname{supp} T^*$. Moreover

$$\bigcup_{n=1}^{\infty} E_n = \{(i, j) \colon t_{ji} \neq 0\}.$$

If in the sequence $E_1, E_2, E_3, \ldots$ some E_{n_0} is empty then the subsequent sets E_n, $n > n_0$, are also empty. The number $h(T)$ of the non-empty sets in the sequence $\{E_n\}$ will be called the height of the graph $G(T)$. We say that the matrix T has the FHG (Finite Height Graph) property if $h(T)$ is finite.

LEMMA 4. *Let $0 \leq T \in \mathscr{L}(l^p)$ have the FHG property, and let $(x_i) \in \mathscr{M}(T)$, $y_j = \sum_i t_{ji} x_i$. Then for each $\varepsilon > 0$ there exists a finite subset I of* **N** *such that*

$$\|T^*(T\mathbf{u})^{p-1}\|_q^q > \|\mathbf{u}\|_q^q - \varepsilon,$$

$$\{i_i\} = I_1 \subset I, \quad \left(T^*\left((T\mathbf{u})^{p-1}\right)\right)_{i_1} > x_{i_1}^{p-1}/2,$$

and for fixed $j_1 \in J_1$ we have $(T\mathbf{u})_{j_1} > y_{j_1}/2$ where

$$u_i = \begin{cases} x_i & \text{if } i \in I \\ 0 & \text{if } i \notin I. \end{cases}$$

Proof. Let $(x_i) \in \mathscr{M}(T)$. Let $y_i = \sum_i t_{ji} x_i$. Fix $\varepsilon > 0$ and $i_1 \in \operatorname{supp} T$. Fix $j_1 \in J_1$. Let $\varepsilon_i > 0$ be such that

$$\left(x_i^{p-1} - 2\varepsilon_i\right)^q > x_i^p - \varepsilon/2^i$$

and

$$\varepsilon_{i_1} < x_{i_1}^{p-1}/4.$$

Let $I_1' = I_1 = \{i_1\}$. We choose a finite subset J_1' of J_1 such that $j_1 \in J_1'$ and

$$\sum_{j \in J_1'} t_{ji_1} y_j^{p-1} > x_{i_1}^{p-1} - \varepsilon_{i_1}.$$

We find $\delta_j > 0$ ($j \in J_1'$) such that $\delta_{j_1} < y_{j_1}/2$ and

$$\sum_{j \in J_1'} t_{ji_1}\left(y_j - \delta_j\right)^{p-1} > x_{i_1}^{p-1} - 2\varepsilon_{i_1}.$$

We choose a finite subset I_2' of I_2 such that

$$\sum_{i \in I_1' \cup I_2'} t_{ji} x_i > y_j - \delta_j$$

for $j \in J_1'$. We choose a finite subset J_2' of J_2 such that

$$\sum_{j \in J_1' \cup J_2'} t_{ji} y_j^{p-1} > x_i^{p-1} - \varepsilon_i$$

for $i \in I_2'$. We find $\delta_j > 0$ ($j \in J_2'$) such that

$$\sum_{j \in J_1' \cup J_2'} t_{ji} \left(y_j - \delta_j \right)^{p-1} > x_i^{p-1} - 2\varepsilon_i$$

$i \in I_2'$. We continue the above process to get (after $h(T)$ steps) a finite sequence $I_1', J_1', I_2', \dots, J_{n_0}'$. Let $I = \bigcup_{n=1}^{n_0} I_n'$ and $J = \bigcup_{n=1}^{n_0} J_n'$. We define

$$u_i = \begin{cases} x_i & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases}$$

Put $\mathbf{v} = (v_j) = T\mathbf{u}$. For $j \in J$ we have

$$v_j = \sum_{i=1}^{\infty} t_{ji} u_i = \sum_{i \in I} t_{ji} x_i > y_j - \delta_j.$$

Hence

$$\sum_{j=1}^{\infty} t_{ji} v_j^{p-1} > \sum_{j \in J} t_{ji} \left(y_j - \delta_j \right)^{p-1} > x_i^{p-1} - 2\varepsilon_i \quad \text{for } i \in I.$$

Therefore we obtain

$$\begin{aligned} \| T^* (T\mathbf{u})^{p-1} \|_q^q = \| T^* \mathbf{v}^{p-1} \|_q^q &\geq \sum_{i \in I} \left[\sum_{j \in J} t_{ji} v_j^{p-1} \right]^q \\ &> \sum_{i \in I} \left(x_i^{p-1} - 2\varepsilon_i \right)^q \\ &> \sum_{i \in I} \left(x_i^p - \frac{\varepsilon}{2^i} \right) \\ &> \| \mathbf{u} \|_p^p - \varepsilon. \end{aligned}$$

Moreover we have

$$\left(T^* \left((T\mathbf{u})^{p-1} \right) \right)_{i_1} \geq \sum_{j \in J_1'} y t_{ji_1} \left(y_j - \delta_j \right)^{p-1} > x_{i_1}^{p-1} - \varepsilon \geq x_{i_1}^{p-1}/2$$

and

$$(T\mathbf{u})_{j_1} \geq \sum_{i \in I_1' \cup I_2'} t_{j_1 i} x_i > y_{j_1} - \delta_{j_1} > y_{j_1}/2.$$

LEMMA 5. *Let $0 \leq T \in \mathscr{L}(l^p)$ have the FHG property and let $\mathscr{M}(T)$ be non-empty. Then $\|T\| = 1$ and all the entries of T are maximal.*

Proof. By Corollary 1 we have $\|T\| \leq 1$. Let $T = (t_{ji})$ have the FHG property and let $(x_i) \in \mathscr{M}(T)$. Suppose, to get a contradiction, that there exists an entry of T which is not maximal. Since the construction of the sequences $I_1, J_1, J_2 \ldots$ can start from every positive entry, we may and do assume that t_{ji} $(i_1 \in I_1 = I_1'$, $j_1 \in J_1)$ is not maximal. Let $\gamma > 0$ be such that $\|S\| \leq 1$, where $S = (s_{ji}) = (t_{ji} + \gamma \delta_{ii_1} \delta_{jj_1})$. Let

$$\beta = \frac{y_{j_1}^{p-1}}{2^{p-1}} \left[\left(1 + \gamma \frac{x_{i_1}}{y_{j_1}} \right)^{p-1} - 1 \right] > 0$$

and

$$\varepsilon = t_{j_1 i_1} \beta q x_{i_1} / 2^q > 0.$$

In view of Lemma 1 there exists $\mathbf{u} = (u_i) \in l^p$ such that if $\mathbf{v} = (v_j) = T\mathbf{u}$, $\mathbf{z} = (z_i) = T^*(\mathbf{v}^{p-1})$ then $\|\mathbf{z}\|_q^q > \|\mathbf{u}\|_p^p - \varepsilon$, and $u_{i_1} = x_{i_1}$, $z_{i_1} > x_{i_1}^{p-1}$, $v_{j_1} > y_{j_1}/2$. We have

$$\left[(v_{j_1} + \gamma x_{i_1})^{p-1} - v_{j_1}^{p-1} \right] \geq v_{j_1}^{p-1} \left[\left(1 + \gamma \frac{x_{i_1}}{y_{j_1}} \right)^{p-1} - 1 \right] \geq 1.$$

Using the mean value theorem we get

$$\left[z_{i_1} + t_{j_1 i_1} \beta \right]^q - z_{i_1}^q \geq t_{j_1 i_1} \beta q z_{i_1}^{q-1} > 2\varepsilon.$$

Therefore we obtain

$$\begin{aligned} \|S^*(S\mathbf{u})^{p-1}\|_q^q &\geq \left\| T^* \left[(T + \gamma \delta_{ii_1} \delta_{jj_1}) \mathbf{u} \right]^{p-1} \right\|_q^q \\ &= \left\| T^*(T\mathbf{u}) + T^* \left[\left((v_{j_1} + \gamma x_{i_1})^{p-1} - v_{j_1}^{p-1} \right) \mathbf{e}_{j_1} \right] \right\|_q^q \\ &\geq \|\mathbf{z} + t_{j_1 i_1} \beta \mathbf{e}_{i_1}\|_q^q \\ &= \|\mathbf{z}\|_q^q + \left(z_{i_1} + t_{j_1 i_1} \beta \right)^q - z_{i_1}^q \\ &\geq \|\mathbf{u}\|_p^p - \varepsilon + 2\varepsilon \geq \|\mathbf{u}\|_p^p. \end{aligned}$$

This contradicts the fact that for arbitrary $R \in \mathscr{P}$ we have

$$\|R^*(R\mathbf{u})^{p-1}\|_q^q \leq \|(R\mathbf{u})^{p-1}\|_q^q = \|R\mathbf{u}\|_p^p \leq \|\mathbf{u}\|_p^p.$$

This shows us that all the entries of T are maximal. Moreover, since $\|S\| > 1$ for each $\gamma > 0$ we have $\|T\| = 1$.

Let $m \in \mathbf{N}$. We define the following maps from the set of all positive contractions which the graph is a tree into the set of matrices having the FHG property by

$$\mathscr{S}_m((t_{ji})) = \begin{cases} t_{ji} & \text{if } (i,j) \in \bigcup_{n=1}^{2m-1} E_n \\ t_{ji}\left[1 - \sum_{k \in J_{m+1}} t_{ki}^p\right]^{-1/p} & \text{if } (i,j) \in E_{2m} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathscr{S}'_m((t_{ji})) = \begin{cases} t_{ji} & \text{if } (i,j) \in \bigcup_{n=1}^{2m-2} E_n \\ t_{ji}\left[1 - \sum_{k \in I_{m+1}} t_{jk}^q\right]^{-1/q} & \text{if } (i,j) \in E_{2m-1} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathscr{T}_m(T) = \mathscr{S}'_m\mathscr{S}_m(T),$$

$$\mathscr{R}_m((t_{ji})) = \begin{cases} t_{ji} & \text{if } (i,j) \in \bigcup_{n=1}^{2m-1} E_n \\ 0 & \text{otherwise.} \end{cases}$$

Note that $h(\mathscr{S}_m(T) \leq 2m$, $h(\mathscr{S}'_m(T)) \leq 2m-1$, $h(\mathscr{T}_m(T)) \leq 2m-1$ and $h(\mathscr{R}_m(T)) \leq 2m-1$. And $\mathscr{R}_m(T) \leq \mathscr{T}_m(T)$.

LEMMA 6. *Let $T \in \mathscr{P}$ have the FHG property, and let all the entries of T be maximal. Then there exists unique (up to a multiplicative constant) sequence $(x_i) \in \mathscr{M}(T)$.*

Moreover, if $h(T) \leq 2m+1$ then all the entries of $\mathscr{S}_m(T)$ are maximal, and if $h(T) \leq 2m$ then all the entries of $\mathscr{S}'_m T$ are maximal.

Proof. Let T satisfy the assumption of the lemma. Our proof is inductive with respect to $h(T)$. First let $h(T) = 1$ i.e., $Te_{i_1} \neq 0$ and $Te_i = 0$ for $i \neq i_i$. Since $\|T\| = 1$ we have $\|Te_{i_1}\| = 1$. It is easy to see that (δ_{ii_1}) is a unique (up to a multiplicative constant) element of $\mathscr{M}(T)$.

Assume that the thesis of the lemma is true for all T with $h(T) \leq N$. Assume that $h(T) = N + 1$. We need to prove that the lemma holds for T. First consider the case when $N = 2m$ is even. Put $(u_{ji}) = \mathscr{S}_m(T)$. For $(i, j) \in E_{2m}$. We define

$$\eta_{ji} = \frac{1}{\sqrt[p]{1 - \sum_{k \neq j} t_{ki}^p}}$$

Note that $1 - \sum_{k \neq j} t_{ki}^p = 1 - \|Te_i\|^p + t_{ji}^p > 0$ since $T \in \mathscr{P}$. We have $h((u_{ji})) = 2N$.

We claim that all the entries of (u_{ji}) are maximal. Indeed, suppose first, to get a contradiction, that the entries of (u_{ji}) are not maximal. We find $\alpha_{ji} \geq 1$ such that all the entries of the matrix $(\alpha_{ji} u_{ji})$ are maximal. Put $\alpha_{ji} = 1$ for $(i, j) \in E_{2m+1}$. By the inductive assumption there exists $(x') \in \mathscr{M}((\alpha_{ji} u_{ji}))$. For every $i \in I_{m+1}$ we denote by j_i the unique element of J_m such that $t_{j_i i} \neq 0$. Now let

$$x'' = \begin{cases} x_i' \eta_{j_i i} & \text{if } i \in I_{m+1} \\ x_i' & \text{otherwise.} \end{cases}$$

It is easy to check that

$$(x_i'') \in \mathscr{M}\big((\alpha_{ji} t_{ji})\big)$$

By Lemma 5, $\|(\alpha_{ji} t_{ji})\| = 1$. Since $t_{ji} \leq \alpha_{ji} t_{ji}$ and all entries of (t_{ji}) are maximal we obtain $\alpha_{ji} = 1$. Now suppose that $\|(u_{ji})\| > 1$. We find $\alpha_{ji} \leq 1$, $(i, j) \in \bigcup_{n=1}^{2n} E_n$, such that all the entries of $(\alpha_{ji} u_{ji})$ are maximal. By inductive assumption there exists $(x') \in \mathscr{M}((\alpha_{ji} u_{ji}))$. Put $\alpha_{ji} = 1$ for $(i, j) \in E_{2m+1}$. It is easy to check that $(x_i'') \in \mathscr{M}((\alpha_{ji} t_{ji}))$, where x_i'' is defined as above. By Lemma 5 the entries of $(\alpha_{ji} t_{ji})$ are maximal, hence all $\alpha_{ji} = 1$. This ends the proof of our claim. Therefore if $h(T) \leq 2m + 1$ then all the entries of $\mathscr{S}_m(T)$ are maximal.

Using inductive assumption we find (unique) $(x_i') \in \mathscr{M}((u_{ji}))$. Put

$$x_i = \begin{cases} x_i' \eta_{j_i i} & \text{if } i \in I_{m+1} \\ x_i' & \text{otherwise.} \end{cases}$$

One can easily verify that $(x_i) \in \mathscr{M}((t_{ji}))$.

Now suppose that $N = 2m - 1$ is odd. Let $(u_{ji}) = \mathscr{S}_m'(T)$. For $(i, j) \in E_{2m-1}$ we let

$$\eta_{ji} = \frac{1}{\sqrt[q]{1 - \sum_{k \neq i} t_{jk}^q}}.$$

By the same argument as in the even case, all the entries of the matrix (u_{ji}) are maximal. Using the inductive assumption we find $(x_i') \in \mathscr{M}((u_{ji}))$. It is not difficult to check that $(x_i) \in \mathscr{M}((t_{ji}))$, where

$$x_i = \begin{cases} \left(t_{j_1 i}\right)^{q/p} \eta_{j_1 i_1}^q t_{j_1 i_1} x_{i_1}' & \text{if } i \in I_{m+1},\ j_1 \in \{j \in J_m\colon t_{j1} \neq 0\}, \\ & \quad i_1 \in \{k \in I_m\colon t_{j_1 k} \neq 0\} \\ x_i' & \text{otherwise} \end{cases}$$

Analogously, if $h(T) \leq 2m$ then all the entries of T are maximal.

Remark 1. (a) From the construction presented in the proof of Lemma 6 it follows that if $\mathscr{M}(\mathscr{S}_m(T)) \neq \varnothing$ ($\mathscr{M}(\mathscr{S}_m'(T)) \neq \varnothing$) then $\mathscr{M}(T) \neq 0$. Therefore, if $\mathscr{M}(\mathscr{T}_m(T)) \neq \varnothing$ then $\mathscr{M}(T) \neq \varnothing$.

(b) We get also that if $\|T\| \leq 1$ then $\|\mathscr{S}_m(T)\| \leq 1$ and $\|\mathscr{S}_m'(T)\| \leq 1$, hence $\|\mathscr{T}_m(T)\| \leq 1$, too.

(c) Let $h(T) \leq 2m + 1$ and let the entries of T are maximal.

If $(x_i') \in \mathscr{M}(T)$, $x_i'' \in \mathscr{M}(\mathscr{T}_m(T))$ and $x_{i_1}' = x_{i_1}'' = 1$ then $x_i' = x_i''$ for $i \in \bigcup_{n=1}^m I_n$.

Although $\mathscr{T}_m$ is not a linear map, it has other useful properties.

LEMMA 7. *Let* $T \in \mathscr{P}$, $m \in \mathbf{N}$.

(a) $\mathscr{T}_m(T) \geq 0$,

(b) $\|\mathscr{T}_m(T)\| \leq 1$,

(c) $(\mathscr{T}_m(T))_{ji} \geq t_{ji}$ *for* $(i, j) \in E_{2m-1}$.

Moreover if $h(T) \leq 2m + 1$ *then*:

(d) *All the entries of* T *are maximal if and only if all the entries of* $\mathscr{T}_m(T)$ *are maximal.*

Proof. (a) and (c) are obvious. (b) follows from Remark 1. For (d), let $h(T) \leq 2m + 1$. Suppose that all the entries of $\mathscr{T}_m(T)$ are maximal. Then in view of Lemma 6, $\mathscr{M}(\mathscr{T}_m(T)) \neq \varnothing$. By Remark 1, $\mathscr{M}(T) \neq \varnothing$. From Lemma 5 all the entries of T are maximal. The reserve implication follows directly from Lemma 6.

4. Proofs of the main lemmas

Let $T \in \mathscr{P}$. We define a family of matrices U^{mk} $(m, k \in \mathscr{N})$ by letting

$$U^{mk} = \mathscr{T}_m \mathscr{T}_{m+1} \mathscr{T}_{m+2} \cdots \mathscr{T}_{m+k-1}(T).$$

By Lemma 7 (b), (c) we obtain $\|U^{mk}\| \leq 1$ and $1 \geq u_{ji}^{m,\, k+1} \geq u_{ji}^{mk} \geq 0$, respectively. Let

$$u_{ji}^{(m)} = \lim_{k \to \infty} u_{ji}^{mk}.$$

We define a map $\mathscr{G}_m$ by

$$\mathscr{G}_m(T) = (u_{ji}^m).$$

We have $\|\mathscr{G}_m(T)\| \leq 1$, since $\|U^{mk}\| \leq 1$. By definition, $U^{mk} = \mathscr{T}_m(U_m^{m+1, k-1})$. Since the function

$$u_{ji}^{mk} = t_{ji}\left[1 - \sum_{a \in I_{m+1}} \left[t_{ja}\left(1 - \sum_{b \in J_{m+1}} \left(u_{ba}^{m+1, k-1}\right)^p\right)^{-1/p}\right]^q\right]^{-1/q}$$

is continuous and increasing in $u_{ba}^{m+1, k-1}$ and

$$\sum_{k \in J_{m+1}} u_{ba}^{(m+1)} < 1, \quad \sum_{a \in I_{m+1}} \cdots < 1 \quad (\|\mathscr{G}(T)\| \leq 1)$$

by passing to the limit as $k \to \infty$ we obtain

$$\mathscr{G}_m(T) = \mathscr{T}_m \mathscr{G}_{m+1}(T).$$

Proof of Lemma 1. Let all the entries of $T \in \mathscr{P}$ be maximal. We claim that all the entries of $\mathscr{G}_m(t)$ are maximal. Indeed we only need to show all the entries of $\mathscr{G}_1(T)$ are maximal, because of Lemma 7 (d) and the fact that

$$\mathscr{G}_1(T) = \mathscr{T}_1 \mathscr{T}_2 \cdots \mathscr{T}_{m-1} \mathscr{G}_m(T).$$

Suppose, to get a contradiction, that the entries of $(u_{ji}^{(1)}) = \mathscr{G}_m(T)$ are not maximal. Let $\alpha_{ji} \geq 1$ $((i, j) \in E_1)$ be such that all the entries of $(\alpha_{ji} u_{ji}^{(1)})$ are maximal. Put $\alpha_{ji} = 1$ for $(i, j) \notin E_1$. Since

$$\left(\alpha_{ji} u_{ji}^1\right) = \mathscr{T}_1 \mathscr{T}_2 \cdots \mathscr{T}_{m-1} \mathscr{G}_m\left(\left(\alpha_{ji} t_{ji}\right)\right),$$

by Lemma 7 (d), all the entries of $\mathscr{G}_m((\alpha_{ji}t_{ji}))$ are maximal, so

$$\left\|\mathscr{G}_m\left((\alpha_{ji}t_{ji})\right)\right\| = 1.$$

Since $0 \le \mathscr{R}_m((\alpha_{ji}t_{ji})) \le \mathscr{T}_m((\alpha_{ji}t_{ji}) \le \mathscr{G}_m((\alpha_{ji}t_{ji}))$ we have $\|\mathscr{R}_m((\alpha_{ji}t_{ji}))\| \le 1$ for all m. Hence $\|(\alpha_{ji}t_{ji})\| \le 1$. But this shows us that the entries of the matrix (t_{ji}) are not maximal. This contradiction proves our claim.

By Lemma 6 and Remark 1 there exist $(x_i^{(m)}) \in \mathscr{M}((u_{ji}^{(m)}))$ for all m. We assume that $x_{i_1}^{(m)} = 1$ for all m. We have $x_i^{(m)} = 0$ for $i \notin \bigcup_{n=1}^m I_n$. From Remark 1(c), if $m < m_1 < m_2$ then

$$x_i^{(m_1)} = x_i^{(m_2)} \neq 0 \quad \text{for } i \in I_m.$$

We put $x_i = x_i^{(m+1)}$ for $i \in I_m$. Now it is easy to see that $(x_i) \in \mathscr{M}(T)$.

Remark 2. Let all the entries of T be maximal. Then if $(x_i) \in \mathscr{M}(T)$, $(x_i^{(m)}) \in \mathscr{M}(\mathscr{G}_m(T))$ and $x_{i_1}^{(m)} = x_{i_1} = 1$ then $x_i = x_i^{(m)}$ for $i \in \bigcup_{n=1}^m I_n$. Moreover,

$$\left(\mathscr{G}_m(T)\right)_{j_0 i_0} = t_{j_0 i_0}^{1/p}\left(y_{j_0}/x_{i_0}\right)^{1/q}$$

for $(i_0, j_0) \in E_{2m-1}$, ((y_j) is a sequence corresponding to $(x_i) \in \mathscr{M}(T)$). Indeed, fix $(i_0, j_0) \in E_{2m-1}$. Let $H = \{(i, j$: the path joining the node i_0 and the edge i, j include the edge $i_0, j_0\}$. Note that $(i, j) \in H$. Put $A = \{i$: $(i, j) \in H\}$. We define a matrix T' by

$$t'_{ji} = \begin{cases} t_{ji} & \text{if } (i,j) \in H \\ \left(\mathscr{G}_m(T)\right)_{ji} & \text{otherwise.} \end{cases}$$

We have $\mathscr{G}_m(T) = \mathscr{G}_m(T')$. Let $(x_i) \in \mathscr{M}(T)$ and $(x_i^{(m)}) \in \mathscr{M}(\mathscr{G}_m(T))$. Put

$$x'_i = \begin{cases} x_i & \text{if } i \in A \cup \bigcup_{n=1}^m I_n \\ 0 & \text{otherwise.} \end{cases}$$

and $y'_j = \sum_i t'_{ji}x'_i$. It is easy to see that $(x'_i) \in \mathscr{M}(T')$. Let $j_1 \in J_{m-1}$ be such that $t_{j_1 i_0} \neq 0$. For $j \in J_m$ we denote a unique $i_j \in I_m$ such that $t_{ji_j} \neq 0$. We have $y_j^{(m)} = (\mathscr{G}_m(T))_{ji_j}x_{i_j}$ for $j \in J_m (x_{i_j} = x_{i_j}^{(m)})$. We have

$$\begin{aligned} x_{i_0}^{p-1} &= t_{j_1 i_0} y_{j_1}^{p-1} + \sum_{j \in J_m} \left(\mathscr{G}_m(T)\right)_{ji_0} y_j^{p-1} \\ &= t_{j_1 i_0} y_{j_1}^{p-1} + \sum_{j \in J_m} \left(\mathscr{G}_m(T)\right)_{ji_0}^p x_{i_0}^{p-1}. \end{aligned}$$

But when we consider the matrix T' we have

$$x_{i_0}^{p-1} = t_{j_1 i_0} y_{j_1}^{p-1} + \sum_{\substack{j \in J_m \\ j \neq j_0}} \left(\mathscr{G}_m(T)\right)_{j i_0}^{p} x_{i_0}^{p-1} + t_{j_0 i_0} y_{j_0}^{p-1}.$$

Hence

$$\left(\mathscr{G}_m(T)\right)_{j_0 i_0}^{p} x_0^{p-1} = t_{j_0 i_0} y_0^{p-1},$$

which ends the proof.

Proof of Lemma 2. We define matrices $U^{2,k} = \mathscr{T}_2 \mathscr{T}_3 \cdots \mathscr{T}_{k+1}(T)$, $(u_{ji}^{(2)}) = \mathscr{G}_2(T)$ and $S^{(k)}$ by

$$s_{ji}^{(k)} = \begin{cases} t_{ji} \dfrac{u_{ji}^{(2)}}{u_{ji}^{2,k}} & \text{if } (i,j) \in E_3 \\ t_{ji} & \text{if } (i,j) \in \bigcup\limits_{\substack{n=1 \\ n \neq 3}}^{2k+3} E_n \\ 0 & \text{otherwise}, \quad k \in \mathbf{N}. \end{cases}$$

We have

$$\mathscr{G}_2(T) = \mathscr{T}_2 \mathscr{T}_3 \cdots \mathscr{T}_{k+1}(S^{(k)})$$

By Lemma 7(d) and the claim in the proof of Lemma 1, all the entries of $S^{(k)}$ are maximal. Let $(x_i) \in \mathscr{M}(T)$, $(x_i^0) \in \mathscr{M}(\mathscr{G}_2(T))$, $(x_i^{(k)}) \in \mathscr{M}(S^{(k)})$ be such that $x_{i_1} = x_{i_1}^0 = x_{i_1}^{(k)} = 1$. By $(y_j), (y_j^0), (y_j^{(k)})$ we denote the corresponding sequences.

Fix $\varepsilon > 0$. Let $\varepsilon_i > 0$ be such that

$$\left(x_i^{p-1} - 2\varepsilon_i\right)^q > x_i^p - \varepsilon 2^{-i-1}, \quad i \in I_1 \cup I_2.$$

Put $I_1' = I_1 = \{i_1\}$. Choose J_1' a finite subset of J_1 such that

$$\sum_{j \in J_1'} t_{j i_1} y_j^{p-1} > x_{i_1}^{p-1} - \varepsilon_{i_1}.$$

Find $\delta_j > 0$ $(j \in J_1')$ such that

$$\sum_{j \in J_1'} t_{j i_1}\left(y_j - \delta_j\right)^{p-1} > x_{i_1}^{p-1} - 2\varepsilon_{i_1}.$$

Choose I_2'' a finite subset of I_2 such that

$$\sum_{i \in I_1' \cup I_2''} t_{ji} x_i > y_j - \delta_j \quad \text{for } j \in J_1'.$$

Let $N_0 = \max\{i \in I_1' \cup I_2'': t_{j_1 i} > 0\}$. Fix $N \geq N_0$. Note that

$$\sum_{i=1}^{N} t_{j_1 i} x_i = \sum_{i \in I_1' \cup I_2'} t_{j_1 i} x_i$$

where $I_2' = I_2'' \cup \{i \in I_2: t_{j_1 i} > 0,\ i \leq N\}$.
Choose J_2' a finite subset of J_2 such that

$$\sum_{j \in J_1' \cup J_2'} t_{ji} y_j^{p-1} > x_i^{p-1} - \varepsilon_i, \quad i \in I_2',$$

and find $\delta_j > 0$ ($j \in J_2'$) such that

$$\sum_{j \in J_1' \cup J_2'} t_{ji} \left(y_j - \sqrt[p-1]{\frac{u_{ji}^{(2)}}{t_{ji}} \delta_j} \right)^{p-1} > x_i^{p-1} - 2\varepsilon_i, \quad i \in I_2'.$$

Let I' be a finite subset of $I_2' \cup \{i \in I_3: t_{ji} \neq 0$ for some $j \in J_2'\}$. Since for all k,

$$\sum_{j \in J_2'} s_{ji}^{(k)} \left(\sum_{n \in I_2 \cup I_3} s_{jn}^{(k)} x_n^{(k)} \right)^{p-1} < x_i^{p-1} \quad (i \in I_2'),$$

there exists $M > 0$ such that

$$\left(S^k(\mathbf{z})\right)_j < M \quad (j \in J_2'),$$

where

$$z_i = \begin{cases} x_i & \text{for } i \in I' \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\sum_{i \in I'} t_{ji} x_i^{(k)} < M$ for $j \in J_2'$. Since $(S^{(k)}(\mathbf{z}))_j$ and $(T\mathbf{z})_j$ differ only for $j \in J_2'$ we have

$$\begin{aligned} \|S^{(k)}\mathbf{z}\|^p - \|T\mathbf{z}\|^p &= \sum_{j \in J_2'} \left|(S^{(k)}\mathbf{z})_j\right|^p - \left|(T\mathbf{z})_j\right|^p \\ &= \sum_{j \in J_2'} \left[\left(\sum_{i \in I'} s_{ji}^{(k)} x_i^{(k)} \right)^p - \left(\sum_{i \in I'} t_{ji} x_i^{(k)} \right)^p \right]. \end{aligned}$$

For $j \in J_2'$ we denote a unique $i_j \in I_2'$ such that $t_{ji_j} \neq 0$. Hence

$$\|S^k \mathbf{z}\|^p - \|T\mathbf{z}\|^p$$
$$\leq \sum_{j \in J_2'} \left\{ \left[\left(s_{ji_1}^{(k)} - t_{ji_1} \right) x_{i_j} + \sum_{i \in I'} t_{ji} x_i^{(k)} \right]^p - \left(\sum_{i \in I'} t_{ji} x^{(k)} \right)^p \right\}$$
$$\leq \sum_{j \in J_2'} pM^{p-1} \left(s_{ji_j}^{(k)} - t_{ji_j} \right) x_{i_j}.$$

Since $s_{ji}^{(k)} \Rightarrow t_{ji}$, there exists k such that

$$\|S^{(k)}\mathbf{z}\|^p - \|T\mathbf{z}\|^p < \varepsilon/2.$$

By Remarks 1(c) and 2 we have $x_i = x_i^0 = x_i^{(k)}$ for $i \in I_2$. Now consider $U^{(2)}$ and $T_a^{(k)} = \mathscr{T}_3 \mathscr{T}_4 \cdots \mathscr{T}_{k+1}(S^{(k)})$. Let $(x_i^{(a)}) \in \mathscr{M}(T_a^{(k)})$. Since $U^{(2)} = \mathscr{T}_2(T_a^{(k)})$, by Remark 2,

$$u_{ji}^{(2)} = \sqrt[p]{s_{ji}^{(k)}} \sqrt[q]{y_j^{(k)}/x_i}$$

for $(i, j) \in E_3$. Since $y_j^0 = u_{ji}^{(2)} x_i$ we get $u_{ji}^{(2)}(y_j^0)^{p-1} = s_{ji}^{(k)}(y_j^{(k)})^{p-1}$. And, since $\mathscr{G}_2(T) = \mathscr{T}_2 \mathscr{T}_3(T)$ we get $u_{ji}^{(2)}(y_j^0)^{p-1} = t_{ji} y_j^{p-1}$.

Now we consider the matrix $(s_{ji}^{(k)})$. We have

$$\sum_{j \in J_1'} s_{ji_1}^{(k)} \left(y_j^{(k)} \right)^{p-1} > \left(x_{i_1}^{(k)} \right)^{p-1} - \varepsilon_{i_1},$$

$$\sum_{j \in J_1'} s_{ji}^{(k)} \left(y_j^{(k)} - \delta_j \right)^{p-1} > \left(x_{i_1}^{(k)} \right)^{p-1} - 2\varepsilon_{i_1},$$

$$\sum_{i \in I_1' \cup I_2'} s_{ji}^{(k)} x_i^{(k)} \geq y_j^{(k)} - \delta_j, \quad j \in J_1',$$

$$\sum_{j \in J_1' \cup J_2'} s_{ji}^{(k)} \left(y_j^{(k)} \right)^{p-1} = \sum_{j \in J_1' \cup J_2'} t_{ji} y_j^{p-1} > \left(x_i^{(k)} \right)^{p-1} - \varepsilon_i, \quad i \in I_2',$$

$$\sum_{j \in J_1' \cup J_2'} s_{ji}^{(k)} \left(y_j^{(k)} - \delta_j \right)^{p-1} = \sum_{j \in J_1' \cup J_2'} s_{ji}^{(k)} \left(\sqrt[p-1]{\frac{t_{ji}}{s_{ji}^{(k)}}}\, y_j - \delta_j \right)^{p-1}$$
$$= \sum_{j \in J_1' \cup J_2'} t_{ji} \left(y_j - \sqrt[p-1]{\frac{s_{ji}^{(k)}}{t_{ji}}}\, \delta_j \right)^{p-1}$$
$$> \left(x_i^{(k)} \right)^{p-1} - 2\varepsilon_i, \quad i \in I_2',$$

Now choose I'_3 a finite subset of I_3 such that

$$\sum_{i\in I'_1\cup I'_2} s_{ji}^{(k)}x_i^{(k)} > y_j^{(k)} - \delta_j, \quad j \in J'_2.$$

Let $\varepsilon_i > 0$, $i \in I_1 \cup I_2$, be such that

$$\left(\left(x_i^{(k)}\right)^{p-1} - 2\varepsilon_i\right)^q > \left(x_i^{(k)}\right)^p - \varepsilon_2^{-i-1}$$

Note that the above inequality holds also for $i \in I'_1 \cup I'_2$. Choose J'_3 a finite subset of J_3 such that

$$\sum_{j\in J'_2\cup J'_3} s_{ji}^{(k)}\left(y_j^{(k)}\right)^{p-1} > \left(x_i^{(k)}\right)^{p-1} - \varepsilon_i, \quad i \in I'_3.$$

Find $\delta_j > 0$ ($j \in J'_3$) such that

$$\sum_{j\in J'_2\cup J'_3} s_{ji}^{(k)}\left(y_j^{(k)} - \delta_j\right)^{p-1} > \left(x_i^{(k)}\right)^{p-1} - 2\varepsilon_i, \quad i \in I'_3.$$

Choose I'_4 a finite subset of I_4 such that

$$\sum_{i\in I'_3\cup I'_4} s_{ji}^{(k)}x_i^{(k)} > y_j^{(k)} - \delta_j, \quad j \in J'_3.$$

We continue the above process for the matrix $S^{(k)}$ to get a finite sequence $I'_1, J'_1, I'_2, \dots, J'_{2k+2}$. Let

$$I = \bigcup_{n=1}^{2k+2} I'_n \quad \text{and} \quad J = \bigcup_{n=1}^{2k+2} J'_n.$$

And let $\mathbf{u}^{(N)} \in l^p$ be defined by

$$u_i^{(N)} = \begin{cases} x_i^{(k)} & \text{if } i \in I \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{v} = (v_j) = S^{(k)}\mathbf{u}^{(N)}$. For $j \in J$ we have

$$v_j = \sum_{i=1}^{\infty} s_{ji}^{(k)}u_i^{(N)} \geq \sum_{i\in I} s_{ji}^{(k)}x_i^{(k)} > y_j^{(k)} - \delta_j.$$

Thus

$$\sum_{j=1}^{\infty} s_{ji}^{(k)} v_j^{p-1} \geq \sum_{j\in J} s_{ji}^{(k)}\left(y_j^{(k)} - \delta_j\right)^{p-1} > \left(x_i^{(k)}\right)^{p-1} - 2\varepsilon_i$$

for $i \in I$. Therefore we obtain

$$\begin{aligned}\left\|\left(S^{(k)}\right)^{*}\left(S^{(k)}\mathbf{u}^{(N)}\right)^{p-1}\right\|_q^q &= \left\|\left(S^{(k)}\right)^{*}\mathbf{v}^{p-1}\right\|_q^q \\ &\geq \sum_{i\in I}\left[\sum_{j\in J} s_{ji}^{(k)} v_j^{p-1}\right]^q \\ &> \sum_{i\in I}\left[\left(x_i^{(k)}\right)^{p-1} - 2\varepsilon_i\right]^q \\ &> \sum_{i\in I}\left[\left(x_i^{(k)}\right)^{p} - \frac{\varepsilon}{2^{i+1}}\right] = \|\mathbf{u}^{(N)}\|_p^p - \frac{\varepsilon}{2}.\end{aligned}$$

Therefore

$$\|S^{(k)}\mathbf{u}^{(N)}\|_p^p > \|\mathbf{u}^{(N)}\|_p^p - \frac{\varepsilon}{2}.$$

We have

$$\|S^{(k)}\mathbf{u}^{(N)}\|_p^p - \|T\mathbf{u}^{(N)}\|_p^p = \|S^{(k)}\mathbf{z}\|_p^p - \|T\mathbf{z}\|_p^p < \frac{\varepsilon}{2}.$$

Thus

$$\|Tu^{(N)}\|_p^p > \|u^{(N)}\|_p^p - \varepsilon.$$

5. Additional remarks on extreme positive l^p-contractions

LEMMA 8. *Let $T \in \mathscr{P}$ and let the graph $G(T)$ be a tree. If all the entries of T are maximal then $(x_i) \in \mathscr{M}(T)$ is unique up to a multiplicative constant.*

Proof. Suppose, to get a contraction, that there exist two different sequences $(x_i'), (x_i'') \in \mathscr{M}(T)$ such that $x_{i_1}' = x_{i_1} = 1$. Then the corresponding sequences (y_j') and (y_j'') differ for some j_1. We may and do assume that $j_1 \in J_1$. Suppose $y_{j_1}'' < y_{j_1}'$. Let $\varepsilon > 0$ be such that

$$\left(\frac{t_{j_1 i_1}}{t_{j_1 i_1} + \varepsilon}\right)^p = t_{j_1 i_1} y_{j_1}''^{\,p-1} + \sum_{\substack{j\in J_1 \\ j\neq j_1}} t_{j i_1} y_j'^{\,p-1}$$

We define a new matrix (t'_{ji}) by

$$t_{ji} = \begin{cases} t_{ji}\dfrac{t_{j_1i_1} + \varepsilon}{t_{j_1i_1}} & \text{if } (i, j) \in E_1 \\ t_{ji} & \text{otherwise.} \end{cases}$$

We have $t'_{ji} \geq t_{ji}$ and $t'_{j_1i_1} > t_{j_1i_1}$. Put $A = \{k$: the path joining nodes k and i_1 includes edge $i_1 j_1\}$. It is easy to see that $(x_i) \in \mathscr{M}((t'_{ji}))$ where

$$x_i = \begin{cases} \dfrac{t_{j_1i_1}}{t_{j_1i_1} + \varepsilon} & \text{for } i = i_1 \\ x''_i & \text{for } i \in A \\ x'_i & \text{otherwise.} \end{cases}$$

Hence, by Corollary 1, $\|(t'_{ji})\| \leq 1$. This contradicts the fact that all the entries of T are maximal. Therefore there are not two linearly independent elements of $\mathscr{M}(T)$.

Example 1. It should be pointed out that for some $T \in \mathscr{P}$ there are more that one linearly independent elements in $\mathscr{M}(T)$ (even if the graph $G(T)$ has no cycle). We define a sequence (a_n) by

$$a_1 = (2^{p+1} - 2)^{1/(p-1)}, \quad a_2 = 2a_1 - 1,$$

$$a_{2n+1} = \left(2a_{2n}^{p-1} - a_{2n-1}^{p-1}\right)^{1/(p-1)}, a_{2n+2} = 2a_{2n-1} - a_{2n} \quad (n \in \mathbf{N}).$$

The sequence (a_n) is increasing. Let $T = (t_{ji})$ be defined as follows:

$$t_{11} = t_{21} = t_{31} = 1/4,$$

$$t_{i,i+1} = t_{i+3,i+1} = 1/2, \quad i \in \mathbf{N},$$

$$t_{ji} = 0 \quad \text{otherwise.}$$

Let

$$x' = \begin{cases} 2 & \text{if } i = 1 \\ a_{2k} & \text{if } i = 3k + 1,\ k \in \mathbf{N} \\ 1 & \text{otherwise} \end{cases}$$

and

$$x_i'' = \begin{cases} 2 & \text{if } i = 1 \\ a_{2k} & \text{if } i = 3k,\ k \in \mathbf{N} \\ 1 & \text{otherwise}. \end{cases}$$

Now it is easy to see that $x', x'' \in \mathscr{M}(T)$.

PROPOSITION 3. *Let $T \in \mathscr{P}$ and let the graph $G(T)$ be a tree. Suppose that all entries of T are maximal. If $t_{j_1 i_1} > 0$, $t_{j_2 i_2} > 0$ and $\alpha \in (0, t_{j_1 i_1})$ then there exists $\beta > 0$ such that $\|(t'_{ji})\| = 1$ and all entries of the matrix (t'_{ji}) are maximal, where $t'_{ji} = t_{ji} - \alpha\delta_{jj_1}\delta_{ii_1} + \beta\delta_{jj_2}\delta_{ii_2}$.*

Proof. Because the graph $G(T)$ is connected we may restrict our attention to the case when $t_{j_1 i_1}$ and $t_{j_2 i_2}$ belong to the same row or column. For instance assume that $j_1 = j_2$. By Lemma 5 there exists $(x_i) \in \mathscr{M}(T)$. Fix $\alpha \in (0, t_{j_1 i_1})$. Choose $\beta > 0$ such that

$$x_{i_1} t_{j_1 i_1} + x_{i_2} t_{j_1 i_2} = \eta^q x_{i_1} t_{j_1 i_1} + \xi^q x_{i_2} t_{j_1 i_2}$$

where

$$\eta^{p-1} = \left(t_{j_1 i_1} - \alpha\right)/t_{j_1 i_1}, \quad \xi^{p-1} = \left(t_{j_1 i_2} + \beta\right)/t_{j_1 i_2}.$$

Let

$$A = \{k\colon \text{the path joining nodes } k \text{ and } j_1 \text{ include the edge } i_1 j_1\},$$

and

$$B = \{k\colon \text{the path joining nodes } k \text{ and } j_1 \text{ include the edge } i_2 j_1\}.$$

Note that $i_1 \in A$, $i_2 \in B$. It is easy to see that $(x_i') \in \mathscr{M}((t_{ji}))$, where

$$x_i' = \begin{cases} \eta x_i & \text{if } i \in A \\ \xi x_i & \text{if } i \in B \\ x_i & \text{otherwise}. \end{cases}$$

Thus $\|(t'_{ji})\| \le 1$ (by Corollary 1). This construction shows us that if some entry of a matrix is not maximal then no entry is maximal. If some entry of (t'_{ji}) is not maximal then doing the reserve operation to that presented above we get that no entry of (t_{ji}) is maximal. Hence the entries of (t'_{ji}) are maximal and $\|(t'_{ji})\| = 1$.

As an immediate consequence we get the following interesting fact.

COROLLARY 2. *For a positive contraction whose graph is a connected tree either all entries are maximal or no entry is maximal.*

Example 2. For $c > 0$ we define an operator T_c by

$$T_c(u_i) = (cu_1, (u_1 + u_2)/2, (u_2 + u_3)/2, (u_3 + u_4)/2, \dots), \quad (u_i) \in l^p.$$

Consider sequences $(x_i), (y_j)$ such that

$$\sum_{i=1}^{\infty} t_{ji} x_i = y_j, \ \sum_{j=1}^{\infty} t_{ji} y_j^{p-1} = x_i^{p-1} \quad \text{with } x_1 = 1.$$

We have $y_1 = c$, $y_2 = 2(1 - c^p)$,

$$x_{n+1} = 2y_{n+1} - x_n \quad (n \geq 1),$$
$$y_{n+1}^{p-1} = 2x_n^{p-1} - y_n^{p-1} \quad (n \geq 2).$$

Let $a_{2n-1} = y_n$ and $a_{2n} = x_n$ $(n \geq 1)$. We have

$$a_{2n+2} - a_{2n+1} = a_{2n+1} - a_{2n} \quad (n \geq 1)$$

and

$$a_{2n+1}^{p-1} - a_{2n}^{p-1} = a_{2n}^{p-1} - a_{2n-1}^{p-1} \quad (n \geq 2).$$

If $c = \sqrt[p]{1/2}$ then $a_n = 1$ for $n \geq 2$. And if $c < \sqrt[p]{1/2}$ then $a_3 - a_2 > 0$ and $a_4 - a_3 > 0$, so (a_n) is increasing. Therefore for $c \in (0, \sqrt[p]{1/2}\,]$ the set $\mathcal{M}(T_c)$ is non-empty and $\mathcal{M}(T_c)$ has exactly one sequence (up to a multiplicative constant). Obviously entries of T_c for $c \in (0, \sqrt[p]{1/2})$ are not maximal, so $T_c \notin \operatorname{ext} \mathcal{P}$. Thus we get an example of non-extreme operator such that an element of $\mathcal{M}(T)$ is unique. Note that the condition (ii) for the operator T_c $(c \in (0, \sqrt[p]{1/2}))$ holds.

Suppose that $\|T_c\| \leq 1$. Then

$$c^p + \sum_{k=1}^{n} \left(\frac{2k+1}{2n}\right)^p = \|T_c \mathbf{u}\| \leq \|\mathbf{u}\| = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^p$$

where

$$\mathbf{u} = \left(1, \frac{n-1}{n}; \frac{n-2}{n}, \dots, \frac{1}{n}, 0, 0, 0, \dots\right).$$

Using the mean-value theorem we obtain

$$\left(k+\frac{1}{2}\right)^p - k^p = \frac{p}{2}\left(k+\frac{1}{2}\xi_k\right)^{p-1} \quad \text{where } 0<\xi<1.$$

Hence

$$c^p \le \sum_{k=1}^{n} \frac{(k+1/2)^p - k^p}{n^p} = \frac{p}{2}\left[\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k+\xi_k/2}{n}\right)^{p-1}\right];$$

when n tends to infinity we obtain

$$c^p \le \frac{p}{2}\int_0^1 t^{p-1}\,dt = \frac{1}{2}$$

Therefore we have $\|T_c\| > 1$ for $c > \sqrt[p]{1/2}$. Hence the entries of T for $c = \sqrt[p]{1/2}$ are maximal.

For $c > \sqrt[p]{1/2}$ we have $a_3 - a_2 < 0$, $a_4 - a_3 < 0$, so (a_n) is decreasing. Because of results presented before there exists n_0 such that $a_n < 0$ for all $n \ge n_0$.

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