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BAILEY CHAINS AND GENERALIZED LAMBERT SERIES: I. FOUR IDENTITIES OF RAMANUJAN

BY

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1. Introduction

In this paper we shall examine the following four identities of Ramanujan [14; p. 264, eqs. (6)–(9), eq. (6) corrected]:

$$\frac{1}{\phi^2(-q)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2q^{\binom{n+1}{2}}}{1+q^n}=\sum_{n=1}^{\infty}\frac{q^{n(2n-1)}(1+q^{2n-1})}{(1-q^{2n-1})^2}; \tag{1.1}$$

$$\frac{1}{\psi^2(q)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)^2q^{n(n-1)}(1+q^{2n-1})}{1-q^{2n-1}} = 1+8\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(n+1)}}{(1+q^n)^2}; \tag{1.2}$$

$$\frac{1}{\psi^2(q)}\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)q^{n(n+1)-1}}{(1-q^{2n-1})^2} = \sum_{n=1}^{\infty}\frac{(-1)^nnq^{n^2}(1+q^{2n})}{1-q^{2n}}; \tag{1.3}$$

$$\frac{1}{\phi^2(-1)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^{\binom{n+1}{2}}(1-q^n)}{(1+q^n)^2}=\sum_{n=1}^{\infty}\frac{nq^{\binom{n+1}{2}}}{1-q^n}; \tag{1.4}$$

where [3; p. 23, Cor. 2.10]

$$\phi(-q)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}=\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1+q^n)}, \tag{1.5}$$

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and

$$(1.6)\qquad \psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^{2n-1})}.$$

Throughout this paper we shall refer to the left-hand side of (1.i) as $L_i(q)$ and to the right-hand side as $R_i(q)$ $(1 \leqq i \leqq 4)$. These four identities are those listed as (6)–(9) in [14; p. 264] except (as noted earlier) we have corrected (1.1) and we have moved $\psi^2(q)$ or $\phi^2(-q)$ to the left-hand side of each identity.

While these identities appear to be closely related to the first five identities of [14; p. 264] and to other results of Ramanujan [1], they seem to be much deeper; at least, the proofs given here require extensive and intricate preparation. It is doubtful that our approach resembles what Ramanujan had in mind at all. The key elements are: (1) series rearrangement, (2) Bailey pairs, (3) q-series transformations including Bailey's nonterminating extension of the q-analog of Whipple's Theorem [9; p. 69, eq. (3)]. Of these topics Ramanujan was a master of (1) and could easily handle (2) in any particular case. However, the formula of Bailey alluded to above (which is crucial to our treatment of (1.3) and (1.4)) was probably not known to Ramanujan.

We should also note that these identities closely resemble formulas of G. Humbert [11] for generating functions $\mathscr{A}(q)$ and $\mathscr{B}(q)$ of class-number related to binary quadratic forms. In particular $R_4(q^2)$ is an instance of the generalized Lambert series in [17; p. 6, eq. (3.02), $x = 0$] while $R_3(-q)$ is an instance of the generalized Lambert series in [17; p. 6, eq. (3.07), $x = 0$]. These facts suggest that the methods developed here may throw some new light on generating functions for class numbers. We plan to return to this question in a subsequent paper in this series.

In Section 2 we consider the necessary background and extensions of classical q-hypergeometric series. In Section 3 we shall develop the Bailey pairs necessary to treat $L_i(q)$ $(1 \leqq i \leqq 4)$. In Section 4 we finally prove (1.1)–(1.4). We close with a look at some of the topics we propose to treat in later work.

2. q-Hypergeometric series

There are several formulas in the literature that we require. These are all identities for certain q-hypergeometric series:

$$(2.1)\qquad {}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r; q, t \\ b_1, \ldots, b_s \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n t^n}{(q, b_1, \ldots, b_s; q)_n},$$

where

$$(2.2) \qquad (A_1, A_2, \ldots, A_r; q)_n = \prod_{i=1}^{r} \prod_{j=0}^{n-1} (1 - A_i q^j),$$

and

$$(A_1, A_2, \ldots, A_r; q)_\infty = \prod_{i=1}^{r} \prod_{j=0}^{n-1} (1 - A_i q^j).$$

The ratio test shows that (2.1) converges absolutely provided $|t| < 1$, $|q| < 1$. Of course the b_i must not be nonpositive integral powers of q to guarantee that each term of the series is well defined. In all our applications the condition $|q| < 1$ will be required for convergence.

We being with Bailey's nonterminating extension of the q-analog of Whipple's theorem [9; p. 69, eq. (3)]

(2.4)

$$
\begin{aligned}
&{}_8\phi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, d, e, f, g, h; q, \dfrac{a^2q^2}{defgh} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{d}, \dfrac{aq}{e}, \dfrac{aq}{f}, \dfrac{aq}{g}, \dfrac{aq}{h} \end{matrix}\right) \\
&= \frac{(aq, aq/fg, aq/fh, aq/gh; q)_\infty}{(aq/f, aq/g, aq/h, aq/fgh; q)_\infty} {}_4\phi_3\left(\begin{matrix} aq/de, f, g, h; q, q \\ \dfrac{aq}{d}, \dfrac{aq}{e}, \dfrac{fgh}{a} \end{matrix}\right) \\
&+ \frac{(aq, aq/de, f, g, h, a^2q^2/(dfgh), a^2q^2/(efgh); q)_\infty}{(aq/d, aq/e, aq/f, aq/g, aq/h, a^2q^2/(defgh), fgh/(aq); q)_\infty} \\
&\times {}_4\phi_3\left(\begin{matrix} aq/gh, aq/fh, aq/fg, a^2q^2/(defgh); q, q \\ aq^2/fgh, a^2q^2/(dfgh), a^2q^2/(efgh) \end{matrix}\right).
\end{aligned}
$$

If in (2.4) we replace h by q^{-N} where N is a nonnegative integer, then the second summand on the right-hand side vanishes due to the argument h in the infinite product portion of the numerator. This yields Watson's q-analog of Whipple's theorem [9; p. 69, eq. (2)]

$$
\begin{aligned}
(2.5) \qquad &{}_8\phi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, d, e, f, g, q^{-N}; q, \dfrac{a^2q^{2+N}}{defg} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{d}, \dfrac{aq}{e}, \dfrac{aq}{f}, \dfrac{aq}{g}, aq^{N+1} \end{matrix}\right) \\
&= \frac{(aq, aq/fg; q)_N}{(aq/f, aq/g; q)_N} {}_4\phi_3\left(\begin{matrix} aq/de, f, g, q^{-N}; q, q \\ aq/d, aq/e, fgq^{-N}/a \end{matrix}\right).
\end{aligned}
$$

Next we have a formula which can be deduced from (2.5) namely the limiting form of Jackson's Theorem [16; p. 96, eq. (3.3.1.3)]

$$(2.6)\qquad {}_6\phi_5\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, d, e, f; q, \dfrac{aq}{def} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{d}, \dfrac{aq}{e}, \dfrac{aq}{f} \end{matrix}\right) = \frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty}.$$

We shall also need in our proof of (1.1) a three term relation among ${}_3\phi_2$ series [15; p. 175, eq. (10.2)]

$$(2.7)\qquad {}_3\phi_2\left(\begin{matrix} a, b, c; q, \dfrac{ef}{abc} \\ e, f \end{matrix}\right) = \frac{(e/a, e/b; q)_\infty}{(e, e/ab; q)_\infty}\,{}_3\phi_2\left(\begin{matrix} a, b, f/c; q, q \\ \dfrac{qab}{e}, f \end{matrix}\right) + \frac{(a, b, f/c, ef/ab; q)_\infty}{(e, ab/e, f, ef/abc; q)_\infty}\,{}_3\phi_2\left(\begin{matrix} e/a, e/b, \dfrac{ef}{abc}; q, q \\ \dfrac{eq}{ab}, \dfrac{ef}{ab} \end{matrix}\right).$$

Finally to our q-hypergeometric compendium we add Heine's transformation, its iterates, and the q-analog of Gauss's summation [3; pp. 38–39, eq. 20]

$$(2.8)\qquad {}_2\phi_1\left(\begin{matrix} a, b; q, t \\ c \end{matrix}\right) = \frac{(b, at; q)_\infty}{(c, t; q)_\infty}\,{}_2\phi_1\left(\begin{matrix} c/b, t; q, b \\ at \end{matrix}\right)$$

$$(2.9)\qquad = \frac{(c/b, bt; q)_\infty}{(c, t; q)_\infty}\,{}_2\phi_1\left(\begin{matrix} abt/c, b; q, \dfrac{c}{b} \\ bt \end{matrix}\right)$$

$$(2.10)\qquad = \frac{(abt/c; q)_\infty}{(t; q)_\infty}\,{}_2\phi_1\left(\begin{matrix} c/a, c/b; q, \dfrac{abt}{c} \\ c \end{matrix}\right)$$

$$(2.11)\qquad {}_2\phi_1\left(\begin{matrix} a, b; q, c/ab \\ c \end{matrix}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}.$$

Our next identity follows from a combination of (2.4) and (2.6) and is essential in our proof of (1.3).

LEMMA 1.

$$(2.12)\quad -\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(f,g;q)_n\left(\frac{aq}{fg}\right)^n}{(1-aq^n)(1-q^n)(aq/f,aq/g;q)_n}$$
$$+\sum_{n=1}^{\infty}\frac{(1-aq^{2n})(d,e,f,g;q)_n\left(\frac{a^2q^2}{defg}\right)^n}{(1-aq^n)(1-q^n)(aq/d,aq/e,aq/f,aq/g;q)_n}$$
$$=\sum_{n=1}^{\infty}\frac{(aq/de,f,g;q)_nq^n}{(1-q^n)(aq/d,aq/e,fg/a;q)_n}$$
$$+\frac{(aq,aq/de,f,g,q,a^2q^2/(dfg),a^2q^2/(efg);q)_\infty}{(aq/d,aq/e,aq/f,aq/g,a^2q^2/(defg),fg/(aq);q)_\infty}$$
$$\times{}_4\phi_3\left(\begin{matrix}aq/g,aq/f,aq/fg,a^2q^2/(defg);q,q\\ aq^2/fg,a^2q^2/(dfg),a^2q^2/(efg)\end{matrix}\right).$$

Proof. Subtract

$$\frac{(aq,aq/fg,aq/fh,aq/gh;q)_\infty}{(aq/f,aq/g,aq/h,aq/fgh;q)_\infty}$$

from both sides of (2.4). The resulting identity is schematically

(2.13)
$$1-\frac{(aq,aq/fg,aq/fh,aq/gh;q)_\infty}{(aq/f,aq/g,aq/h,aq/fgh;q)_\infty}+{}_8\phi_7^*(\ )$$
$$=\frac{(aq,aq/fg,aq/gh,aq/gh;q)_\infty}{(aq/f,aq/g,aq/h,aq/fgh;q)_\infty}{}_4\phi_3^*(\ )$$
$$+\frac{(aq,aq/de,f,g,h,a^2q^2/(dfgh),a^2q^2/(efgh);q)_\infty}{(aq/d,aq/e,aq/f,aq/g,aq/h,a^2q^2/(defg),fgh/(aq);q)_\infty}{}_4\phi_3(\ ).$$

The asterisks on the ${}_8\phi_7(\ )$ and ${}_4\phi_3(\ )$ mean that the sums start from $n=1$ instead of $n=0$.

We now use (2.6) to write

$$(2.14)\qquad 1-\frac{(aq,aq/fg,aq/fgh,aq/gh;q)_\infty}{(aq/f,aq/g,aq/h,aq/fh;q)_\infty}$$
$$= -{}_6\phi_5^*\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, f, g, h; q, \dfrac{aq}{fgh} \\ \sqrt{a}, -\sqrt{a}, aq/f, aq/g, aq/h \end{matrix}\right),$$

and we substitute this $-{}_6\phi_5^*$ into the left-hand side of (2.13). Now every term in the resulting identity has $(1-h)$ as a factor. We divide both sides by $(1-h)$ and then we set $h=1$. The result is precisely (2.12). □

LEMMA 2.

$$(2.15)\qquad \sum_{n=1}^{\infty}\frac{(1-aq^{2n})}{(1-aq^n)(1-fq^n)}\frac{(f;q)_n a^n q^{\binom{n+1}{2}}(-1)^{n-1}}{(aq/f;q)_n f^n}$$
$$+\sum_{n=1}^{\infty}\frac{(1-aq^{2n})}{(1-aq^n)(1-q^n)}\frac{(d,e,f;q)_n(-1)^n q^{\binom{n+1}{2}+n}a^{2n}}{(aq/d,aq/e,aq/f;q)_n(def)^n}$$
$$=\sum_{n=1}^{\infty}\frac{(aq/de,f;q)_n\left(\frac{aq}{f}\right)^n}{(1-q^n)(aq/d,aq/e;q)_n}$$

Proof. In Lemma 1 we replace g by q^{-N} where N is a nonnegative integer. The second term on the right of (2.12) consequently vanishes. We then let $N\to\infty$ and the result is (2.15). □

This completes our q-hypergeometric arsenal.

3. Bailey chains

Our object in this section is to derive formulas for each of the $L_i(q)$ that eliminate the appearances of $\phi^2(-q)$ and $\psi^2(q)$. Our first step is to recall a weak version of Bailey's Lemma [6; pp. 25–26, eq. (3.27) and then $n\to\infty$ in eqs. (3.28)–(3.30)].

If for each $n \geqq 0$,

$$(3.1)\qquad \beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}},$$

then

$$(3.2)\qquad \sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\beta_n$$
$$=\frac{(aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/\rho_1\rho_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1,aq/\rho_2;q)_n}$$

subject to the convergence of the infinite series and products. In each relevant instance we need only $|q|<1$ as always. The sequences α_n,β_n are said to form a Bailey pair if they satisfy (3.1).

LEMMA 3. *With $a=1$ in (3.1), then*

$$(3.3)\qquad \alpha_n=\begin{cases}(-1)^n\left(z^nq^{\binom{n}{2}}+z^{-n}q^{\binom{n+1}{2}}\right), & n>0,\\ 1, & n=0,\end{cases}$$

and

$$(3.4)\qquad \beta_n=\frac{(z,q/z;q)_n}{(q;q)_{2n}}$$

form a Bailey pair.

Proof. We must verify (3.1) with $a=1$.

$$(3.5)\qquad \sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(q;q)_{n+r}}$$
$$=\frac{1}{(q;q)_n^2}+\sum_{r=1}^{n}\frac{(-1)^r\left(z^rq^{\binom{r}{2}}+z^{-r}q^{\binom{r+1}{2}}\right)}{(q;q)_{n-r}(q;q)_{n+r}}$$
$$=\sum_{r=-n}^{n}\frac{(-1)^rz^rq^{\binom{r}{2}}}{(q;q)_{n-r}(q;q)_{n+r}}$$
$$=\frac{(z,q/z;q)_n}{(q;q)_{2n}}$$

by [13; p. 75] (cf. [10]), which is the desired β_n. □

We now differentiate (3.3) and (3.4) with respect to z multiply by -1 and then set $z=1$. This operation preserves the fact that the results again form a

Bailey pair:

$$\alpha_n = (-1)^{n-1} n q^{\binom{n}{2}}(1-q^n), \tag{3.6}$$

$$\begin{aligned}\beta_n &= -\left[\frac{d}{dz}\frac{(1-z)(zq;a)_{n-1}(q/z;q)_n}{(q;q)_{2n}}\right]_{z=1} \\ &= \begin{cases} 0, & n=0, \\ \dfrac{(q;q)_{n-1}(q;q)_n}{(q;q)_{2n}}, & n>0. \end{cases}\end{aligned} \tag{3.7}$$

LEMMA 4.

$$\begin{aligned} L_4(q) &= \sum_{j=1}^{\infty} \frac{(-q,-q,q;q)_{j-1}(q;q)_j q^j}{(q;q)_{2j}} \\ &= \sum_{j=1}^{\infty} \frac{(-q,-q,q;q)_{j-1}(q;q)_j q^j}{(q,-q,q^{1/2},-q^{1/2};q)_j}. \end{aligned} \tag{3.8}$$

Proof. Set $a=1$, $\rho_1=\rho_2=-1$ in (3.2) and insert the Bailey pair (3.6) and (3.7). After dividing both sides by 4, this yields

$$\begin{aligned} &\sum_{n=1}^{\infty}(-q;q)_{n-1}^2 q^n \frac{(q;q)_{n-1}(q;q)_n}{(q;q)_{2n}} \\ &\quad = \frac{1}{\phi^2(-q)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1} n q^{\binom{n+1}{2}}(1-q^n)}{(1+q^n)^2} \\ &\quad = L_4(q). \end{aligned}$$

□

We now apply the operator

$$-\frac{d}{dz}z\frac{d}{dz}$$

to the Bailey pair in Lemma 3 and then set $z=1$. The result is a new Bailey pair $(\alpha_n^{(1)}, \beta_n^{(1)})$:

$$\begin{aligned}\alpha_n^{(1)} &= -\left[(-1)^n\left(n^2 z^{n-1} q^{\binom{n}{2}} + n^2 z^{-n-1} q^{\binom{n+1}{2}}\right)\right]_{z=1} \\ &= (-1)^{n-1} n^2 q^{\binom{n}{2}}(1+q^n). \end{aligned} \tag{3.9}$$

Note that

$$(3.10)\quad \left[\frac{d}{dz}z\frac{d}{dz}(1-z)F(z)\right]_{z=1} = \left[\frac{d}{dz}(z(1-z)F'(z) - zF(z))\right]_{z=1} = -F(1) - 2F'(1).$$

Therefore $\beta_0^{(1)} = 0$ and for $n > 0$

$$(3.11)\quad \beta_n^{(1)} = -\left[\frac{d}{dz}z\frac{d}{dz}(1-z)\frac{(zq;q)_{n-1}(q/z;q)_n}{(q;q)_{2n}}\right]_{z=1}$$
$$= \frac{(q;q)_{n-1}(q;q)_n}{(q;q)_{2n}} + 2\frac{(q;q)_{n-1}(q;q)_n}{(q;q)_{2n}}\left\{\sum_{j=1}^{n-1}\frac{-q^j}{1-q^j} + \sum_{j=1}^{n}\frac{q^j}{1-q^j}\right\}$$
$$= \frac{(q;q)_{n-1}(q;q)_n}{(q;q)_{2n}} + 2\frac{(q;q)_{n-1}(q;q)_n}{(q;q)_{2n}}\frac{q^n}{1-q^n}$$
$$= \frac{(q;q)_{n-1}^2}{(1-q^n)(q;q)_{2n-1}}.$$

LEMMA 5.

$$(3.12)\quad L_1(q) = \sum_{n=1}^{\infty}\frac{(-q,-q,q,q;q)_{n-1}q^n}{(1-q^n)(q;q)_{2n-1}}.$$

Proof. Set $a = 1$, $\rho_1 = \rho_2 = -1$ in (3.2) and insert the Bailey pair (3.9) and (3.11). After dividing both sides by 4 this yields

$$\sum_{n=1}^{\infty}(-q;q)_{n-1}^2 q^n \frac{(q;q)_{n-1}^2}{(1-q^n)(q;q)_{2n-1}} = \frac{1}{\phi^2(-q)}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2 q^{\binom{n+1}{2}}}{(1+q^n)} = L_1(q).$$

□

To treat $L_2(q)$ and $L_3(q)$ we require a further Bailey pair:

LEMMA 6. *With q replaced by q^2 in* (3.1) *and $a = q^2$, then*

$$(3.13)\quad \alpha_n = (-1)^{n-1}\left(z^{n+1}q^{n^2+n} - z^{-n}q^{n^2+n}\right)/(1-q^2)$$

and

$$\beta_n = \frac{(z;q^2)_{n+1}(q^2/z;q^2)_n}{(q^2;q^2)_{2n+1}} \tag{3.14}$$

Proof. We must verify (3.1) with q replaced by q^2 and $a = q^2$.

$$\begin{aligned} &\sum_{r=0}^{n} \frac{\alpha_r}{(q^2;q^2)_{n-r}(q^4;q^2)_{n+r}} \\ &= \sum_{r=0}^{n} \frac{(-1)^r\left(z^{-r}q^{r^2+r} - z^{r+1}q^{r^2+r}\right)}{(q^2;q^2)_{n-r}(q^2;q^2)_{n+r+1}} \\ &= \sum_{r=-n-1}^{n} \frac{(-1)^r z^{-r}q^{r^2+r}}{(q^2;q^2)_{n-r}(q^2,q^2)_{n+r+1}} \\ &= \frac{(z;q^2)_{n+1}(q^2/z;q^2)_n}{(q^2;q^2)_{2n+1}} \end{aligned} \tag{3.15}$$

by [13; p. 75] (cf. [10]), which is the desired β_n. □

We now differentiate (3.13) and (3.14) with respect to z, and then set $z = 1$. As before we still have a Bailey pair:

$$\alpha_n^{(3)} = (-1)^{n-1}(2n+1)q^{n^2+n}\big/(1-q^2) \tag{3.16}$$

$$\begin{aligned} \beta_n^{(3)} &= \left[\frac{d}{dz}(1-z)\frac{(zq^2,q^2/z;q^2)_n}{(q^2;q^2)_{2n+1}}\right]_{z=1} \\ &= -\frac{(q^2,q^2;q^2)_n}{(q^2;q^2)_{2n+1}}. \end{aligned} \tag{3.17}$$

LEMMA 7.

$$L_3(q) = -\frac{q}{1-q^2}\,{}_4\phi_3\left(\begin{matrix} q, q, q^2, q^2; q^2, q^2 \\ -q^2, q^3, -q^3 \end{matrix}\right). \tag{3.18}$$

Proof. Replace q by q^2 in (3.2), then set $a = q^2$, $\rho_1 = \rho_2 = q$. Finally insert the Bailey pair (3.16) and (3.17). This yields after multiplication by q:

$$-\sum_{n=0}^{\infty} (q;q^2)_n^2 q^{2n+1} \frac{(q^2;q^2)_n^2}{(q^2;q^2)_{2n+1}}$$

$$= \frac{(q^2,q^3;q^2)_\infty}{(q^4,q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(1-q)^2 q^{2n+1}}{(1-q^{2n+1})^2} \frac{(-1)^{n-1}(2n+1)q^{n^2+n}}{(1-q^2)}$$

$$= \frac{1}{\psi^2(q)} \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)q^{n^2+n-1}}{(1-q^{2n-1})^2}$$

$$= L_3(q). \qquad \square$$

For the Bailey pair required for $L_2(q)$, we replace z by zq^2 in Lemma 6, multiply the results by $z^{-1/2}$, then apply the operator $(d/dz)z(d/dz)$ and finally set $z = 1$. This yields

$$\alpha_n^{(2)} = \left[\frac{d}{dz} z \frac{d}{dz} (-1)^{n-1}\left(z^{n+1/2} q^{(n+2)(n+1)} - z^{-n-1/2} q^{n^2-n}\right)\right]_{z=1} \Big/ (1-q^2) \tag{3.19}$$

$$= (-1)^n \left(n + \tfrac{1}{2}\right)^2 q^{n^2-n}(1-q^{4n+2}) \big/ (1-q^2).$$

Applying (3.10) again, we find $\beta_0^{(2)} = 1/4$, and, for $n > 0$,

$$\beta_n^{(2)} = \left[\frac{d}{dz} z \frac{d}{dz} (1-z)\left(\frac{-z^{-3/2}(zq^2;q^2)_{n+1}(q^2/z;q^2)_{n-1}}{(q^2;q^2)_{2n+1}}\right)\right]_{z=1} \tag{3.20}$$

$$= \frac{(q^2;q^2)_{n+1}(q^2;q^2)_{n-1}}{(q^2;q^2)_{2n+1}} \left(-2 + 2\sum_{j=1}^{n+1} \frac{(-q^{2j})}{1-q^{2j}} + 2\sum_{j=1}^{n-1} \frac{q^{2j}}{1-q^{2j}}\right)$$

$$= \frac{2(q^2;q^2)_{n+1}(q^2;q^2)_{n-1}}{(q^2;q^2)_{2n+1}} \left(-1 - \frac{q^{2n}}{1-q^{2n}} - \frac{q^{2n+2}}{1-q^{2n+2}}\right)$$

$$= \frac{2(q^2;q^2)_{n+1}(q^2;q^2)_{n-1}}{(q^2;q^2)_{2n+1}} \left(\frac{-1+q^{4n+2}}{(1-q^{2n})(1-q^{2n+2})}\right)$$

$$= \frac{-2(q^2;q^2)_{n-1}^2}{(q^2;q^2)_{2n}}. \qquad \square$$

LEMMA 8.

$$L_2(q) = 1 - 8 \sum_{n=1}^{\infty} (q;q^2)_n^2 q^{2n} \frac{(q^2;q^2)_{n-1}^2}{(q^2;q^2)_{2n}} \tag{3.21}$$

Proof. Replace q by q^2 in (3.2), then set $a = q^2$, $\rho_1 = \rho_2 = q$. Finally insert the Bailey pair (3.19) and (3.20). This yields after multiplication by 4

$$\begin{aligned}
1 &- 8 \sum_{n=1}^{\infty} (q;q^2)_n^2 q^{2n} \frac{(q^2;q^2)_{n-1}^2}{(q^2;q^2)_{2n}} \\
&= \frac{(q^3,q^3;q^2)_\infty}{(q^4,q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(1-q)^2 q^{2n}}{(1-q^{2n+1})^2} \frac{(-1)^n(2n+1)^2 q^{n^2-n}(1-q^{4n+2})}{(1-q^2)} \\
&= \frac{1}{\psi^2(q)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}(2n+1)^2(1+q^{2n+1})}{(1-q^{2n+1})} \\
&= L_2(q). \qquad \square
\end{aligned}$$

The Bailey pairs from Lemmas 3 and 6 have other applications which we shall discuss briefly in the Conclusion.

4. Ramanujan's four identities

Section 3 provides useful representations of each of the $L_i(q)$. To prove Ramanujan's identities we shall transform each $R_i(q)$ in such a way that the desired result follows from an instance of some identity in Section 2.

THEOREM 1. *Equation* (1.1) *is valid*; *i.e.*,

$$L_1(q) = R_1(q). \tag{4.1}$$

Proof. We transform $R_1(q)$ as follows:

$$\begin{aligned}
R_1(q) &= \sum_{n=0}^{\infty} \frac{q^{(n+1)(2n+1)}(1+q^{2n+1})}{(1-q^{2n+1})^2} \\
&= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n+1}}{(q^2;q^2)_n(1-q^{2n+1})}
\end{aligned} \tag{4.2}$$

by (2.5) with q replaced by $q^2, e, N \to \infty$, $a = q^2$, $d = f = g = q$. Also by (3.12)

$$(4.3)\quad L_1(q) = \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n^2 q^{n+1}}{(1-q^{n+1})(q^2;q^2)_n(q;q^2)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n^2 q^{n+1}(1+q^{n+1})}{(1-q^{2n+2})(q^2;q^2)_n(q;q^2)_{n+1}}$$

$$= \frac{q}{(1-q)(1-q^2)}\left({}_3\phi_2\left(\begin{matrix} q^2, q^2, q^2; q^2, q \\ q^3, q^4 \end{matrix}\right) + q\,{}_3\phi_2\left(\begin{matrix} q^2, q^2, q^2; q^2, q^2 \\ q^3, q^4 \end{matrix}\right)\right)$$

$$= \frac{q}{(1-q)(1-q^2)} \frac{(q^2, q^2, q^2, q^3; q^2)_\infty}{(q^3, q, q^4, q; q^2)_\infty}\, {}_3\phi_2\left(\begin{matrix} q, q, q; q^2, q^2 \\ q, q^3 \end{matrix}\right)$$

(by (2.7) with q replaced by q^2, $a = b = c = q^2$, $e = q^3$, $f = q^4$)

$$= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n+1}}{(q^2;q^2)_n(1-q^{2n+1})}$$

$$= R_1(q)$$

by (4.2). □

THEOREM 2. *Equation* (1.2) *is valid*; *i.e.*,

$$(4.4)\qquad L_2(q) = R_2(q).$$

Proof. In Lemma 2 replace q by q^2, divide both sides by $(1-f)$, then set $a = f = 1$, $d = -1$, $e = -q$. This yields

$$(4.5)\qquad \sum_{n=1}^{\infty} \frac{(1+q^{2n})(-1)^{n-1} q^{n^2+n}}{(1-q^{2n})^2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(1-q^{2n})^2}$$

$$= \sum_{n=1}^{\infty} \frac{(q;q^2)_n (q^2;q^2)_{n-1} q^{2n}}{(1-q^{2n})(-q^2,-q;q^2)_n}.$$

Algebraically combining the sums on the left-hand side term by term we find

$$(4.6)\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+n}}{(1+q^n)^2}=\sum_{n=1}^{\infty}\frac{(q;q^2)_n^2(q^2;q^2)_{n-1}^2q^{2n}}{(q,q^2,-q^2,-q;q^2)_n}$$
$$=\tfrac{1}{8}(1-L_2(q)),$$

by (3.21). Hence

$$(4.7)\qquad L_2(q)=1+8\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}}{(1+q^n)^2}$$
$$=R_2(q).\qquad\square$$

THEOREM 3. *Equation* (1.3) *is valid*; *i.e.*,

$$(4.8)\qquad L_3(q)=R_3(q).$$

Proof. In Lemma 1 replace q by q^2, divide both sides by $(1-f)$, then set $a=f=1$, $g=q$, $e=-1$, $d=-q$. This yields

$$(4.9)\qquad -\sum_{n=1}^{\infty}\frac{q^n(1+q^{2n})}{(1-q^{2n})^2}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}$$
$$=\sum_{n=1}^{\infty}\frac{(q;q^2)_n(q^2;q^2)_{n-1}q^{2n}}{(1-q^{2n})(-q,-q^2;q^2)_n}$$
$$-\frac{q}{1-q^2}{}_4\phi_3\begin{pmatrix}q,q,q^2,q^2;q^2,q^2\\ q^3,-q^2,-q^3\end{pmatrix}$$
$$=\sum_{n=1}^{\infty}\frac{(q;q^2)_n(q^2;q^2)_{n-1}q^{2n}}{(1-q^{2n})(-q,-q^2;q^2)_n}+L_3(q),$$

by (3.18).

We can replace the series on the right-hand side of (4.9) with the expression on the left-hand side of (4.6). Hence

$$(4.10)\qquad L_3(q)=-\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}+\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}}{(1+q^n)^2}.$$

To conclude our proof we must show that $R_3(q)$ is equal to the right-hand side of (4.10). Now

$$(4.11)\qquad \begin{aligned} R_3(q) &= \sum_{n=1}^{\infty} \frac{(-1)^n n q^{n^2}}{1-q^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n n q^{n^2+2n}}{1-q^{2n}} \\ &= \sum_{n=1}^{\infty} \frac{2nq^{4n^2}}{1-q^{4n}} - \sum_{n=1}^{\infty} \frac{(2n+1)q^{4n^2+4n+1}}{1-q^{4n+2}} \\ &\quad + \sum_{n=1}^{\infty} \frac{2nq^{4n^2+4n}}{1-q^{4n}} - \sum_{n=1}^{\infty} \frac{(2n+1)q^{4n^2+8n+3}}{1-q^{4n+2}} \\ &= S_1(q) - S_2(q) + S_3(q) - S_4(q). \end{aligned}$$

We examine these terms separately. To do so we require the following simple summation which is obtained by differentiating the finite geometric series:

$$(4.12)\qquad \sum_{n=1}^{m} nx^n = \frac{x - x^{m+1}}{(1-x)^2} - \frac{mx^{m+1}}{1-x} \quad \text{for } m \geqq 0.$$

Consequently

$$(4.13)\qquad \begin{aligned} S_1(q) &= 2\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} nq^{4n^2+4nm} \\ &= 2\sum_{n=1}^{\infty}\sum_{m=n}^{\infty} nq^{4nm} \\ &= 2\sum_{m=1}^{\infty}\sum_{n=1}^{m} nq^{4nm} \\ &= 2\sum_{m=1}^{\infty}\left(\frac{q^{4m}-q^{4m(m+1)}}{(1-q^{4m})^2} - \frac{mq^{4m(m+1)}}{1-q^{4m}}\right) \\ &= 2\sum_{m=1}^{\infty}\frac{q^{4m}-q^{4m(m+1)}}{(1-q^{4m})^2} - S_3(q), \end{aligned}$$

and

$$(4.14)\quad S_2(q) - \sum_{n=0}^{\infty} \frac{q^{(2n+1)^2}}{1-q^{4n+2}}$$
$$= 2\sum_{n=0}^{\infty} \frac{nq^{(2n+1)^2}}{1-q^{4n+2}}$$
$$= 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} nq^{(2n+1)(2m+1+2n)}$$
$$= 2\sum_{n=0}^{\infty}\sum_{m=n}^{\infty} nq^{(2n+1)(2m+1)}$$
$$= 2\sum_{m=0}^{\infty} q^{(2m+1)}\sum_{n=0}^{m} nq^{4(m+2)^n}$$
$$= 2\sum_{m=0}^{\infty} q^{(2m+1)}\left(\frac{q^{4m+2}-q^{(m+1)(4m+2)}}{\left(1-q^{4m+2}\right)^2} - \frac{mq^{(m+1)(4m+2)}}{\left(1-q^{4m+2}\right)}\right)$$
$$= 2\sum_{m=0}^{\infty} \frac{q^{6m+3}\left(1-q^{4m^2+2m}\right)}{\left(1-q^{4m+2}\right)^2} - S_4(q) + \sum_{m=0}^{\infty}\frac{q^{(2m+1)(2m+3)}}{1-q^{4m+2}}.$$

Obtaining $S_1(q) + S_3(q)$ from (4.11) and $S_2(q) + S_4(q)$ from (4.14) and substituting the results into (4.11), we find

$$(4.15)\quad R_3(q) = 2\sum_{m=1}^{\infty}\frac{\left(q^{4m}-q^{4m(m+1)}\right)}{\left(1-q^{4m}\right)^2} - \sum_{n=0}^{\infty}\frac{q^{(2n+1)^2}}{1-q^{4n+2}}$$
$$- 2\sum_{m=0}^{\infty}\frac{q^{6m+3}\left(1-q^{4m^2+2m}\right)}{\left(1-q^{4m+2}\right)^2} - \sum_{m=0}^{\infty}\frac{q^{(2m+1)(2m+3)}}{1-q^{4m+2}}$$
$$= 2\sum_{m=1}^{\infty}\frac{q^{4m}}{\left(1-q^{4m}\right)^2} - 2\sum_{m=0}^{\infty}\frac{q^{6m+3}}{\left(1-q^{4m+2}\right)^2}$$
$$- 2\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+2)}}{\left(1-q^{2n}\right)^2}$$
$$- \sum_{n=0}^{\infty}\frac{q^{(2n+1)^2}\left(1+q^{4n+2}\right)}{\left(1-q^{4n+2}\right)}$$
$$= 2T_1(q) - 2T_2(q) - 2T_3(q) - T_4(q).$$

Now

$$(4.16)\quad \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{4n+2}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q^{(2n+1)(2m+1)}$$
$$= \left(\sum_{n=0}^{\infty}\sum_{m=0}^{n} + \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\right) q^{(2n+1)(2m+1)}$$
$$= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} q^{(2n+1)(2m+1)} + \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} q^{(2n+1)(2m+1)}$$
$$= \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{1-q^{4m+2}} + \sum_{n=0}^{\infty} \frac{q^{(2n+1)^2+4n+2}}{1-q^{4n+2}}$$
$$= T_4(q),$$

and

$$(4.17)\quad \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^n)^2} = \sum_{n=1}^{\infty} \frac{(-q)^n q^{n^2+n}(1-q^n)^2}{(1-q^{2n})^2}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}(1+q^{2n})}{(1-q^{2n})^2} - 2T_3(q).$$
$$= -\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} - 2T_3(q),$$

by Lemma 2 wherein we replace q by q^2, divide by $(1-f)$ and set $a = f = d = e = 1$. Also

(4.18)

$$-\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} = -\sum_{n=1}^{\infty} \frac{q^n(1-2q^n+q^{2n})}{(1-q^{2n})^2}$$
$$= -\sum_{n=1}^{\infty} \frac{q^n}{(1-q^{2n})^2} + 2\sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2}$$
$$+ 2\sum_{n=0}^{\infty} \frac{q^{4n+2}}{(1-q^{4n+2})^2}$$
$$- \sum_{n=1}^{\infty} \frac{q^{6n}}{(1-q^{4n})^2} - \sum_{n=0}^{\infty} \frac{q^{6n+3}}{(1-q^{4n+2})^2}$$
$$= -\sum_{n=1}^{\infty} \frac{q^n}{(1-q^{2n})^2} + 2T_1(q) + 2\sum_{n=0}^{\infty} \frac{q^{4n+2}}{(1-q^{4n+2})^2}$$
$$- \sum_{n=1}^{\infty} \frac{q^{6n}}{(1-q^{4n})^2} - T_2(q).$$

Utilizing (4.16), (4.17) and (4.18) to eliminate $T_4(q)$, $T_3(q)$ and $T_1(q)$ respectively in (4.15), we find

(4.19)

$$\begin{aligned}
R_3(q) = &-\sum_{n=1}^{\infty}\frac{q^n}{(1+q^n)^2}+\sum_{n=1}^{\infty}\frac{q^n}{(1-q^{2n})^2}\\
&-2\sum_{n=1}^{\infty}\frac{q^{4n+2}}{(1-q^{4n+2})^2}+\sum_{n=1}^{\infty}\frac{q^{6n}}{(1-q^{4n})^2}-T_2(q)\\
&+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2+n}}{(1+q^n)^2}+\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}-\sum_{n=1}^{\infty}\frac{q^{2n+1}}{1-q^{4n+2}}\\
= &\, L_3(q)+\sum_{n=1}^{\infty}\frac{q^n}{(1-q^{2n})^2}-2\sum_{n=0}^{\infty}\frac{q^{4n+2}}{(1-q^{4n+2})^2}\\
&+\sum_{n=1}^{\infty}\frac{q^{6n}}{(1-q^{4n})^2}-\sum_{n=0}^{\infty}\frac{q^{6n+3}}{(1-q^{4n+2})^2}+\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\\
&-\sum_{n=0}^{\infty}\frac{q^{2n+1}(1-q^{4n+2})}{(1-q^{4n+2})^2}
\end{aligned}$$

(by (4.10))

$$\begin{aligned}
= &\, L_3(q)+\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{4n})^2}-2\sum_{n=0}^{\infty}\frac{q^{4n+2}}{(1-q^{4n+2})^2}\\
&+\sum_{n=1}^{\infty}\frac{q^{6n}}{(1-q^{4n})^2}+\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\\
= &\, L_3(q)+\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{4n})^2}-\sum_{n=0}^{\infty}\frac{q^{4n+2}}{(1-q^{4n+2})^2}\\
&+\sum_{n=1}^{\infty}\frac{q^{6n}}{(1-q^{4n})^2}+\sum_{n=1}^{\infty}\frac{q^{4n}}{(1-q^{4n})^2}\\
= &\, L_3(q)+\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{4n})^2}-\left(\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}-\sum_{n=1}^{\infty}\frac{q^{4n}}{(1-q^{4n})^2}\right)\\
&+\sum_{n=1}^{\infty}\frac{q^{6n}}{(1-q^{4n})^2}+\sum_{n=1}^{\infty}\frac{q^{4n}}{(1-q^{4n})^2}\\
= &\, L_3(q)+\sum_{n=1}^{\infty}\frac{q^{2n}(1+2q^{2n}+q^{4n})}{(1-q^{4n})^2}-\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}\\
= &\, L_3(q).
\end{aligned}$$

□

THEOREM 4. *Equation* (1.4) *is valid*; *i.e.*,

$$L_4(q) = R_4(q). \tag{4.20}$$

Proof. From (3.8) we see that

$$L_4(q) = \frac{q}{(1-q^2)}\,{}_4\phi_3\left(\begin{matrix}-q,-q,q,q;q,q\\ -q^2,q^{3/2},-q^{3/2}\end{matrix}\right). \tag{4.21}$$

Now in (2.4) set $a = q$, $f = d = h = q^{1/2}$, $g = -g^{1/2}$, $e = -q$. After simplification this yields

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(1-q^{1/2})^2q^n}{(1-q^{n+1/2})^2} &= \frac{(q^2;q^2)_\infty^2}{(1-q)(1+q^{1/2})(q^3;q^2)_{\infty}^2}\sum_{n=0}^{\infty}\frac{(1-q^{1/2})(q;q^2)_nq^n}{(1-q^{n+1/2})(q^2;q^2)_n}\\ &\quad+\frac{(1-q^{1/2})^2(1+q^{1/2})}{(1-q^2)(1+q^{-1/2})}\,{}_4\phi_3\left(\begin{matrix}-q,q,-q,q;q,q\\ -q^{3/2},q^{3/2},-q^2\end{matrix}\right).\end{aligned} \tag{4.22}$$

Multiplying by $q^{1/2}(1-q^{1/2})^{-2}$ yields

$$\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{(1-q^{n+1/2})^2} = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{(q;q^2)_nq^{n+1/2}}{(1-q^{n+1/2})(q^2;q^2)_n} + L_4(q) \tag{4.23}$$

by (4.21).

We now dissect the terms making up (4.23). First

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{(1-q^{n+1/2})^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} mq^{m(n+1/2)}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} 2mq^{m(2n+1)}\\ &\quad+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(2m+1)q^{m(2n+1)+n+1/2}\\ &= 2\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(1-q^{2n+1})^2} + q^{1/2}H(q).\end{aligned} \tag{4.24}$$

Next

(4.25)
$$\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{n+1/2}}{(1-q^{n+1/2})(q^2;q^2)_n}$$
$$=\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(q;q^2)_n}{(q^2;q^2)_n}q^{m(n+1/2)}$$
$$=\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\sum_{m\geqq 1}\frac{(q;q^2)_n}{(q^2;q^2)_n}q^{m(2n+1)}+q^{1/2}K(q)$$
$$=\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n+1}}{(1-q^{2n+1})(q^2;q^2)_n}+q^{1/2}K(q).$$

We now regard (4.23) as an identity for functions of $q^{1/2}$ and we extract the even portions using (4.24) and (4.25). Thus

(4.26)
$$2\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(1-q^{2n+1})^2}$$
$$=\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n+1}}{(1-q^{2n+1})(q^2;q^2)_n}+L_4(q).$$

Now in (2.5) we replace q by q^2, we let e and $N\to\infty$, $d=f=g=q$, $a=q^2$. Upon multiplication by $q(1+q)(1-q)^{-2}$ this yields

(4.27)
$$\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})q^{(2n+1)(n+1)}}{(1-q^{2n+1})^2}$$
$$=\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n+1}}{(q^2;q^2)_n(1-q^{2n+1})}.$$

Combining (4.26) and (4.27) we obtain

(4.28)
$$L_4(q)=2\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(1-q^{2n+1})^2}-\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})q^{(2n+1)(n+1)}}{(1-q^{2n+1})^2}.$$

We must now identify $R_4(q)$ with the right-hand side of (4.28). Now

$$R_4(q) = \sum_{n=1}^{\infty} \frac{nq^{n(n+1)/2}}{1-q^n} = \sum_{n=1}^{\infty} \frac{2nq^{n(2n+1)}}{1-q^{2n}} + \sum_{n=0}^{\infty} \frac{(2n+1)q^{(n+1)(2n+1)}}{1-q^{2n+1}} = U_1(q) + U_2(q). \tag{4.29}$$

Now

$$\begin{aligned} U_1(q) &= \sum_{n=1}^{\infty} 2nq^{n(2n-1)} \sum_{m=1}^{\infty} q^{2nm} \\ &= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} 2nq^{n(2m+1)} \\ &= 2\sum_{m=1}^{\infty} \sum_{n=1}^{m} nq^{n(2m+1)} \\ &= 2\sum_{m=0}^{\infty} \left(\frac{q^{2m+1} - q^{(2m+1)(m+1)}}{\left(1-q^{2m+1}\right)^2} - \frac{mq^{(2m+1)(m+1)}}{1-q^{2m+1}} \right) \qquad \text{(by (4.12))} \\ &= 2\sum_{m=0}^{\infty} \left(\frac{q^{2m+1} - q^{(2m+1)(m+1)}}{\left(1-q^{2m+1}\right)^2} \right) - U_2(q) \\ &\quad + \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}}{1-q^{2m+1}}. \end{aligned} \tag{4.30}$$

Combining (4.29) and (4.30), we find

(4.31)

$$\begin{aligned} R_4(q) &= 2\sum_{m=0}^{\infty} \frac{q^{2m+1}}{\left(1-q^{2m+1}\right)^2} - 2\sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}}{\left(1-q^{2m+1}\right)^2} \\ &\quad + \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}\left(1-q^{2m+1}\right)}{\left(1-q^{2m+1}\right)^2} \\ &= 2\sum_{m=0}^{\infty} \frac{q^{2m+1}}{\left(1-q^{2m+1}\right)^2} - \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}\left(1+q^{2m+1}\right)}{\left(1-q^{2m+1}\right)^2} \\ &= L_4(q), \end{aligned}$$

by (4.28). □

Conclusion

In previous papers we examined some applications of q-hypergeometric series to number theory and generalized Lambert series [2], [7]. In light of our comments in the introduction it is clearly plausible that the methods developed here may reveal new results for the class-number generating functions and related number-theoretic problems.

The Bailey pairs arising in Section 3 also pose surprising questions. For example, if we insert the Bailey pair from (3.6) and (3.7) into (3.2) with $a = 1$, $\rho_1, \rho_2 \to \infty$ we obtain

$$\begin{aligned}(5.1)\quad \sum_{n=1}^{\infty} \frac{q^{n^2}(1-q)(1-q^2)\cdots(1-q^{n-1})}{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n})} &= \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{n=1}^{\infty} (-1)^{n-1} n q^{n(3n-1)/2}(1-q^n) \\ &= q + q^3 + q^4 + 2q^7 + q^9 + q^{12} + 2q^{13} + q^{16} + 2q^{19} + \cdots .\end{aligned}$$

We calculated the first 10000 coefficients on the computer; they are all small nonnegative integers and only 2299 are positive. Most surprising of all, the coefficients are multiplicative. This is quite reminiscent of the phenomenon treated at length in [8] for Ramanujan's series

$$(5.2)\qquad \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)}.$$

It turns out that the mystery of (5.1) can be explained by identifying that function with

$$\begin{aligned}(5.3)\quad \sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{(1-q^{3n+1})} - \frac{q^{3n+2}}{(1-q^{3n+2})}\right) &= \sum_{n=1}^{\infty} \frac{\left(\frac{n}{3}\right)q^n}{1-q^n} \\ &= \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{d}{3}\right)\right)q^n.\end{aligned}$$

This latter expression has turned up in number-theoretic work by Kloosterman [12] and others [5], and indeed appears in [14; Ch. 21, p. 11] in other identities. We shall subsequently examine this topic. Also Lemmas 3 and 6 imply the main results in [4].

Finally we cannot resist remarking that there are numerous rather surprising q-series identities that flow from our results. Indeed by (4.1), (4.2), and (4.3)

$$(5.4)\quad \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n q^n}{(1-q^{n+1})(q;q^2)_{n+1}} = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(1-q^{2n+1})(q^2;q^2)_n}$$
$$= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2(1-q)}\, {}_2\phi_1\left(\begin{matrix} q, q; q^2, q^2 \\ q^3 \end{matrix}\right).$$

Thus the arsenal of Heine transformations (listed as (2.8)–(2.11)) may be applied. Perhaps the most elegant result follows by applying (2.10):

$$(5.5)\quad \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n q^n}{(1-q^{n+1})(q;q^2)_{n+1}} = \psi(q) \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n q^n}{(q;q^2)_{n+1}}.$$

Presumably the interest of further applications has been adequately suggested by the above brief sketch.

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