ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 2, Summer 1992

ALMOST-ATOMIC SPACES

BY

M.L. Reese

1. Introduction

An infinite dimensional metric linear space X is *atomic* if every infinite dimensional subspace has trivial dual—the most functionally bereft space conceivable. The question of the existence of such a space was first posed by Pelczyński around 1960 (cf. [6]). Although it has often appeared in the literature during the intervening years (e.g., see [2], [3], [4], [8]), it remains unanswered. (This question is known as the Atomic Space problem [3].)

The standard example of a space which itself has trivial dual is L_p, $0 \leq p < 1$. Recently Bastero [1] showed that any subspace of L_p, $0 < p < 1$, contains a (further) subspace that is isomorphic to a space with many continuous linear functionals—namely, l_q for some $p \leq q \leq 2$. Thus no subspace of L_p is atomic. Note the resemblance of this to an alternate version of Pelczyński's question [8]: Must every m.l.s. contain an infinite dimensional subspace with a separating family of functionals? If there are no atomic spaces, we can build (in any m.l.s.) an appropriate subspace: In X pick an infinite dimensional subspace S_1 that admits a continuous linear functional Λ_1 with $\Lambda_1(x_1) \neq 0$, for some $x_1 \in S_1$. Then consider $\ker(\Lambda_1)$. It is not atomic so we can find an infinite dimensional subspace with a nontrivial continuous functional. And so on. Then $\{\Lambda_i: i \in \mathbf{N}\}$ is a separating family for the span of $\{x_i: i \in \mathbf{N}\}$.

In [5] Kalton and Shapiro show that an F-space (a complete m.l.s.) contains a basic sequence if and only if it contains a *closed* infinite dimensional subspace with a separating family of continuous linear functionals. They prove that in any non-minimal space there is a basic sequence, a *minimal* F-space being one that admits no weaker (Hausdorff) vector topology; the only known example is ω, the space of all sequences. A complete atomic space would be another. Drewnowski in [2] worked with the idea of quotient-minimal (every quotient is minimal). Again ω is quotient-minimal and so would be any complete atomic space. (See [3], Chapter 4.) It is unknown whether or not the completion of an atomic space is atomic.

Received September 10, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 46A16; Secondary 46A35, 46A45.

Yet a third formulation of the concept of atomic space—a space in which every proper closed subspace is finite dimensional—both accounts for the name and offers another view of the structure. The equivalence is apparent. Suppose X has only finite dimensional proper closed subspaces. Since the kernel of a functional on any infinite dimensional subspace S of X is closed and infinite dimensional, S has trivial dual. Conversely, if every infinite dimensional subspace of X has trivial dual and S is any proper closed subspace, choose $x \in X$ that is independent of S. Then $\Lambda(y) := 1$ if $y = x$ and $\Lambda(y) := 0$ if $y \in S$ defines a continuous linear functional on the span of $S \cup \{x\}$. So S must be finite dimensional.

Atomicity can be extended to general (Hausdorff) topological vector spaces; however, this leads us back to metric spaces. Recall that any t.v.s. with a countable base of neighborhoods of 0 is metrizable. So for an atomic t.v.s. (X, τ), we could obtain such a topology by taking $\tau_0 \subset \tau$ to be the topology generated by an appropriate nested sequence of 0-neighborhoods. Then (X, τ_0) is the image of the continuous (identity) map from (X, τ). Thus (X, τ_0) would be necessarily atomic. This suggests the (additional) metric structure is appropriate; so with it and the third definition, we begin our study of the problem.

Since an atomic space would be separable, it would have to contain a sequence of independent vectors with dense linear span. Thus it would be reasonable (in an attempt to construct an atomic space) to devise an atomic space topology on the set V of sequences of real numbers that are eventually zero. If there is no such topology on V, then there are no atomic spaces. In any infinite dimensional subspace of V there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with the property that, for each $n \in \mathbf{N}$, $x_n(i) \neq 0$ and $x_{n+1}(j) \neq 0$ imply $i < j$. So we may reduce the Atomic Space Problem to determining the existence of a m.l.s. topology on V with the property: If $\langle n_k \rangle_{k=1}^{\infty}$ is an increasing sequence of natural numbers (a *blocking* of $\mathbf{N}$) and $\langle x_k \rangle_{k=1}^{\infty}$ is a sequence of vectors with

$$x_k \in \operatorname{span}(e_n : n_1 + \cdots + n_{k-1} + 1 \leq n \leq n_1 + \cdots + n_k)$$

and nonzero for infinitely many k's, then

$$\overline{\operatorname{span}}(x_k : k \in \mathbf{N}) = V.$$

An approach to proving that atomic spaces do not exist would be to try to show that for a given sequence of natural numbers the above property cannot hold in V with a m.l.s. topology. Thus we define an m.l.s. X to be *almost-atomic* if it has a sequence $\langle V_n \rangle_{n=1}^{\infty}$ of independent subspaces with $\dim V_n \to$

∞ so that for any choice of $x_n \in V_n$ (with infinitely many nonzero) $\{x_n\}$ has dense linear span (in X). In this paper we shall prove:

THEOREM 1. *There exists an almost-atomic space. Moreover, for any increasing sequence* $\langle n_k \rangle_{k=1}^{\infty}$ *of natural numbers, there is an almost-atomic space with* $\dim V_k = n_k$.

The kernel of any linear functional has codimension 1. So it intersects (nontrivially) infinitely many of the subspaces V_k. The subspace generated by these intersections is dense (according to the definition). Consequently, every almost-atomic space has trivial dual.

Theorem 1 shows that the above approach to solving the Atomic Space Problem does not ask enough about the structure of an atomic space. Any atomic space would necessarily be 'almost-atomic' for *every* sequence of its (independent) finite-dimensional subspaces.

2. The construction

Since every m.l.s. topology can be given by an F-norm [4], we define the topology on V via one. If X is a (real) vector space, then a map $\|\cdot\|$: $X \to [0, \infty)$ is an *F-seminorm* if

$$(1) \qquad \|\alpha x\| \leq \|x\|, \quad |\alpha| \leq 1, x \in X$$

$$(2) \qquad \|\alpha x\| \to 0 \text{ as } \alpha \to 0, \quad x \in X$$

$$(3) \qquad \|x + y\| \leq \|x\| + \|y\|, \quad x, y \in X$$

If we also have that $\|x\| = 0$ implies $x = 0$, then $\|\cdot\|$ is an *F-norm*.

For an increasing sequence $\langle n_k \rangle_{k=1}^{\infty}$ of natural numbers, set $I_1 = \{2, \ldots, n_1\}$ and, for $N = 2, 3, \ldots,$ set $I_N = \{n_1 + \cdots + n_{N-1} + 1, \ldots, n_1 + \cdots + n_N\}$. (This is a *blocking* of $\mathbf{N}$ into intervals.) For each $N \in \mathbf{N}$, let V_N be the linear span of $\{e_n\colon n \in I_N\}$. These will be the V_N's of the desired space V. For each N, let

$$S_N = \{x \in V_N\colon \|x\|_{l_1} = 1\}.$$

The following (F-norm) construction will yield for any sequence $\langle x_n \rangle_{n=1}^{\infty}$ (with $x_n \in V_n$ and nonzero for infinitely many n), $N \in \mathbf{N}$ and $\varepsilon > 0$, an element y of the linear span of $\{x_n\colon x_n \in \mathbf{N}\}$ with

$$\|e_N - y\| < \varepsilon.$$

That is, each e_N is in the closed span of $\langle x_n \rangle_{n=1}^{\infty}$. We shall recycle the F-norm construction technique used by Roberts in his production of a rigid

F-space. [4]

LEMMA. *If $K \subset V$ and $w\colon K \to [0, \infty)$, then*

$$\|x\| = \inf\left(\left\|x - \sum_{i=1}^{M} \beta_i k_i\right\|_{l_1} + \sum_{i=1}^{M} w(k_i)\colon M \geq 0,\ k_i \in K,\ |\beta_i| \leq 1\right)$$

defines an F-seminorm on V.

This can (of course) be done in greater generality; however, this version suffices for our purposes.

The elements of K will be of the form $e_N - y$, where y is some linear combination of elements of (distinct) S_n's. The elements of K will be called *atoms*. Although an arbitrary selection will yield an F-seminorm with the desired properties, it is necessary to have $\|x\| \neq 0$ for all $x \neq 0$ in (at least) an infinite dimensional subspace. We shall make a selection that yields an F-norm on V. In choosing these atoms inductively, we shall define an increasing sequence $\langle b_N \rangle$ of natural numbers, functions $\sigma, a\colon \mathbf{N} \to \mathbf{N}$, and a sequence $\langle K_n \rangle$ of closed, convex sets.

Let Q_N and T_N be operators restricting an element of V to I_N and $\{1\} \cup I_1 \cup \cdots \cup I_{N-1}$, respectively. Define $\langle \lambda_n \rangle$ to be an increasing sequence of natural numbers with $\lambda_1 = 1$ and $\lambda_n > 4n^2(1 + \lambda_1 + \cdots + \lambda_{n-1})^2$, for each n. Each S_N can be written as a finite union of closed, convex sets—the faces of the unit ball in $l_1(I_N)$. Let $b_1 = 0$.

Let b_2 be the number of faces of S_1 and for $1 \leq i \leq b_2$, define

$$K_i = \lambda_i F_i,$$

where F_i is the i-th face (in some ordering). Also set $\sigma_i = a_i = 1$.

Assume b_N and K_i, σ_i, a_i (for $1 \leq i \leq b_N$) have been determined. For each $i \leq b_N$, write K_i as a union (not necessarily disjoint) of finitely many closed, convex subsets

$$K_i = \bigcup_j H_{i,j}$$

each of l_1-diameter at most $\varepsilon_N = 1/4N^2$. Let $\langle F_k \rangle$ be a listing of the faces of S_N. Match triples (i, j, k) with integers starting at $b_N + 1$. If n is matched with (i, j, k) we assign

$$K_n = H_{i,j} + \lambda_n F_k,$$
$$\sigma_n = \sigma_i + 1,$$
$$a_n = a_i.$$

When this process is completed at some integer p, match pairs (k, m), for $1 \le m \le N$, with integers starting at $p + 1$. If n is matched with (k, m), set

$$K_n = \lambda_n F_k,$$
$$\sigma_n = 1,$$
$$a_n = m$$

This ends at b_{N+1}. We may assume that σ_i is non-increasing for $b_N < i \le b_{N+1}$. (Reordering may be necessary.) Note that σ_i and a_n are less than or equal to N on this interval.

If $x \in K_n$, for $b_N < n \le b_{N+1}$, then

$$\|Q_N(x)\|_{l_1} = \lambda_n$$
$$\|T_N(x)\|_{l_1} \le 1 + \lambda_1 + \cdots + \lambda_{n-1}.$$

Also if $x' \in K_N$,

$$\|T_N(x) - T_N(x')\|_{l_1} \le \varepsilon_N = \frac{1}{4N^2}.$$

Define an *n-atom* z to be an element of the form $e_{a_n} - x$, where $x \in K_n$. Assign the weight $w(z) = \sigma_n^{-1}$ to any n-atom. Note that if $T_N(z) \ne 0$, then it is a k-atom, for some $k \le n$.

For $x \in V$, define $\|x\|$ to be the infimum of

$$\left\| x - \sum \alpha_j z_j \right\|_{l_1} + \sum w(z_j)$$

over all finite collections of atoms z_j and all α_j with $|\alpha_j| \le 1$.

By the lemma, this defines an F-seminorm on V. However, since each K_n is convex, we can combine all n-atoms with α's of the same sign into a scalar (possibly greater than 1) multiple of a single n-atom. Let $\hat{\alpha}$ be the least integer greater than or equal to α. Then $\|x\|$ can be calculated by minimizing the following sum over all finite n, all $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \ge 0$, and all k-atoms z_k and z'_k for $1 \le k \le n$:

$$\left\| x - \sum_{k=1}^{n} (\alpha_k z_k - \beta_k z'_k) \right\|_{l_1} + \sum_{k=1}^{n} \left(\hat{\alpha}_k + \hat{\beta}_k \right) \sigma_k^{-1}.$$

We may assume that either α_n or β_n is greater than 0. In what follows, we shall show that $\|\cdot\|$ is actually an F-norm on V. For $x \in V_1 + \cdots + V_M$, for

$M \geq 3$, with $\|x\|_{l_1} = 1$, we assert that

$$\|x\| \geq \frac{1}{M}.$$

Otherwise there would be a representation (as above) with least possible n with sum less than $1/M$. Find $N \in \mathbf{N}$ so that $b_N < n \leq b_{N+1}$. Now $\sigma_k \leq N$, for $b_N < k \leq n$, so N must be greater than M. Also $\hat{\alpha}_k \cdot \sigma_k^{-1} < 1/M$; hence

$$\alpha_k \leq \hat{\alpha}_k < \frac{\sigma_k}{M} \leq \frac{N}{M} \leq N.$$

Likewise $\beta_k < N$.

Because of the symmetry of the l_1-norm, we may assume that $\alpha_n \geq \beta_n$. We first consider the case $\beta_n > 0$ and show that then we can find an even smaller sum by replacing $\alpha_n z_n - \beta_n z_n'$ by $T_N(\gamma z_n)$, where $\gamma = \alpha_n - \beta_n \geq 0$, and replacing atoms z_k and z_k' by atoms $T_N(z_k)$ and $T_N(z_k')$, for $b_N < k \leq n-1$. (These atoms have greater weights.) The new representation is

$$\left\| x - \sum_{k=1}^{b_N} (\alpha_k z_k - \beta_k z_k') - \sum_{k=b_N+1}^{n-1} (\alpha_k T_N(z_k) - \beta_k T_N(z_k')) - T_N(\gamma z_n) \right\|_{l_1}$$
$$+ \sum_{k=1}^{b_N} (\hat{\alpha}_k + \hat{\beta}_k)\sigma_k^{-1} + \sum_{k=b_N+1}^{n-1} (\hat{\alpha}_k + \hat{\beta}_k)(\sigma_k - 1)^{-1} + \hat{\gamma}(\sigma_n - 1)^{-1}.$$

Let us estimate the difference D of this and the original sum

$$D \leq \| T_N(\gamma z_n) - T_N(\alpha_n z_n - \beta_n z_n') \|_{l_1} + \sum_{k=b_N+1}^{n-1} (\hat{\alpha}_k + \hat{\beta}_k)\left(\frac{1}{\sigma_k - 1} - \frac{1}{\sigma_k} \right)$$
$$+ \hat{\gamma}\left(\frac{1}{\sigma_n - 1} \right) - (\hat{\alpha}_n + \hat{\beta}_n)\sigma_n^{-1}$$
$$= \| \beta_n T_N(z_n - z_n') \|_{l_1} + \sum_{k=b_N+1}^{n-1} (\hat{\alpha}_k + \hat{\beta}_k)\left(\frac{1}{\sigma_k - 1} - \frac{1}{\sigma_k} \right)$$
$$+ \hat{\gamma}\left(\frac{1}{\sigma_n - 1} \right) - (\hat{\alpha}_n + \hat{\beta}_n)\sigma_n^{-1}$$
$$< N \cdot \varepsilon_N + \sum_{k=b_N+1}^{n-1} (\hat{\alpha}_k + \hat{\beta}_k)\left(\frac{1}{\sigma_k - 1} - \frac{1}{\sigma_k} \right)$$
$$+ \hat{\gamma}\left(\frac{1}{\sigma_n - 1} \right) - (\hat{\alpha}_n + \hat{\beta}_n)\sigma_n^{-1}$$
$$= \frac{1}{4N} + \sum_{b_N+1}^{n-1} (\hat{\alpha}_k + \hat{\beta}_k)\left(\frac{1}{\sigma_k(\sigma_k - 1)} \right) + \frac{\hat{\gamma}}{\sigma_n - 1} - (\hat{\alpha}_n + \hat{\beta}_n)\sigma_n^{-1}$$

which, using the fact that σ_k is decreasing on I_N, is

$$\begin{aligned}
&\le \frac{1}{4N} + \frac{1}{\sigma_n - 1}\left(\sum_{b_N+1}^{n} (\hat{\alpha}_k + \hat{\beta}_k)\sigma_k^{-1}\right) + \frac{\hat{\gamma} - (\hat{\alpha}_n + \hat{\beta}_n)}{\sigma_n - 1} \\
&< \frac{1}{4N} + \frac{1}{\sigma_n - 1} \cdot \frac{1}{M} - \frac{\hat{\beta}_n}{\sigma_n - 1} \\
&\le \frac{1}{4N} + \frac{1}{\sigma_n - 1}\left(\frac{1}{M} - 1\right) \\
&\le \frac{1}{4(\sigma_n - 1)} + \frac{1}{\sigma_n - 1}\frac{1 - M}{M} \\
&< 0
\end{aligned}$$

for $M \ge 2$. This contradicts the minimality of n since we have just demonstrated a smaller sum for $n - 1$. Thus we may assume that $\beta_n = 0$.

In this case we write

$$\alpha_n = \frac{1}{\lambda_n} \| Q_N(\alpha_n z_n) \|_{l_1},$$

since $\|Q_N(z_n)\|_{l_1} = \lambda_n$. Again we replace all z_k and z'_k by $T_N(z_k)$ and $T_N(z'_k)$, for $b_N < k \le n - 1$, and this time delete z_n. Our result is

$$\begin{aligned}
&\left\| x - \sum_{k=1}^{b_N} (\alpha_k z_k - \beta_k z'_k) - \sum_{k=b_N+1}^{n-1} (\alpha_k T_N(z_k) - \beta_k T_N(z'_k)) \right\|_{l_1} \\
&\quad + \sum_{k=1}^{b_N} (\hat{\alpha}_k + \hat{\beta}_k)\frac{1}{\sigma_k} + \sum_{k=b_N+1}^{n-1} (\hat{\alpha}_k + \hat{\beta}_k)\frac{1}{\sigma_k - 1}.
\end{aligned}$$

Once more we estimate the difference:

$$\begin{aligned}
D &\le \| T_N(\alpha_n z_n) \|_{l_1} + \sum_{b_N+1}^{n-1} (\hat{\alpha}_k + \hat{\beta}_k)\left(\frac{1}{\sigma_k - 1} - \frac{1}{\sigma_k}\right) - \hat{\alpha}_n \sigma_n^{-1} \\
&\le \alpha_n \| T_N(z_n) \|_{l_1} + \frac{1}{\sigma_n - 1}\left(\frac{1}{M} - 1\right).
\end{aligned}$$

The last inequality is similar to the one before. Now consider α_n.

$$\begin{aligned}\alpha_n &= \lambda_n^{-1}\|Q_N(\alpha_n z_n)\|_{l_1}\\ &= \lambda_n^{-1}\left\|\sum_{b_N+1}^{n}(\alpha_k Q_N(z_k) - \beta_k Q_N(z_k'))\right.\\ &\qquad \left.- \sum_{b_N+1}^{n-1}(\alpha_k Q_N(z_k) - \beta_k Q_N(z_k'))\right\|_{l_1}\\ &< \lambda_n^{-1}\left(\frac{1}{M} + \left\|\sum_{b_N+1}^{n-1}(\alpha_k Q_N(z_k) - \beta_k Q_N(z_k'))\right\|_{l_1}\right)\\ &\le \lambda_n^{-1}\left(1 + \sum_{b_N+1}^{n-1}(N\lambda_k + N\lambda_k)\right)\end{aligned}$$

since $\|Q_N(z_k)\|_{l_1} = \lambda_k$ and $\alpha_k, \beta_k \le N$. Recalling that $\lambda_n > 4n^2(1 + \lambda_1 + \cdots + \lambda_{n-1})^2$, we have

$$\alpha_n < \frac{N}{2n^2(1 + \lambda_1 + \cdots + \lambda_{n-1})}.$$

Consequently,

$$\begin{aligned}D &\le \alpha_n(1 + \lambda_1 + \cdots + \lambda_{n-1}) + \frac{1}{\sigma_n - 1}\left(\frac{1}{M} - 1\right)\\ &\le \frac{1}{2N} + \frac{1}{\sigma_n - 1}\left(\frac{1}{M} - 1\right).\end{aligned}$$

Therefore $D < 0$, for $M \ge 3$. This again contradicts the minimality of n. It follows that $\|\cdot\|$ is indeed an F-norm on V.

Proof of Theorem 1. Suppose $\langle x_N\rangle_{N=1}^{\infty}$ is a sequence in V with $x_N \in V_N$ nonzero for infinitely many N's. It suffices to show that for each m, $e_m \in \overline{\operatorname{span}}(x_N\colon N \in \mathbf{N})$. Fix $m \in \mathbf{N}$ and let $\varepsilon > 0$ be given. We may assume that if $x_N \neq 0$, then $x_N \in S_N$; that is, $\|x_N\|_{l_1} = 1$. Also assume that M is a natural number with $1/M < \varepsilon$.

We claim that there exists

$$y = \sum_{N=1}^{\infty} \mu_N x_N,$$

where μ_N is a nonnegative integer (for each N), $y \in K_p$ (for some p), $\sigma_p = M$, and $a_p = m$.

Recall that, for each $N \in \mathbf{N}$ and $b_N < n \leq b_{N+1}$, the values of both σ_n and a_n are $1, 2, \ldots, N$. Let $\langle x_{N_i} \rangle$ be a subsequence of $\langle x_N \rangle$ so that $x_{N_i} \neq 0$, for each i, and $N_1 \geq m$. Set $\mu_k = 0$, if $k \neq N_i$, for some $1 \leq i \leq M$. We must choose $\mu_{N_1}, \ldots, \mu_{N_M}$.

Set $\mu_{N_1} = \lambda_{p_1}$, where $b_{N_1} < p_1 \leq b_{N_1+1}$, $a_{p_1} = m$, and $\sigma_{p_1} = 1$. For $j = 2, \ldots, M$, inductively set $\mu_{N_j} = \lambda_{p_j}$, where $b_{N_j} < p_j \leq b_{N_j+1}$, and $\mu_{N_1} x_{N_1} + \cdots + \mu_{N_j} x_{N_j} \in K_{p_j}$, $a_{p_j} = m$, and $\sigma_{p_j} = \sigma_{p_{j-1}} + 1 = j$. Thus

$$y = \sum_{N=1}^{\infty} \mu_N x_N = \lambda_{p_1} x_{N_1} + \cdots + \lambda_{p_M} x_{N_M}$$

is a linear combination of the x_N's, and $e_m - y$ is a p_M-atom. Therefore

$$\|e_m - y\| \leq \frac{1}{\sigma_{p_M}} = \frac{1}{M} < \varepsilon.$$

Acknowledgements. The results in this paper are contained in the author's Ph.D. thesis, completed at the University of South Carolina under the direction of Professor J.W. Roberts, to whom I am greatly indebted. Many thanks to the referee for suggestions which have been incorporated.

References

1. J. Bastero, *l_q-subspaces of stable p-Banach spaces*, $0 < p \leq 1$, Arch. Math., vol. 40 (1983), pp. 538–544.
2. L. Drewnowski, *On minimal topological linear spaces and strictly singular operators*, Comment. Math., Special Issue II (1979), pp. 89–106.
3. N.J. Kalton, *The Maharam problem*, Seminaire Initation a L'Analyse (G. Choquet, G. Godefroy, M. Rogalski, J. Saint Raymond, eds.), Exposé 18, Université de Paris VI/VII, 1988/9.
4. N.J. Kalton, N.T. Peck and J.W. Roberts, *An F-space sampler*, Cambridge University Press, Cambridge, 1984.
5. N.J. Kalton and J.H. Shapiro, *Bases and basic sequences in F-spaces*, Studia Math., vol. 56 (1976), pp. 47–61.
6. V.L. Klee, *Exotic topologies for linear spaces*, Symp. on General Topology and Its Relations to Modern Analysis and Algebra, Prague, 1961, pp. 238–249.
7. M.L. Reese, *An example related to the Atomic Space Problem*, Ph.D. Thesis, University of South Carolina, 1989.
8. S. Rolewicz, *Metric linear spaces*, PWN–Polish Scientific Publishers, Warszawa, 1972, 1984.

St. Olaf College
Northfield, Minnesota