ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 4, Winter 1991

THE SCHWARZ-PICK LEMMA FOR CIRCLE PACKINGS

BY

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Connections between circle packings and analytic functions were suggested by William Thurston at the International Symposium in Celebration of the Proof of the Bieberbach Conjecture, Purdue University, March 1985. He conjectured that the conformal mapping of a simply connected plane domain Ω to the unit disc Δ could be approximated by manipulating hexagonal circle configurations lying in Ω. His idea is illustrated in Figure 1: First, approximate Ω with a uniform hexagonal circle packing P as in 1(a). Now, repack P to obtain a certain combinatorially equivalent extremal circle packing P_a lying in Δ, as shown in 1(b). Finally, define a piecewise affine mapping from P_a to P by identifying centers of corresponding circles in the two configurations. He conjectured that as the sizes of the circles in P go to zero (and assuming certain natural normalizations), the mappings so defined would converge uniformly on compact subsets of Δ to the conformal (analytic) mapping of Δ onto Ω. This conjecture was subsequently proven by Burt Rodin and Dennis Sullivan [6].

Thurston termed the conjectured result a "Finite Riemann Mapping Theorem", the intuition being, at least in part, that since the conformal map carries infinitesimal circles to infinitesimal circles, one might approximate it by mapping real circles to real circles. Our interest is in developing this analogy—rather than consider increasingly fine packings and the approximation question, we will study individual circle packings P and how they compare to their extremal repackings P_a. Our main result is a natural analogue for the classical Schwarz Lemma, in the invariant form due to Pick, which we term the "Discrete Schwarz-Pick Lemma" or DSPL. In the definitions and results along the way, further parallels with classical complex analysis emerge. There are several intriguing results on circle packings in Chapter 13 of Thurston's notes [10], and certain of the key ideas here (e.g., parameterized hyperbolic structures and some monotonicity results) occur there, though without our complex function theory slant.

We want to state our main result somewhat informally here at the beginning before introducing more technical definitions and notation. To that

Received January 29, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 30C80; Secondary 51M99, 31C20.

[1]The second author gratefully acknowledges support of the National Science Foundation and the Tennessee Science Alliance.

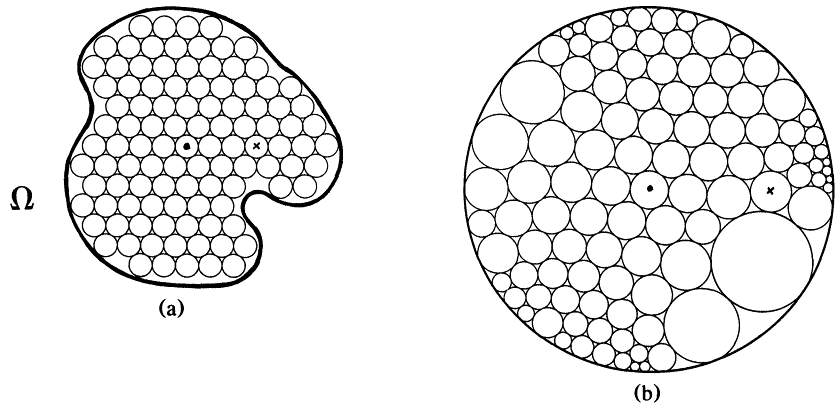


FIG. 1 Approximate conformal mapping.

end, suppose we have a configuration P of circles in the hyperbolic plane as shown in Figure 2(a). We will later describe the combinatorics underlying P using an abstract complex; however, at this point simply note that the circles of P have disjoint interiors, that the interstices between circles are triangular, and that the union of the circles and interstices is simply connected. (Note also that we place no hexagonal restriction on the combinatorics.) By results of Andreev [2, Theorem 2], interpreted for circles by Thurston [10, Ch. 13], there is a combinatorially isomorphic circle packing in Δ having all its boundary circles internally tangent to the unit circle. This will be termed the Andreev packing P_a, and in this instance is illustrated in 2(b). There are some minor additional hypotheses needed on P, but those are best left for later.

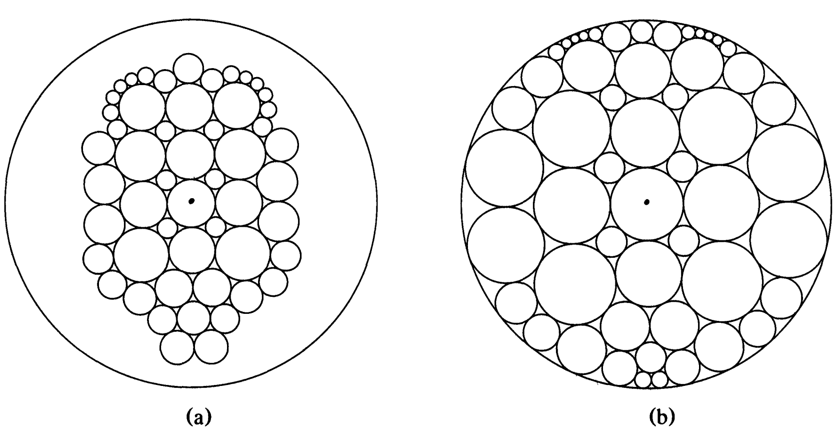


FIG. 2 A packing and its Andreev packing.

THE DISCRETE SCHWARZ-PICK LEMMA. *Let P be a circle packing in the hyperbolic plane, P_a its Andreev packing. Then:*

(a) Each circle of P has a hyperbolic radius which is less than or equal to that of the corresponding circle in P_a.

(b) The hyperbolic distance between (centers of) circles in P is less than or equal to the hyperbolic distance between the corresponding circles in P_a. Moreover, a single instance of (finite) equality in either (a) or (b) implies equality in every instance, i.e., P and P_a are hyperbolically congruent.

If the mapping from P_a to P is interpreted, as we believe it may be, as a discrete analytic self-map of the hyperbolic plane, then the DSPL states that it is either a strict contraction or an isometry—precisely the classical statement.

We have tried to keep the proofs as elementary and geometric as possible, though we have felt obliged, in anticipation of future developments, to put the technical details on a rigorous foundation. A key feature of our work is the use of hyperbolic rather than euclidean geometry; and it has been a constant source of pleasure to see how faithfully the "discrete" situations in this setting appear to mimic their classical models. The astute reader will recognize analytic continuation, harmonic and subharmonic functions, the Perron method, the monodromy theorem, and more. In the concluding section of the paper, we summarize some of the parallels; in our opinion, they are more than shallow analogies, but rather suggest a fundamental geometric rigidity in the discrete situations which underlies the rigidity of analytic functions.

Here, briefly, is how the paper proceeds: In Section 1, we introduce appropriate simplicial complexes to represent the combinatoric information in circle configurations. In Section 2 we impose hyperbolic structures on the complexes, and thereafter work in the setting of "hyperbolic complexes" rather than with actual circle configurations. The technical meaning of "packing" is defined, and we prove that hyperbolic complexes which are packings may be immersed in the hyperbolic plane. Section 3 contains our statement of Andreev's result, followed by several examples of packings (with illustrations) which we feel will be helpful to the reader's intuition. Monotonicity results (and the only real computations) are gathered in Section 4. In Section 5 we solve a boundary value problem for hyperbolic complexes using "subpackings" and the Perron method, famous for its use in solving the classical Dirichlet problem. The DSPL is formally stated and proved in Section 6. Not until the concluding section are we in position to suggest further connections between our discrete considerations and the classical continuous case—we commend this section particularly to the reader's attention. Since the original draft of this work, several individuals have suggested that an independent proof of what we are calling Andreev's Theorem might follow via induction. This is indeed the case, and an appendix is provided for that purpose, the separate treatment arising from the need for slightly more general complexes than those treated in the body of the paper.

A number of papers on circle packing have been written since [6], largely in the euclidean setting and with hexagonal combinatorics. In [7] and [8], Rodin introduces a version of Schwarz's Lemma in which he compares packings to N-generational regular hexagonal packings. This is not directly related to our work here. Though the DSPL does show that the constant a in [7, Theorem 5.1] is strictly greater than one, Rodin's interest is in an upper bound on a independent of N. The reader should be aware that the discrete potential theory in [7] is quite different in character from that which occurs here: the euclidean radii in a hexagonal setting satisfy a submean value property, permitting use of classical discrete potential theory. This fails for more general combinatorics, so our references to subharmonic functions, the Perron method, and so forth, are by way of analogy. More substantial connections exist, but lie much deeper in the geometry—they are developed and exploited in [9]. Also to be noted is the solution of boundary value problems in the euclidean setting by Carter and Rodin [5]; they use a somewhat different analogue of the Perron method, formalizing a computational algorithm suggested by Thurston.

1. Simplicial complexes

The abstract representation of packing combinatorics has typically been a graph; but it seems best to rely instead on an abstract 2-dimensional simplicial complex, K. Its vertices correspond to circles, with an edge between vertices if and only if their circles are intended to be tangent. The faces are triangular, corresponding to triangular interstices between triples of circles. We have at our disposal not only the topology of K, but all the terminology associated with the elementary theory of simplicial complexes; (cf. [4]).

Throughout the paper, then, K will denote a fixed simplicial 2-complex; we make the requirements explicit in this definition.

DEFINITION. K will be a simplicial 2-complex which is isomorphic (i.e., simplicially equivalent) to a finite triangulation of the closed unit disc and which has these additional properties: every boundary vertex shares an edge with at least one interior vertex; every pair of interior vertices may be connected by an edge path whose edges have only interior vertices as endpoints.

Among the consequences of this definition: K is an oriented manifold; its boundary is a simple closed curve with at least three vertices; and each interior vertex is the vertex for at least three faces. The additional properties placed on K are helpful in avoiding situations inconsistent with circle packings or in sidestepping minor pathologies. The final condition, for example, is the discrete analogue of "connected interior", and will be needed in proving the case of equality in our main result. (Only in the appendix will we consider more general complexes.)

We will be imposing a hyperbolic structure on K by specifying hyperbolic radii for its vertices. The hyperbolic plane will be represented as the unit disc Δ equipped with the Poincaré metric, $\rho(\cdot, \cdot)$. (See [1] and [3].) Circles in this metric happen to be euclidean circles in Δ; however, in referring to centers and radii, we always intend the hyperbolic quantities. Horocycles, which are euclidean circles internally tangent to the unit circle, may be regarded as hyperbolic circles, each having the point of tangency as its center and infinite hyperbolic radius. Hyperbolic geodesics lie along arcs of euclidean circles which are orthogonal to the unit circle. The geodesic between the centers of tangent circles will pass through their point of tangency, even if one or both has center at infinity. The isometries (or rigid motions) of Δ consist of Möbius transformations and their complex conjugates. Hyperbolic quantities are invariant under isometries, so most of the statements we make should be taken in the invariant sense. For example, we have the important, though elementary result:

LEMMA 1. *Given a triple of hyperbolic radii $\{a, b, c\}$, $0 < a, b, c \leq \infty$, there exists a unique triple of circles with these radii which have disjoint interiors and are mutually tangent, and a unique hyperbolic triangle formed by the circle centers.*

Here, the configuration of circles and the triangle itself are unique only up to isometries. Note however that the **hyperbolic structure** of the triangle—its angles, edge lengths, area—is uniquely determined by the three radii. These triples of radii will arise as the radii assigned to vertices of faces of K. We illustrate in Figure 3, using a labeling to which we refer frequently in the sequel. Note that since K has an orientation, we will treat these as ordered triples, the corresponding circles being placed in Δ in a counterclockwise

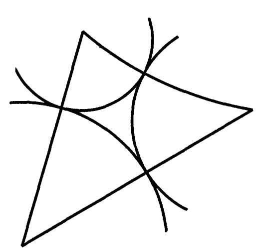


FIG. 3 A typical triangle.

manner. The resulting configuration of circles and the hyperbolic triangle are then determined up to Möbius transformations.

2. Hyperbolic structures on K

In this section we move away from the collections of circles themselves, focusing instead on the hyperbolic structures which they impose on the faces of K. Let $R = \{r_0, r_1, \ldots, r_{k-1}\}$ denote a collection of hyperbolic radii, one for each of the k vertices of K. The three vertices of each face f_j of K determine (along with the orientation of the face) an ordered triple of radii, and hence, via Lemma 1, a hyperbolic triangle T_j. By identifying f_j and T_j pointwise so that corresponding vertices match, the hyperbolic structure of T_j induces a hyperbolic structure on f_j. If two faces, say f_1 and f_2, are contiguous, then they share two of their three radii and the corresponding configurations of circles may be arranged to share two of their three circles. The resulting triangles T_1, T_2 will be contiguous in Δ, and the identifications with f_1 and f_2 may be adjusted near their common edge to be consistent—that is, so the hyperbolic structures induced on f_1 and f_2 are compatible across their shared edge. In this way, the collection R induces a hyperbolic structure on K, with the possible exception of its vertices. Put another way, the collection R determines a hyperbolic metric of constant curvature -1 on K with singularities at the vertices.

DEFINITION. The complex K with the hyperbolic structure determined by R is termed a **hyperbolic complex** and is denoted $K(R)$. If v is a vertex of K, then $v(R)$ denotes the radius in R associated with v. We make the standing assumption that $v(R) < \infty$ for interior vertices v, while the radius may be finite or infinite for boundary vertices. The collections R form a directed set, where we write $R_1 \leq R_2$ if $v(R_1) \leq v(R_2)$ for every vertex $v \in K$.

We are interested in the extent to which $K(R)$ looks, locally and globally, like the hyperbolic plane. More precisely, we will discuss **locally isometric immersions** of $K(R)$ in Δ—continuous maps which are local isometries.

First, consider the local situation: Each face f of $K(R)$ has the hyperbolic structure of a triangle (including orientation); since local isometries preserve geodesic curves, its image in Δ under a local isometry must be an actual hyperbolic triangle T. Moreover, as we have observed before, faces sharing an edge will give triangles sharing an edge. Thus, as far as the local situation goes, only the geometry near interior vertices remains at issue. Were there a locally isometric immersion of $K(R)$ in Δ, the singularities at interior vertices would necessarily be removable; that is the hyperbolic metric would extend across each singularity with curvature -1. At an interior vertex v, this condition has to do with the way in which the star of faces at v fits together. We need to study these, as well as the stars at boundary vertices, so let us establish some uniform notation.

DEFINITION. Let $K(R)$ be a hyperbolic complex, v a vertex of K. Each face f of K in the star of v has the hyperbolic structure of a triangle and hence forms an angle $\theta(v, R, f)$ at v. The **angle sum** at v, denoted $\theta_v(R)$, is defined by

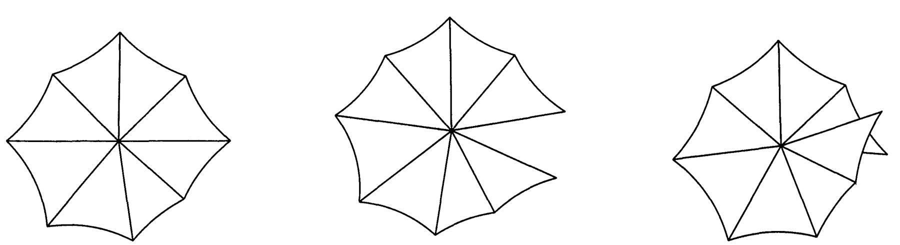
$$\theta_v(R) = \sum_{j=1}^{n} \theta(v, R, f_j),$$

where $f_1, f_2, \ldots, f_n$ comprise the faces in the star of v.

Note that $\theta_v(R) \geq 0$, with equality if and only if v is a boundary vertex having infinite radius.

Consider, now, an interior vertex v. Because K is a manifold, the faces in the star of v form a linearly ordered chain $f_1, f_2, \ldots, f_n$, each contiguous to the next and f_n contiguous to f_1. (Note also that $n \geq 3$.) If one fixes a location in Δ for v and successively lays down the hyperbolic triangles associated with these faces, each sharing an edge with its predecessor, then one of three things occurs: (a) If $\theta_v(R) = 2\pi$, then the (interiors of the) triangles do not overlap, and the final triangle shares an edge with the initial one. (b) If $\theta_v(R) < 2\pi$, then the triangles again do not overlap, but the final triangle does not reach the initial one. Lastly, (c) if $\theta_v(R) > 2\pi$, then the triangles overlap. We illustrate these in Figure 4.

It is clear from considerations of the circumference of infinitesimal circles about v in the metric of $K(R)$ that the singularity at v is removable with curvature -1 if and only if the first situation pertains, i.e., if and only if $\theta_v(R) = 2\pi$. This therefore gives necessary and sufficient conditions for an isometric immersion in the neighborhood of an interior vertex.



(a) (b) (c)

FIG. 4 The star at v.

DEFINITION. The hyperbolic complex $K(R)$ is said to be a **local packing** at a vertex v if $\theta_v(R) = 2\pi$. It is termed a packing if it is a local packing at every interior vertex.

Now for the global situation: the process of laying down the pieces of a local isometry one after the other is roughly analogous to the process of analytic continuation or to a "developing" map in Thurston's terminology [10]. The local-to-global step is contained in the following result.

THEOREM 1. *Let $K(R)$ be a hyperbolic complex. A necessary and sufficient condition for the existence of a locally isometric immersion ϕ: $K(R) \to \Delta$ is that $K(R)$ be a packing. In this case the immersion is unique up to isometries of Δ.*

Note that if $v(R) = \infty$ for a boundary vertex v, then the immersion will map v to a point of the ideal boundary of Δ rather than Δ itself; this technicality should cause no confusion. Also note that we will write $\phi = \phi_R$, even though ϕ is determined only up to isometries.

Proof. We have already proved necessity. For sufficiency, we assume $K(R)$ is a packing and show how to define ϕ_R: $K(R) \to \Delta$.

It has been observed that each face f of $K(R)$ is identified with an actual triangle T in Δ, but that the location of T is determined only up to isometries. To define ϕ_R, therefore, it is enough to describe the placement of the faces. There is one simple idea underlying our approach: Suppose $C = \{f_1, \ldots, f_n\}$ is a **chain** of (not necessarily distinct) faces of K—meaning a sequence in which each face shares an edge of K with its predecessor. Choose a location for the triangle T_1 associated with f_1. There is then a unique location for the triangle T_2 associated with f_2, since it shares a certain edge and an orientation with T_1. Proceeding inductively, we place triangles $T_3, T_4, \ldots$ associated with faces of the chain until we have placed the triangle T_n associated with f_n. We will say that the location of T_n was obtained from that of T_1 by a "development" along C.

We use this notion to define the locations of all triangles as follows: Designate one face as f_0 and place its triangle T_0 at a specific location in Δ. Given any other face f, choose a chain

$$C = \{f_0, f_1, \ldots, f_n\} \quad \text{with } f_n = f$$

and obtain the location of T_n by a development along C. Since K is connected, it is clear that this defines the desired immersion if we can show that the location determined for T_n is independent of the chain from f_0 to f.

Of course, this is simply a version of the monodromy theorem. It clearly suffices to consider closed chains $C = \{f_0, f_1, \ldots, f_n\}$, $f_n = f_0$, and to show that the triangle T_n obtained by a development along C is identical to T_0. Briefly, here's how one might proceed using homotopies:

We consider the class of closed chains $C = \{f_0, f_2, \dots, f_n\}$ at f_0. A subchain $f_j, \dots, f_k$ of C is said to be "local at vertex v" if its faces belong to the star of v. A new closed chain C' is obtained from C by a "local modification" if this local subchain at v is replaced by another local subchain at v having the same first and last faces. We say that closed chains C_1 and C_2 are "homotopic" if one can be obtained from the other by a succession of local modifications. Among the local modifications are ones which reverse the direction in which a subchain passes around some vertex, as well as ones which carry out these pattern simplifications:

$$\begin{gathered} \{\dots, f, f, \dots\} \to \{\dots, f, \dots\} \\ \{\dots, f, g, f, \dots\} \to \{\dots, f, \dots\}. \end{gathered} \tag{1}$$

Now, consider developments along chains: Suppose $c = \{g_1, g_2, \dots, g_m\}$ is a local chain at v. As we saw earlier, since $K(R)$ is a local packing at v, placing a single triangle of its star uniquely determines the locations for the remaining triangles. Therefore, a development along c gives a location for T_m which depends only on T_1 and is independent of the intervening elements $g_2, \dots, g_{m-1}$. This means that if c were a subchain of a closed chain C, local modification of c would not affect the development along C. We conclude that developments along homotopic closed chains will place their final triangles at identical locations. Finally, we need to prove that all closed chains are homotopic to the "null" chain $\{f_0\}$. Given a chain C, draw a closed curve γ starting in f_0 and passing successively through the faces of the chain (avoiding vertices). Since K is homeomorphic to a triangulation of the closed disc, we may count the number $N(C)$ of vertices which γ separates from ∂K. Applying local modifications, one can obtain a homotopic chain C' with $N(C') < N(C)$. Repeating this a finite number of times, we obtain a chain which separates no vertices from ∂K. Such a chain is homotopic to $\{f_0\}$ via a finite number of local modifications of type (1) indicated above. The details are left to the reader.

Note that our immersion is completely determined by the placement and orientation of the initial triangle T_0. Since this is unique up to an isometry, the last statement of the theorem follows. ■

Necessary and sufficient conditions for ϕ_R to be an embedding will clearly be quite difficult to formulate, being roughly analogous to conditions for univalence of locally univalent analytic functions. However, as certain important packings are embeddable, we felt that some terminology would be helpful.

DEFINITION. The hyperbolic complex $K(R)$ is said to be a **planar packing** if it is a packing and if its locally isometric immersion ϕ_R is an embedding, i.e., if ϕ_R is globally one-to-one.

3. Packing examples

It would be comforting to know that circle packings actually exist for a given complex K. In fact, they are very abundant; but for now we'll introduce the important Andreev packings and a few additional circles packings as illustrations. Among other things, these show how (with the help of Theorem 1) one can move easily between circle configurations and our hyperbolic complex formulation.

Let us begin with what we have been referring to as Andreev's Theorem. This is a special case of more general results, as shown by Thurston [10, Ch. 13]. In our terminology, the statement is this:

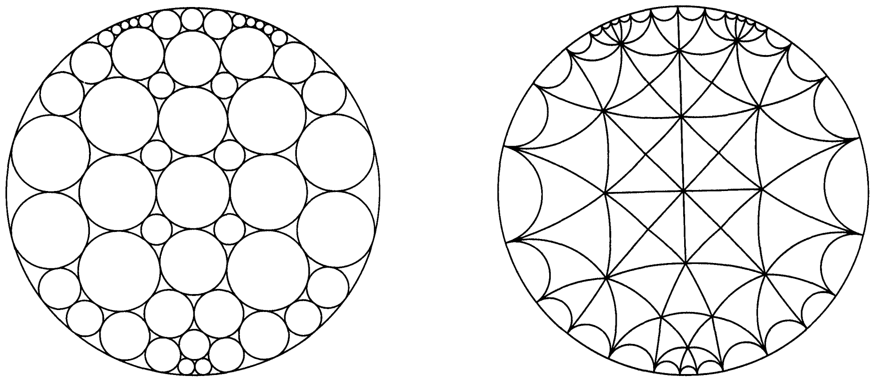
THEOREM 2 (ANDREEV'S THEOREM). *Given the complex K, there exists a unique collection of radii, denoted R_a, for which $K(R_a)$ is a packing and for which all boundary vertices have infinite radius. Moreover, $K(R_a)$ is a planar packing.*

This theorem has been the main ingredient in the work on circle packing, yet it makes no direct mention of circles! A rendition more faithful to earlier usage might be this:

Given K, there exists a configuration of circles in Δ, one associated with each vertex of K, so that the circles have mutually disjoint interiors, circles are tangent if their vertices share an edge in K, and the circles associated with boundary vertices are horocycles; the configuration is unique up to hyperbolic isometries.

Andreev's original work concerned triangulations of the sphere; and following Thurston, one removes a vertex and normalizes in order to apply the results to arbitrary finite triangulations of the closed unit disc. Recall, however, that our complexes have added regularity properties. Though the uniqueness of R_a comes out of our work, we initially rely on the existence portion of Andreev's Theorem. Only in the appendix do we show how to avoid this, and hence provide an independent proof of Andreev's Theorem.

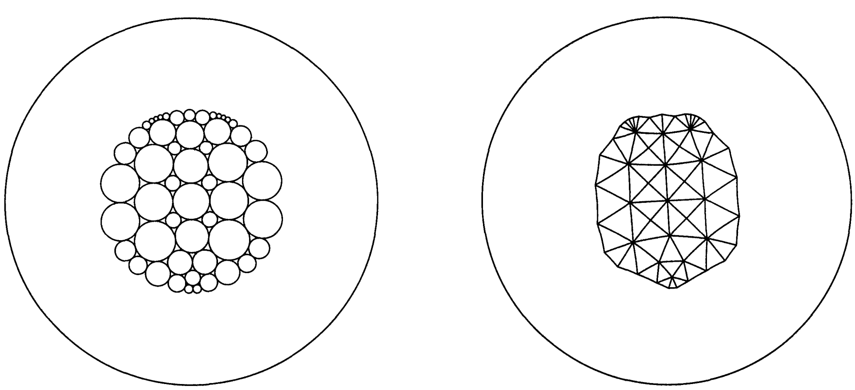
Figure 5 illustrates an Andreev packing, (a) being the circle configuration and (b) the (embedded) hyperbolic complex $K(R_a)$. Figure 5(a) clearly leads to the planar packing in Figure 6 via the euclidean scaling $z \mapsto z/2$, though this scaling effects the hyperbolic structure in a rather complicated way.



(a) (b)

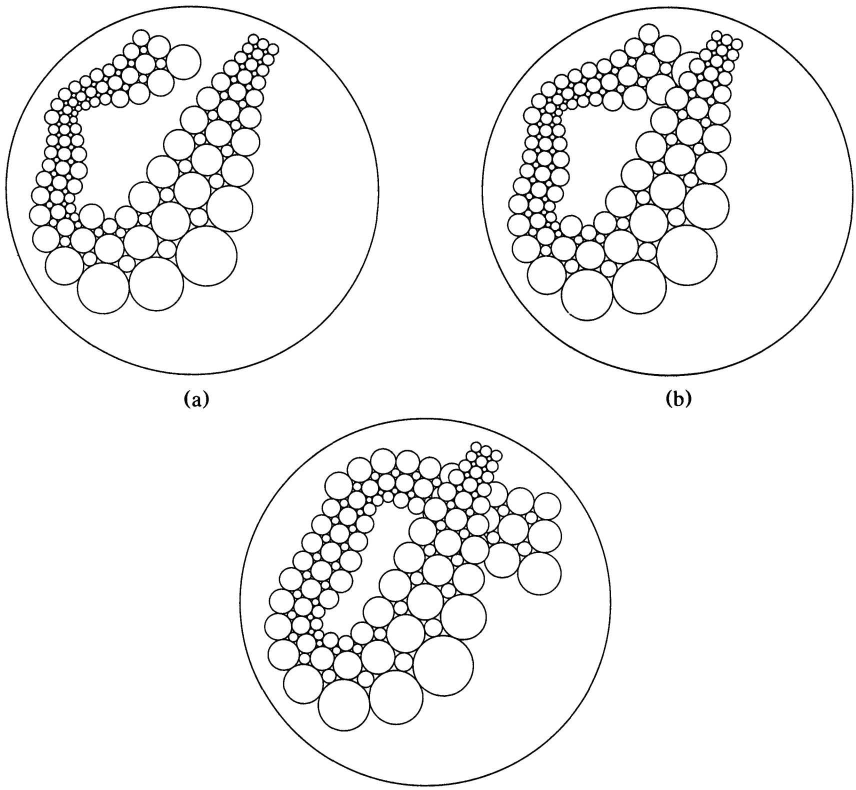
FIG. 5 An Andreev packing.

Three packings based on another complex are illustrated in Figure 7. Figure 7(a) is clearly a planar packing. Figure 7(b) is intended to illustrate a minor subtlety regarding complexes and their circle configurations: Namely, if the faces of the embedded complex were shown in 7(b), they would not overlap, even though the circles do. Thus, 7(b) is a planar packing, showing that the hyperbolic structures determined by the circles are important, rather than the circles themselves. Figure 7(c) is clearly a non-planar packing. Here, faces fit together locally, but there are global overlaps in the immersed complex. The overlapping regions are analogous to the separate sheets in the image Riemann surface of an analytic function which is locally but not globally univalent.



(a) (b)

FIG. 6 Scaling an Andreev packing.



(a)

(b)

(c)

FIG. 7 Three packing of the same complex.

4. Monotonicity results in hyperbolic geometry

Our study of hyperbolic complexes relies on several "monotonicity" results —results which show how the structures are affected by changes in the radii. Our primary tool will be the Cosine Rule; refer to Figure 3 for notation (cf. [Ch. 7, 3]).

COSINE RULE. *Assume a, b, and c are finite. Then*

$$\cos \alpha = \frac{\cosh(a+b)\cosh(a+c) - \cosh(b+c)}{\sinh(a+b)\sinh(a+c)}.$$

The obvious modifications to this rule are valid when some (or all) of a, b, c are infinite.

This, as with many results in the hyperbolic setting, seems a close parallel with the euclidean case. However, the added rigidity of the hyperbolic plane can enter in striking fashion. For example, in the hyperbolic setting, the angles of a triangle add up to less than π. Similarity implies congruence for hyperbolic triangles; and the angles alone (or the sides alone) are sufficient for solving a triangle. The hyperbolic area of the triangle of Figure 3 is $\pi - (\alpha + \beta + \gamma)$. More generally:

POLYGONAL AREA RULE. *If P is a hyperbolic polygon of $n \geq 3$ sides, with interior angles $\theta_1, \theta_2, \dots, \theta_n$, then*

$$\text{Area}(P) = (n-2)\pi - (\theta_1 + \dots + \theta_n).$$

We begin with the study of hyperbolic triangles formed by triples of circles. We are interested in what happens when the radius of one of the circles is changed. (See [10, Ch. 13].)

LEMMA 2. *Consider the configuration of Figure 3. Assume that the hyperbolic radii b and c are held constant. Then the angles α, β, γ and* Area(T) *are continuous functions of a. Moreover, we have the following monotonic behavior:*

(i). $\alpha(a)$ is strictly decreasing, with $\lim_{a\uparrow\infty} \alpha(a) = 0$ and $\lim_{a\downarrow 0} \alpha(a) = \pi$.

(ii). $\beta(a)$ (respectively $\gamma(a)$) is increasing; monotonicity is strict if $b < \infty$ (respectively $c < \infty$).

(iii). Area(T) *is strictly increasing.*

For a point p on the edge of T opposite to v, let $l_p(a)$ denote the distance from v to p. Then:

(iv). $l_p(a)$ is continuously differentiable. Moreover, $dl_p/d\alpha$ is bounded below by a positive function $\tau(a)$ which is continuous in a but independent of p.

Recall that all the quantities under consideration are invariant, so we may modify the configuration of Figure 3 with isometries, if desired. Our monotonicity properties are not difficult to see, at least when $b < \infty$: simply place the vertex for β at the origin in Δ, the vertex for γ on the positive real axis, and then apply geometric reasoning.

Proof. Continuity in (i)–(iii) follows from the Cosine Rule, though adjustments must be made if one or both of b, c are infinite. We will do some representative computations; it is convenient to specify new quantities $y = e^{2b}$, $z = e^{2c}$, and new variable $x = e^{2a}$:

(i). Assume first that b, c are finite; by the Cosine Rule,

$$\cos\alpha = \frac{(xy+1)(xz+1) - 2x(yz+1)}{(xy-1)(xz-1)}.$$

A computation gives

$$\frac{d\cos\alpha}{dx} = 2\frac{(x^2yz-1)(y-1)(z-1)}{(xy-1)^2(xz-1)^2}.$$

Since this is strictly positive, α is strictly decreasing with x, hence also with a. If, say, b is finite but c is infinite, then the Cosine Rule becomes

$$\cos\alpha = \frac{x(xy+1)-2xy}{x(xy-1)},$$

while if both are infinite,

$$\cos\alpha = \frac{x-2}{x}.$$

Again, computations show

$$\frac{d\cos\alpha}{dx} > 0,$$

so strict monotonicity follows. As for the behavior of α when a goes to 0 or infinity, these limits are geometrically evident.

(ii). If $b=\infty$, then $\beta=0$, independent of a. Therefore assume $b<\infty$. If $c<\infty$, the Cosine Law gives

$$\cos\beta = \frac{(yx+1)(yz+1)-2y(xz+1)}{(yx-1)(yz-1)}.$$

A computation shows

$$\frac{d\cos\beta}{dx} = \frac{2y(1-y)(1-z)(1-yz)}{(yx-1)^2(yz-1)^2} < 0,$$

so β is strictly increasing with a. The computation in case $c=\infty$ is similar. Interchanging the roles of b and c completes the proof of (ii).

(iii). Area(T) $=\pi-(\alpha+\beta+\gamma)$. If $b=c=\infty$, then this area is strictly increasing by (i). On the other hand, suppose b, say, is finite; the monotonicities in (i) and (ii) now compete in their effects on Area(T). Nonetheless, one can see geometrically that the area increases: Place the angle β at the origin and γ on the positive real axis. The side between these remains fixed as a changes, but the angles open up. In particular, the angle β is strictly increasing, so area is strictly increasing.

(iv). Assume for the moment that $b, c<\infty$. Let $l=l_p(a)$ and let t be the hyperbolic distance from p to w. The Cosine Law for T and for the triangle

(w, p, v) gives

$$\cosh l = \cosh(a+b)\cosh t - \cos\beta \sinh(a+b)\sinh t,$$
$$\cosh(a+c) = \cosh(a+b)\cosh(b+c) - \cos\beta\sinh(a+b)\sinh(b+c).$$

Eliminating $\cos\beta$ gives

$$\begin{aligned}\cosh l &= \cosh(a+b)\cosh t \\ &\quad + \sinh t\left[\frac{\cosh(a+c) - \cosh(a+b)\cosh(b+c)}{\sinh(b+c)}\right] \\ \Rightarrow \cosh l &= \frac{\cosh(a+c)\sinh t + \cosh(a+b)\sinh(b+c-t)}{\sinh(b+c)} \\ \Rightarrow \frac{dl}{da} &= \frac{\sinh(a+c)\sinh t + \sinh(a+b)\sinh(b+c-t)}{\sinh(b+c)\sinh l}.\end{aligned}$$

Define $m = \max\{a+b, a+c\}$,

$$\tau(a) = \min\left\{\frac{\sinh(a+c)}{\sinh m}, \frac{\sinh(a+b)}{\sinh m}\right\}.$$

Recalling that $0 < t < b+c$, we see that $dl/da > \tau(a) > 0$, as desired.

Taking limits as c increases gives the result when $c = \infty$ (or by symmetry, when $b = \infty$). The result is similar when both b and c are infinite; this case will not arise, so the computations are left to the reader. This completes the proof of (iv) and the lemma. ∎

Next, we consider all triangles in the star of a vertex of K. A collection R of radii for K associates with each face a unique hyperbolic triangle. For a particular vertex v, the way in which the triangles comprising its star will fit around v in the hyperbolic plane depends on the angle sum $\theta_v(R)$. Of course, this angle sum only depends upon the radii of v and its immediate neighbors; the next result summarizes how it is affected by these radii.

LEMMA 3. *Let v be a vertex of K with neighboring vertices $w_1, \ldots, w_n$. Let r, $0 < r < \infty$, and $r_1, \ldots, r_n$, $0 < r_j \le \infty$, respectively, denote their hyperbolic radii and let*

$$\theta = \theta(r, r_1, \ldots, r_n)$$

be the resulting angle sum at v.

(*a*) *θ is continuous and strictly decreasing in r.*

(*b*) *θ is continuous and strictly increasing in each of r_j, $j = 1, 2, \ldots, n$.*

(*c*) *Fix radii $r_1, \ldots, r_n$. Given any number Θ, $0 \le \Theta < n\pi$, there exists a unique radius r, $0 < r \le \infty$ for which $\theta = \Theta$.*

(*d*) *If $\theta \ge 2\pi$, then $r \le \sqrt{n}$.*

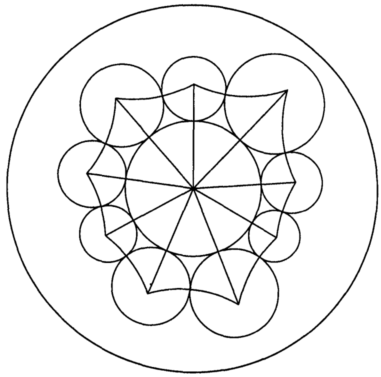


FIG. 8 A typical flower.

Proof. (a) and (b) are just restatements for the star at v of results established in Lemma 2(i) and 2(ii) for individual triangles. Likewise, (c) follows from Lemma 2(i) and the intermediate value theorem. In light of (a), it suffices to consider the case of equality, $\theta = 2\pi$, in the proof of (d). In this case, the configuration of circles can be placed in the disc to form a "flower", as in Figure 8. Let P be the polygon obtained by connecting the centers of the outer circles by geodesics. Then Area$(P) \leq (n-2)\pi$. The circle at v of radius r has area $4\pi \sinh^2(r/2)$ (cf. [Ch. 7, 3]) and lies inside P. We have

$$\pi r^2 = 4\pi\left(\frac{r}{2}\right)^2 \leq 4\pi \sinh^2\left(\frac{r}{2}\right) \leq (n-2)\pi \leq n\pi.$$

Part (d) follows. (More careful analysis gives the sharp upper bound $r \leq -\log(\sin(\pi/n))$.) ■

As we shall see, the bound on r obtained above is one of those advantages gained by working in the hyperbolic plane rather than the euclidean plane, where no such bound exists.

The final monotonicity result compares distances in hyperbolic complexes. Write $\rho_R(\cdot,\cdot)$ for distance in $K(R)$, where as usual, the distance between points is the infimum of the lengths of paths connecting them. We have:

LEMMA 4. *Let $K(R')$ and $K(R)$ be hyperbolic complexes, $R' \leq R$. Then*

$$\rho_{R'}(v_1, v_2) \leq \rho_R(v_1, v_2) \tag{2}$$

for every pair of vertices v_1, v_2. If v_1 satisfies $v_1(R') < v_1(R)$ and if $\rho_R(v_1, v_2)$ is finite, then the inequality is strict.

Proof. Throughout the following, v_1 and v_2 will be fixed, distinct vertices of K, while Γ denotes the collection of paths between them. Quantities associated with $K(R')$ will be marked with a prime (′) to distinguish them from those associated with $K(R)$. For example, $l(\cdot)$ will denote length in $K(R)$ while $l'(\cdot)$ denotes length in $K(R')$. It suffices to work under the assumption that R and R' differ only for a single vertex v. We let $r = v(R)$ and $r' = v(R')$ with $0 < r' < r \leq \infty$.

Fix $\gamma \in \Gamma$. Without affecting the infimum of lengths over Γ, we may assume that γ is simple and that it passes through a finite chain $C = \{f_0, f_1, \dots, f_m\}$ of (not necessarily distinct) faces. It is not appropriate to compare $l(\gamma)$ and $l'(\gamma)$, since we have not specified the point maps by which structures have been placed on K. Instead, (2) will follow by building a path $\gamma' \in \Gamma$, depending perhaps on R', for which

$$l'(\gamma') \leq l(\gamma). \tag{3}$$

Our approach involves successively laying down triangles for the faces of C and then measuring lengths of paths via their immersed images in Δ. To this end, suppose $T_0, T_1, \dots, T_m$ and $T'_0, T'_1, \dots, T'_m$ are triangles corresponding to the faces in C in the structures on K. Carry out developments along the chain using each of these two sets of triangles; we may loosely refer to the developments as D and D'. The path γ is immersed in Δ in the development D. (With a slight abuse of notation, we use the same symbols for quantities in K and in the development). We will build γ' in D'.

Let $S \subseteq K$ be the closed star of v. We need to start under the assumption that all radii for the vertices of S are finite—we will see how to jiggle the picture to accomplish this at the end. Let us start by breaking γ into segments γ_j, $j = 1, 2, \dots, N$, classified as "good" or "bad". Corresponding segments γ'_j will be built in D' and concatinated to form γ'. The (maximal connected) segments of γ which lie in the complement of S are termed good. Triangles associated with faces not in S are congruent under the structures of $K(R)$ and $K(R')$, so a good segment γ_j may be transplanted via a hyperbolic isometry to a segment γ'_j in D' having the same length. The bad segments require more work.

Fix attention on a bad segment γ_j, letting $c = \{g_1, g_2, \dots, g_n\}$ be the subchain of C through which it passes, with corresponding triangles $t_1, t_2, \dots, t_n$ and $t'_1, t'_2, \dots, t'_n$. In each of our two developments, the triangles associated with c all have a common vertex (the one corresponding to v) in Δ, so they lay out like a "fan" about that point. Caution is in order, since there is not a well-defined immersion of all of S when the packing condition fails at v. Thus, the fan of triangles may wrap around v and overlap itself, or a face which occurs more than once in c may give rise to distinct (though congruent) triangles in the development. (It is not difficult to arrange, for example, that the shortest path between two points of the same face is not

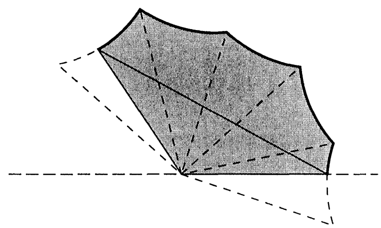


FIG. 9 The fan about v.

the geodesic curve between them in that face but rather a path which leaves the face and goes around v.)

Now, by its maximality and connectedness, γ_j either connects v to a point p which is the endpoint of a good segment (this can happen if $v = v_1$ or $v = v_2$) or else it connects points x and y which are both endpoints of good segments. Endpoints of good segments have well-defined locations in D', since we have already transplanted those segments. So, we have the endpoints of the path γ_j' we are trying to build. The two situations we face require slightly different treatments.

Case 1. γ_j connects v to p. p lies on the edge of t_n which is opposite to v. Since the triangles $t_1, t_2, \ldots, t_n$ have a common vertex, it is evident that the length of γ_j is at least the hyperbolic length of the geodesic segment from v to p in t_n. By Lemma 2(iv), the corresponding segment in t_n' will be shorter, so it is our choice for γ_j'.

Case 2. γ_j connects x to y. Here x lies on the edge of t_1 opposite to v while y lies on the edge of t_n opposite to v. Without loss of generality, we may assume that the two developments of c have v placed at the origin and x placed along the positive real axis. If γ_j passes through v, we are done by two applications of Case 1; so assume $v \notin \gamma_j$. Since γ_j does not pass through the origin, we may consider the argument of a point (as a complex number) moving from x to y along γ_j. Let $\Theta > 0$ denote the change in argument. If $\Theta \geq \pi$, then the shortest path from x to y (through $t_1, \ldots, t_n$ in succession) would pass through the origin. Again, we would be done by Case 1.

We are left, then, with the case $0 < \Theta < \pi$, a situation illustrated in Figure 9. Let

$$P = [x, w_0, w_1, \ldots, w_{n-1}, y]$$

denote the polygonal path from x to y along the outer edges of the triangles, with Ω the polygonal region consisting of the rays from the origin to points of P. It is clear that the shortest path from x to y within our development of triangles must lie in Ω. It is clear that this picture evolves continuously as r

decreases. When we arrive at the development D', where the radius of v has dropped from r to r', we know by earlier monotonicity results how this picture will have changed:

— the angles the triangles form at v will be larger
— the distances $\rho(0, x)$ and $\rho(0, y)$ will be smaller
— the interior angles ψ_j formed by the edges of P (see Figure 9) will be smaller.

The edges of the triangles opposite to v will not change in length, since they are shared with triangles not in S. Likewise, the locations of x and y as endpoints of good segments determine the locations of x' and y' in D'. Our task is to find a path γ_j' in Ω' connecting x' to y' and having length no greater than $l(\gamma_j)$.

We start with the simplest situation: suppose that the geodesic $[x, y]$ lies, except for its endpoints, in the interior of Ω, so we may take $\gamma_j = [x, y]$. By convexity of the triangles, we see that the corner points $w_0, w_1, \ldots, w_k$ of P lie on one side of and a positive distance away from the geodesic through x and y. If we decrease r, the edge lengths of P remain unchanged, but the interior angles ψ_i at which they meet decrease. It is not difficult, using for example the Cosine Law and an induction on the number of edges in P, that the distance between the endpoints of P, the distance $\rho(x, y)$, decreases and, for small decreases in r, the geodesic segment $[x, y]$ remains in Ω. (This depends on the fact that P lies on one side of the geodesic through its endpoints—it may fail otherwise—and also on the assumption of finite radii, which implies the angles ψ_i are positive.) Specifically, there exists $\delta > 0$ so that if $r - \delta < r' < r$, then the segment $\gamma_j' = [x', y']$ lies in Ω' and satisfies

$$l'(\gamma_j') = \rho(x', y') \leq \rho(x, y) = l(\gamma_j).$$

In this case, we are done. In case $[x, y]$ does not lie in Ω, or we want to decrease r by more than δ, we must consider paths with geodesic segments which pass through the interior as well as segments (or points) in $\partial\Omega$. We may apply the reasoning above to the former, replacing each by a segment in D' with corresponding endpoints but whose length is no greater. Meanwhile, those segments in $\partial\Omega$ may be left unchanged, since their lengths are the same in D and D'. The result is a new path γ_j' with $l'(\gamma_j') \leq l(\gamma_j)$.

To complete Case 2, we must consider the situation when one or more of the vertices of S has infinite radius. The triangles in the development of c are determined by circles whose hyperbolic radii are specified. A vertex with infinite radius gives a horocycle, and it is evident that by replacing that radius with a sufficiently large but finite value, one may jiggle the immersion of γ_j by no more than some preassigned small euclidean amount. The previous reasoning applies to the resulting finite situation, and our inequality follows by a standard approximation argument.

The good and bad segments of γ have now given us segments in D' of no greater length which link together to form γ'. This proves (3), and taking the infimum over Γ gives (2).

Finally, the case of equality: This occurs when the radius at v_1 decreases, so we may assume that $v = v_1$ above. The assumption $\rho_R(v_1, v_2) < \infty$ means $r = v(R) < \infty$ and allows us to consider only paths $\gamma \in \Gamma$ having finite length. In particular, the initial segment γ_1 of γ connects v to a point p on the opposite edge of some triangle, as in Case 1. Replacing r by the strictly smaller value r' and applying the full strength of Lemma 2(iv), there exists $\eta > 0$, independent of p and hence independent of γ, so that the geodesic segment γ_1' in D' satisfies

$$l'(\gamma_1') < l(\gamma_1) - \eta.$$

Proceeding as before with the remaining segments, we obtain a path $\gamma' \in \Gamma$ with $l'(\gamma') < l(\gamma) - \eta$. Taking the infimum over Γ gives $\rho_{R'}(v_1, v_2) < \rho_R(v_1, v_2) - \eta$; i.e., strict inequality in (2). ■

5. The boundary value problem

In this section we prove the existence of packings $K(R)$ with specified boundary radii. We rely on a class of hyperbolic complexes broader than the class of packings.

DEFINITION. The hyperbolic complex $K(R)$ is said to be a **local subpacking** at a vertex v if $\theta_v(R) \geq 2\pi$. It is termed a **subpacking** if it is a local subpacking at every interior vertex.

The terminology derives from that of potential theory. Think of a packing as the analogue of a harmonic function. In a subpacking, each interior vertex feels upward pressure on its radius—its radius is too small to meet the packing condition (see Lemma 3(a)). Increasing its radius, its interior neighbors feel additional upward pressure (see Lemma 3(b)), so they increase, and so forth. Repeated adjustments cause the radii to converge monotonically to the radii of a packing; this is essentially the method of "relaxation" for solving the discrete Dirichlet problem and could be used in the next proof. We have chosen, instead, to follow (quite precisely) the line of the classical Perron method.

THEOREM 3. *Given the complex K, let $w_1, \ldots, w_q$ be its boundary vertices, and suppose corresponding hyperbolic radii r_j, $0 < r_j \leq \infty$ are specified, $j = 1, \ldots, q$. Then there exists a unique collection $\widetilde{R}$ of hyperbolic radii for which the hyperbolic complex $K(\widetilde{R})$ is a packing with $w_j(\widetilde{R}) = r_j$, $j = 1, \ldots, q$.*

Of course, the Andreev packing results if all boundary radii are infinite. However, in the proof we need to know of the existence of some packing for K, so we use the existence portion of Andreev's Theorem; only in the Appendix do we show how an inductive argument can get around this.

Incidentally, one can show by the topological argument principle that $K(\widetilde{R})$ is a planar packing: Let $\phi_a : K(\widetilde{R}) \longrightarrow \Delta$ be a locally isometric immersion. A face of K containing a boundary edge e is mapped to a hyperbolic triangle in Δ with edge $\phi_a(e)$ a complete geodesic (starting and ending on the unit circle). One can easily define a homotopy of that triangle which fixes the points of the other two edges while pushing $\phi_a(e)$ out to the unit circle. Applying this process to all such faces, one sees that ϕ_a is homotopic to a continuous discrete mapping of K to $\overline{\Delta}$ which carries ∂K to $\partial\Delta$. This mapping is locally one-to-one on the interior, so the argument principle implies it is globally one-to-one. Undoing the homotopy, this implies ϕ_a is globally one-to-one, so $K(\widetilde{R})$ is a planar packing.

Proof. Define the family $\mathscr{R}$ to consist of those collections R for which the hyperbolic complex $K(R)$ is a subpacking and for which $w_j(R) \leq r_j$ for every boundary vertex w_j, $j = 1, \ldots, q$. Define $\widetilde{R}$ to be the supremum of $\mathscr{R}$; that is,

$$v(\widetilde{R}) = \sup_{R \in \mathscr{R}} \{v(R)\},$$

for every vertex v of K.

We show successively that $\mathscr{R}$ is nonempty, that $w_j(\widetilde{R}) = r_j$, $j = 1, \ldots, q$, while $v(\widetilde{R}) < \infty$ for interior vertices v, that $\widetilde{R} \in \mathscr{R}$, and finally that $K(\widetilde{R})$ is a packing. These steps will prove existence; uniqueness will be shown with arguments involving area.

Step I. $\mathscr{R}$ nonempty. Let R_a denote the radii for the Andreev packing of K, whose existence follows from Theorem 2. As noted, the hyperbolic complex $K(R_a)$ may be isometrically embedded in the hyperbolic plane Δ, giving rise to an associated configuration of circles. Choose a positive t such that $W = \{|z| < t\}$ has hyperbolic diameter smaller than any of the given boundary radii and shrink the configuration of circles with the map $z \mapsto tz$. The resulting circles give a packing $K(R_0)$ which is embedded in W, so R_0 is an element of $\mathscr{R}$. (See Figure 5(a) and Figure 6, which give the circle configurations of an Andreev packing before and after shrinking.)

Step II. Radii in $\widetilde{R}$. Here is where our monotonicity properties enter. Let $K(R)$ be a subpacking and suppose we increase the hyperbolic radius in R corresponding to some boundary vertex w, giving a new collection R'. By Lemma 3, $K(R')$ is also a subpacking. To be precise, the angle sum at w will be less in $K(R')$ than in $K(R)$, the angle sums at neighboring vertices will be more, and the remaining angle sums will be unchanged. In particular, at interior vertices v,

$$\theta_v(R') \geq \theta_v(R) \geq 2\pi,$$

so $K(R')$ is a subpacking.

Now, suppose we start with the collection $R_0 \in \mathscr{R}$ determined previously and we successively increase the boundary radii to their prescribed values r_j. Each change leaves us with a subpacking. Having made all adjustments, we arrive at a collection $R \in \mathscr{R}$ with $w_j(R) = r_j$, $j = 1, \ldots, q$. Consequently, $w_j(\widetilde{R}) = r_j$, $j = 1, \ldots, q$.

Concerning the radii at interior vertices, note that a vertex of K has less than k neighbors, k the number of vertices in K. If $K(R)$ is a subpacking, then $\theta_v(R) \geq 2\pi$ for any interior vertex v, so by Lemma 3(d), $v(R) \leq \sqrt{k} < \infty$. Thus we have a universal finite upper bound on the radii of interior vertices for any subpacking for K. The definition of $\widetilde{R}$ implies $v(\widetilde{R}) \leq \sqrt{k}$.

Step III. $\widetilde{R} \in \mathscr{R}$. As with the classical Perron method, the essential ingredient here is the fact that $\mathscr{R}$ is a net—that is, if $R_1, R_2 \in \mathscr{R}$, then $R = \max\{R_1, R_2\} \in \mathscr{R}$. The boundary radii of R clearly satisfy the required inequality, so we need only verify that $K(R)$ is a subpacking. As this is a local condition, fix attention on an arbitrary interior vertex v and its immediate neighbors. Suppose R_1 has the larger radius at v. Computing $\theta_v(R)$, note that the radius at v is that of R_1, while all the neighboring radii are at least as large as in R_1. By Lemma 3(b), $\theta_v(R) \geq \theta_v(R_1) \geq 2\pi$, as desired.

Using the fact that $\mathscr{R}$ is a net and that angle sums are continuous functions of the radii involved, the proof that $K(\widetilde{R})$ is a subpacking becomes elementary. Since $\widetilde{R}$ has the correct boundary values, we see $\widetilde{R} \in \mathscr{R}$.

Step IV. $K(\widetilde{R})$ is a packing. We need to show that at an interior vertex v, equality holds in $\theta_v(\widetilde{R}) \geq 2\pi$. Suppose this were a strict inequality. The inequality would persist if we made a small increase in the radius at v, and by Lemma 3(b) the angle sums at neighbors would only get larger. We would therefore obtain a collection of radii in $\mathscr{R}$ which would have a radius at v strictly larger than $v(\widetilde{R})$, contradicting the definition of $\widetilde{R}$. Therefore, $K(\widetilde{R})$ is a packing.

We complete our proof with an area comparison which establishes the uniqueness of the packing $K(\widetilde{R})$. Write $\text{Area}_R(\cdot)$ for area in the metric of $K(R)$. This may be computed as follows: Let $v_1, v_2, \ldots, v_p$ denote the interior vertices of K, $w_1, w_2, \ldots, w_q$ the boundary vertices, and N the number of faces. The area of each face in the metric of $K(R)$ is π minus the sum of its three angles. Each of these angles contributes to the angle sum of a vertex, so by adding the areas for all faces and reorganizing the expression we obtain

$$\text{Area}_R(K) = N\pi - \sum_{k=1}^{p} \theta_{v_k}(R) - \sum_{k=1}^{q} \theta_{w_k}(R).$$

Furthermore, if $K(R)$ is a packing, then $\theta_{v_k}(R) = 2\pi$, $k = 1, 2, \ldots, p$, giving

$$\text{Area}_R(K) = N\pi - 2p\pi - \sum_{k=1}^{q} \theta_{w_k}(R). \tag{4}$$

Similarly, we have

$$\text{Area}_{\tilde{R}}(K) = N\pi - 2p\pi - \sum_{k=1}^{q} \theta_{w_k}(\widetilde{R}). \tag{5}$$

Successively replace each radius $v_k(R)$ by the radius $v_k(\widetilde{R})$ and compare the results: Since the radii of $\widetilde{R}$ are at least as large as those of R, the monotonicity result of Lemma 2(iii) implies that

$$\text{Area}_{\tilde{R}}(K) \geq \text{Area}_R(K),$$

with equality only if $R = \widetilde{R}$. On the other hand, the radii of the boundary vertices themselves are unchanged, while those of interior neighbors may increase, so the monotonicity result of Lemma 3(b) implies that

$$\theta_{w_k}(\widetilde{R}) \geq \theta_{w_k}(R), \qquad k = 1, 2, \ldots, q.$$

These two inequalities contradict (4) and (5) unless $R = \widetilde{R}$, proving uniqueness. ■

Figure 7 illustrates the solutions of three boundary value problems for the same complex; in fact, the three have precisely the same boundary radii, save for a single vertex. The changes in that single circle (it can be seen on the inside of the large bend of the configuration) force changes throughout the interior and hence have a major effect on the overall collection.

6. The discrete Schwarz-Pick Lemma

We are now ready for the formal statement of our main result. Note that we write $\rho_R(\cdot, \cdot)$ and $\text{Area}_R(\cdot)$ for distance and area in the hyperbolic complex $K(R)$. These are shortened to $\rho_a(\cdot, \cdot)$ and $\text{Area}_a(\cdot)$ for the Andreev packing $K(R_a)$.

THE DISCRETE SCHWARZ-PICK LEMMA. *Let K be a simplicial complex with Andreev radii R_a, and let R be any collection of radii for which $K(R)$ is a packing. Then:*

(a) $R \leq R_a$; that is, $v(R) \leq v(R_a)$ for every vertex v of K.

(b) $\rho_R(v_1, v_2) \leq \rho_a(v_1, v_2)$ for every pair of vertices v_1, v_2 of K.

(c) $\text{Area}_R(f) \leq \text{Area}_a(f)$ for every face f of K.

Furthermore, if equality holds for a single interior vertex in (a), for a single pair of interior vertices in (b), or for a single face in (c), then $R = R_a$.

We have not stated this result in the greatest generality, hoping instead to maintain a fairly close parallel to the classical situation in complex analysis. That parallel may remain a mystery to the reader at this point, but we will say more about it shortly. As for the proof, however, we may proceed with more general hypotheses.

THEOREM 4. *Let K be a simplicial complex with boundary vertices $w_1, \ldots, w_q$ and assume $K(\widetilde{R})$ is a packing. If $K(R)$ is any subpacking with $w_j(R) \leq w_j(\widetilde{R})$, $j = 1, \ldots, q$, then:*

(a) $R \leq \widetilde{R}$; that is, $v(R) \leq v(\widetilde{R})$ for every vertex v of K.

(b) $\rho_R(v_1, v_2) \leq \rho_{\widetilde{R}}(v_1, v_2)$ for every pair of vertices v_1, v_2 of K.

(c) $\text{Area}_R(f) \leq \text{Area}_{\widetilde{R}}(f)$ for every face f of K.

Furthermore, if equality holds for a single interior vertex in (a) or a single face in (c), or if (finite) equality holds for a pair of vertices in (b), at least one of which is interior, then $R = \widetilde{R}$.

Proof. (a) This inequality is immediate from the proof of Theorem 3: $\widetilde{R}$ is the supremum of the net $\mathscr{R}$ of collections R for which $K(R)$ is a subpacking and $w_j(R) \leq w_j(\widetilde{R})$, $j = 1, \ldots, q$.

The case for equality depends on conditions we have placed on K: Recall that any two interior vertices of K may be joined by an edge path which goes through only interior vertices, and that every boundary vertex has an interior neighbor. Suppose, now, that v is an interior vertex. The angle sum $\theta_v(R) \geq 2\pi$ and is a strictly increasing function of the neighboring radii. Since $\theta_v(\widetilde{R}) = 2\pi$ and any neighboring vertex w satisfies $w(R) \leq w(\widetilde{R})$, the equality $v(R) = v(\widetilde{R})$ would imply $w(R) = w(\widetilde{R})$ for all these neighbors. It is immediate that, with the two conditions mentioned on K, this equality propagates to all vertices, giving $R = \widetilde{R}$.

(b) This inequality follows from (a) and Lemma 4. As for equality, suppose we have $\rho_R(v_1, v_2) = \rho_{\widetilde{R}}(v_1, v_2) < \infty$, were v_1 is an interior vertex. By the last statement of Lemma 4, we have $v_1(R) = v_1(\widetilde{R})$; this gives equality in (a), implying $R = \widetilde{R}$.

(c) Every face has at least one interior vertex v, so this inequality follows from the monotonicity result of Lemma 2(iii). Since monotonicity is strict there, equality in (c) would imply $v(R) = v(\widetilde{R})$, hence $R = \widetilde{R}$ by (a). ∎

7. Conclusion

We have alluded to connections between our work in the discrete setting of circle packing and the classical continuous setting of complex analysis. Some of these—analytic continuation, the monodromy theorem, the Perron method, boundary value problems—are apparent and, in fact, may seem rather superficial. However, as the topic develops, the authors feel that the deeper and more fundamental parallels will become clear. Whether any of that depth is apparent now depends in part on how the classical results are viewed.

From a geometric standpoint, the classical Schwarz-Pick result is a statement about contractions and extremal mappings or about curvatures and extremal metrics. For example, in the classical setting, an analytic self map of the disc is a contraction or an isometry. The Discrete Schwarz-Pick Lemma (DSPL) implies that if $K(R)$ is a packing (or even a subpacking), then the identity map from $K(R_a)$ to $K(R)$ is a contraction (on vertices) or an isometry. Alternately, the classical result states that the Poincaré metric is the maximal ultrahyperbolic metric on Δ (cf. [1]). In our setting, we have the underlying complex K in place of Δ and we have the metrics induced by the various hyperbolic structures, what one might term "simplicial" hyperbolic structures. When $K(R)$ is a subpacking, the corresponding metric is ultrahyperbolic. The DSPL implies that $K(R_a)$ has the unique maximal ultrahyperbolic metric among these.

Satisfying as these analogies might be, the reader may well be asking: Where are the analytic functions? It is perhaps premature to settle on an explicit definition. Nonetheless, we will suggest one possibility—a definition which at least justifies the name of the paper, while indicating some additional parallels with the classical continuous setting.

As throughout the paper, K is a fixed simplicial complex. Each R for which $K(R)$ is a packing determines (up to an isometry) a locally isometric immersion $\phi_R\colon K(R) \longrightarrow \Delta$. For purposes of normalization, fix an interior vertex v_0 and a neighboring vertex v_1 of K and require

$$\phi_R(v_0) = 0, \qquad \phi_R(v_1) > 0.$$

In case $R = R_a$, we have the Andreev packing, and the corresponding immersion ϕ_a is an embedding. Therefore, $K(R_a)$ is isometrically isomorphic with its image $\Delta_K = \phi_a(K) \subseteq \Delta$; it is convenient in what follows to identify K with this concrete realization.

Now, we define the class $\mathcal{F}$ of functions $F\colon \Delta_K \to \Delta$ of the form

$$F \equiv \psi \circ \phi_R \circ \phi_a^{-1},$$

where ψ is a (classical) hyperbolic isometry of Δ and $K(R)$ is a packing. In the setting of this paper, the functions of $\mathcal{F}$ would be termed **discrete analytic functions**. They are open, continuous, and locally univalent. Their common domain Δ_K plays the role of the unit disc. Each F maps Δ_K to an immersed image of some hyperbolic complex $K(R)$—essentially this immersed image is the image Riemann surface of F. F is univalent if and only if $K(R)$ is a planar packing, in which case the image of F is isometrically isomorphic with $K(R)$. F would be called an isometry if $F \equiv \psi|_{\Delta_K}$ for some hyperbolic isometry ψ of Δ.

Let us interpret the results of the DSPL: Given vertices $v, w \in K$, the distance $\rho_R(v, w)$ is the infimum of the lengths of paths from v to w in $K(R)$. Working instead in the hyperbolic plane, this may be interpreted as the infimum of the hyperbolic lengths of paths from $\phi_R(v)$ to $\phi_R(w)$ in an immersed image of $K(R)$. In particular,

$$\rho(\phi_R(v), \phi_R(w)) \le \rho_R(v, w).$$

In the case of the Andreev packing, ϕ_a is an embedding and the image Δ_K is a convex polygon; consequently

$$\rho(\phi_a(v), \phi_a(w)) = \rho_a(v, w).$$

Identifying K with Δ_K and applying inequality (b) of the DSPL, we have this discrete version of the classical contraction principle:

THEOREM OF PICK. *Suppose $F \in \mathcal{F}$ and $v, w \in \Delta_K$ are distinct vertices of K. Then*

$$\rho(F(v), F(w)) \le \rho(v, w).$$

Finite equality holds if and only if F is an isometry.

In the classical setting, the infinitesimal version of this result concerns the factor by which an analytic function contracts distances locally, i.e., the magnitude of its derivative. In our discrete setting, this local contraction factor at an interior vertex v would seem to be the ratio of $v(R)$ to $v(R_a)$.

Applying inequality (a) of the DSPL, we have a discrete version of the local contraction principle:

THEOREM OF SCHWARZ. *Suppose $F \in \mathscr{F}$ and $v \in \Delta_K$ is an interior vertex. Then*

$$\frac{v(R)}{v(R_a)} \leq 1.$$

Equality holds if and only if F is an isometry.

The notation $|F'(v)| = v(R)/v(R_a)$ might be appropriate. With this in mind, a "development" along a chain (see Section 2) is more akin to a line integral than to analytic continuation. What about $\arg(F')$? Well, that will have to await further study. But think for a moment about a polygonal path between two vertices along edges of the immersed complex Δ_K. Decreases in those edge lengths under F would not account for the inequality of Pick's theorem unless there were closely coupled changes in the angles at which they meet. Suffice it to say that the particular hyperbolic structures which we have imposed on K seem to embody a discrete analogue of the Cauchy-Riemann equations.

Of course, these are only analogies; the functions of $\mathscr{F}$ are not analytic in the classical sense—indeed, they are simplicial. It has been pleasing, therefore, to see how faithfully the results mimic the classical models when appropriate discrete notions are found. The discerning reader may foresee other opportunities, e.g. branch points (angle sums which are multiples of 2π) and multiply-connected complexes. Indeed, since the topic's inception, with Thurston's Purdue talk, the possibility of such analogies has been very appealing—at least to those who view complex analysis as a geometric subject. Our objective in this paper was not simply to prove the discrete version of the Schwarz-Pick lemma, but also to provide a framework for the development of further parallels. The hyperbolic setting, the notion of the "hyperbolic complex", the contraction property, and the solution of boundary value problems seem quite important in this regard. The authors expect that, with further research, these and other tools will lead to a wide range of analogies with complex analysis—ultimately, perhaps, to a discrete complex analysis.

Appendix

Although the existence portion of Andreev's theorem (Theorem 2) was used in the proof of Theorem 3, we now present an inductive argument in its place. Coupled with the consideration of more general complexes, this furnishes an independent proof of Andreev's Theorem.

It is precisely because of the need for general complexes that we have left these arguments for an appendix. To avoid ambiguity, we use "complex" here to refer to any 2-complex which is isomorphic to a finite triangulation of the closed disc, while we term as "special" those meeting the additional requirements used earlier (see the definition in Section 1). Note that if K has boundary vertices lacking interior neighbors, or if the interior vertices of K comprise more than one component, then the conclusions regarding equality in the Discrete Schwarz-Pick Lemma and Theorem 4 may fail. These are among the reasons that the special complexes of the body of the paper more closely model the continuous complex setting.

To prove Theorem 2 with K any complex, we use induction on k, the number of its vertices. Note that the topological boundary of K is a simple closed curve γ passing through all boundary vertices. If $k \leq 3$, our result is evident; henceforth assume $k \geq 4$ and assume that our conclusion holds for complexes having fewer than k vertices.

Suppose first that K is not a special complex. It is not difficult to find a pair v, w of boundary vertices connected by an edge not belonging to γ. In particular, any two components of interior vertices will be separated by such an edge and any vertex without an interior neighbor can be separated from the rest of the complex by such an edge. Removing that edge breaks K into two smaller complexes, K_1 and K_2, which share only the vertices v and w. By the induction hypothesis, there are Andreev packings for K_1 and K_2. Let P_1 and P_2 denote corresponding circle configurations. The circles for v and w will be tangent horocycles in each configuration, so applications of Möbius transformations allow us to assume that these circles are

$$C_v = \{|z - \tfrac{1}{2}| = \tfrac{1}{2}\} \quad \text{and} \quad C_w = \{|z + \tfrac{1}{2}| = \tfrac{1}{2}\}$$

in each case. Orientation considerations show that the remaining circles for K_1 will lie on the opposite side of the real axis from those for K_2. Therefore, superimposing the two configurations provides an Andreev packing for K. Uniqueness is evident from the hypothesized uniqueness of the Andreev packings for K_1 and K_2. This completes the induction step if K is not special.

If K is special, vertices v, w as above cannot exist and we proceed differently. Let v be any boundary vertex of K. Removing v and the edges and faces of its star from K yields a reduced complex K', with former neighbors of v being boundary vertices of K'. By the induction hypothesis, there is an Andreev packing for K', and we denote by P' a corresponding configuration of circles in Δ. By considering our situation on the Riemann sphere, we will use the circles of P' along with the exterior of Δ, call that C, to reconstitute K.

The argument is this: Pick boundary vertices u, w of K' which are neighbors both in K' and in K (i.e., so that u, v, w are not the vertices of a face of

K). Let C_u and C_v denote the corresponding horocycles in P', and choose a disc $\Delta(x, r)$ having its closure in the interstice formed by C_u, C_w, and C. Apply a Möbius transformation which carries x to infinity and $\partial\Delta(x, r)$ to the unit circle. With a slight abuse of notation, we continue to use P' for the now transformed configuration, again in Δ. The (transformed) circle C will be identified with v and denoted C_v. It along with P' gives a collection of circles in Δ with mutually disjoint interiors which represents a packing for K. Indeed, the circles of P' represent a packing for K'; moreover, C_v is tangent to all the boundary circles of P'. Disregarding the extraneous tangencies, C_v is tangent to all the circles associated with its neighboring vertices in the original complex K. Thus the packing condition holds for any of those which happen to be interior vertices of K. Conclusion: we have obtained a packing for K.

Now consider the proof of Theorem 3, noting that K is one of the special complexes hypothesized there. The packing we obtained for K is all we need to show that the family $\mathscr{R}$ of subpackings there is nonempty. The proof yields an Andreev packing for K (and its uniqueness). This completes the induction step for special complexes, and hence the proof of Andreev's Theorem for complexes.

On a practical note, a technique for computing the radii of packings by an iterative process was suggested by Thurston in his Purdue talk. It relies on Lemma 3(c) and consists of repeated adjustments to individual radii to achieve local packing. A version of this technique produced the pictures in this paper on a SUN microcomputer; the algorithm works especially nicely in the hyperbolic setting, where the monotonicity results, the bound of Lemma 3(d), and finite area are available. Even on a home computer, one can compute and display hyperbolic complexes and their circle packings, allowing for geometric experiments while producing some very beautiful and intricate pictures.

Added in proof. Steffen Rohde has brought to our attention a reference which has been overlooked in the circle packing literature; namely, what has been termed the Andreev or Andreev-Thurston Theorem (Theorem 2 above) was proven by Paul Koebe in "Kontaktprobleme der konformen Abbildung", Ber. Sächs. Akad. Wiss. Leipzig, Math,-phys. Kl., vol. 88 (1936), 141–164.

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