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HOLOMORPHIC FUNCTIONS WITH POSITIVE REAL PART ON THE UNIT BALL OF C^n

BY

JOHN N. MCDONALD

Consider the set $\mathscr{P}$ of holomorphic functions on the open unit ball B of C^n which have positive real part and take the value 1 at 0. Except in the case where $n = 1$, the problem of identifying the extreme elements of the convex set $\mathscr{P}$ is unsolved. Some results on this interesting and natural question have been obtained by Forelli in papers mentioned below and there is a discussion of it in the book of Rudin [7]. It seems, however, that a complete and satisfactory solution is not close at hand.

In this paper we study the relationship between the extreme elements of $\mathscr{P}$ and the extreme elements of the closed unit ball $\mathscr{U}$ of the space $H^\infty(B)$ via the representation

$$f(z) = (1 + g(z))/(1 - g(z)), \tag{1}$$

where g is a member of $\mathscr{U}$ which vanishes at 0. Forelli has shown that the function (1) is an extreme point of $\mathscr{P}$ in the cases where

$$g(z) = g(z_1, z_2, \dots, z_n) = z_1^2 + z_2^2 + \cdots + z_n^2$$

and

$$g(z) = cz^\alpha = cz_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n},$$

where the greatest common divisor of the positive integers α_j is 1 and c is a constant chosen so that

$$\|g\| = \sup\{|g(z)| : z \in B\} = 1.$$

See [1], [3]. Forelli has also produced sufficient conditions on a homogeneous polynomial p in order that $(1 + p)/(1 - p)$ be extreme in $\mathscr{P}$ [3]. One of our main results implies that, if g is a homogeneous polynomial of degree $k \geq 1$ which is also an extreme point of $\mathscr{U}$, then there exists a polynomial r of degree $\leq k - 1$ such that $(1 + g + r)/(1 - g)$ is an extreme point of $\mathscr{P}$. We also use our results to derive the examples of Forelli described above, as well as some new examples of extreme members of $\mathscr{P}$.

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Main results

THEOREM 1. *Suppose that an extreme element f of $\mathcal{U}$ is written in the form* (1). *Then g is an extreme element of $\mathcal{U}$.*

Proof. Suppose that g is not an extreme point of $\mathcal{P}$. Then, by results due to R. Phelps [4, Lemma 3.1 and Corollary 3.2], there exists a non-zero function h in $H^\infty(B)$ such that

$$|g^2| + |h| \leq 1.$$

Replacing $h(z)$ by $z_1 h(z)$ if necessary, we may assume that $h(0) = 0$. We will show that f is not extreme by showing that

$$0 \leq \mathrm{Re}\left(\frac{1 + g \pm \frac{1}{2}h}{1 - g}\right). \tag{2}$$

To verify (2), we first observe that

$$\mathrm{Re}\left(\frac{1 + g \pm \frac{1}{2}h}{1 - g}\right) = 1 - |g|^2 \pm \tfrac{1}{2}\frac{\mathrm{Re}(h(1 - \bar{g}))}{|1 - g|^2}.$$

Since $\mathrm{Re}(h(1 - \bar{g})) \leq 2|h|$, it follows that

$$1 - |g|^2 \pm \tfrac{1}{2}\mathrm{Re}(h(1 - \bar{g})) \geq 1 - |g|^2 - |h| \geq 0.$$

Remarks. It is clear that the proof above works for more general domains.

Another necessary condition on extreme points of $\mathcal{P}$ is given by Forelli in [2].

The next result amounts to an observation: namely, that a theorem of Rochberg concerning positive linear operators on the disc algebra [6] can be rephrased as a theorem about holomorphic functions with positive real part on the unit disc D in the complex plane. The proof is almost word for word the same as the one given by Rochberg for his result.

THEOREM 2. *Suppose that F is holomorphic and has positive real part on D and that $F(0) = 1$. Let*

$$F(\lambda) = 1 + 2\sum_{n=1}^{\infty} a_n \lambda^n$$

be the Taylor series expansion of F. Then, for $n, m \geq 1$,

$$|a_{n+m} - a_n a_m| \leq 4(1 - |a_n|)^{1/4}.$$

Proof. By Herglotz's Theorem there exists a measure μ on the unit circle T such that

$$F(\lambda) = \int_T \frac{x+\lambda}{x-\lambda}\,d\mu(x).$$

Also, we have

$$a_j = \int_T \bar{x}^j\,d\mu(x),$$

for $j = 1, 2, \ldots$. Replacing $F(\lambda)$ by $F(e^{i\alpha}\lambda)$ for appropriate α if necessary, we may assume that a_n is a positive real number. Let

$$S = \left\{x \in T\colon \operatorname{Re} x^n \le a_n - (1-a_n)^{1/2}\right\}.$$

Since a_n is real, we have

$$a_n = \int_S \operatorname{Re} x^n\,d\mu(x) + \int_{T\setminus S} \operatorname{Re} x^n\,d\mu(x).$$

Thus,

$$\begin{aligned} a_n &\le \mu(S)\left(a_n - (1-a_n)^{1/2}\right) + \mu(T\setminus S) \\ &\le \mu(S)\left(a_n - 1 - (1-a_n)^{1/2}\right) + 1. \end{aligned}$$

It follows that

$$\text{(2)} \qquad \begin{aligned} \mu(S) &\le (1-a_n)/\left(1 - a_n + (1-a_n)^{1/2}\right) \\ &\le (1-a_n)^{1/2}. \end{aligned}$$

Next we observe that

(3)

$$\begin{aligned} |a_{n+m} - a_n a_m| &= \left|\int_T (\bar{x}^n - a_n)\bar{x}^m\,d\mu(x)\right| \\ &\le \left|\int_S (\bar{x}^n - a_n)\bar{x}^m\,d\mu(x)\right| + \left|\int_{T\setminus S} (\bar{x}^n - a_n)\bar{x}^m\,d\mu(x)\right| \\ &\le 2\mu(S) + \sup\{|\bar{x}^n - a_n| : x \in T\setminus S\}. \end{aligned}$$

Also, for $x \in T \setminus S$ we have

$$(4)\qquad \begin{aligned} |\bar{x}^n - a_n|^2 &= 1 - 2a_n \operatorname{Re} \bar{x}^n + |a_n|^2 \\ &\le 1 - 2a_n\left(a_n - (1 - a_n)^{1/2}\right) + a_n^2 \\ &\le 2(1 - a_n) + 2(1 - a_n)^{1/2} \\ &\le 4(1 - a_n)^{1/2}. \end{aligned}$$

The theorem now follows from (2), (3), and (4).

We recall that each f in $\mathscr{P}$ has a unique expansion of the form

$$f(z) = 1 + 2\sum_{j=1}^{\infty} f_j(z),$$

where f_j is a homogeneous polynomial of degree j with $|f_j(z)| \le 1$ for $z \in B$.

THEOREM 3. *Suppose that f is in $\mathscr{P}$ and that k is a positive integer. If f_k is an extreme point of $\mathscr{U}$, then there exists a polynomial q of degree $\le k - 1$, such that*

$$f = \frac{1 + f_k + q}{1 - f_k}.$$

Proof. For fixed $z \in B$, let $F(\lambda) = f(\lambda z)$, where $\lambda \in D$. Then

$$F(\lambda) = 1 + 2\sum_{j=1}^{\infty} f_j(z)\lambda^j.$$

Hence, by the previous theorem, we have

$$|(f_{k+m}(z) - f_k(z)f_m(z))/4|^4 \le 1 - |f_k(z)|$$

for $m = 0, 1, 2, \ldots$. Let $g(z) = (4^{-1}(f_{k+m}(z) - f_k(z)f_m(z)))^4$. Then, since $|g(z)| + |f_k(z)| \le 1$, it follows that $f_k \pm g \in \mathscr{U}$. Since f_k is an extreme element of $\mathscr{U}$, we must have $g(z) = 0$. Hence, $f_{k+m}(z) = f_k(z)f_m(z)$ for $m = 0, 1, 2, \ldots$. Returning to the homogeneous expansion for f, we find that

$$\begin{aligned} f(z) &= 1 + 2\sum_{j=1}^{k}\sum_{m=0}^{\infty} f_{j+mk}(z) \\ &= 1 + 2\sum_{j=1}^{k}\sum_{m=0}^{\infty} f_j(z)(f_k(z))^m \\ &= \frac{1 + f_k(z) + q(z)}{1 - f_k(z)}, \end{aligned}$$

where $q = \sum_{j=1}^{k-1} 2f_j$.

Consider a polynomial p which belongs to $\mathscr{U}$ and is homogeneous of degree $k \geq 1$. Let $\mathscr{F}(p) = \{f \in \mathscr{P}\colon f_k = p\}$. Note that $\mathscr{F}(p)$ is closed with respect to the topology of uniform convergence on compact subsets of B and is convex. If p happens to be an extreme point of $\mathscr{U}$, then $\mathscr{F}(p)$ is also a *face* of $\mathscr{P}$, i.e., if $cf_1 + (1-c)f_2 \in \mathscr{F}(p)$, where f_1 and f_2 belong to $\mathscr{P}$ and $0 < c < 1$, then f_1 and f_2 belong to $\mathscr{F}(p)$. Since $(1+p)/(1-p) \in \mathscr{F}(p)$, it follows from the Krein-Milman Theorem that $\mathscr{F}(p)$ always contains extreme elements of $\mathscr{P}$. As the following shows, even more is true.

COROLLARY. *If p is a homogeneous polynomial of degree $k \geq 1$ which is also an extreme point of $\mathscr{U}$, then there exists a polynomial q of degree $\leq k-1$ such that*

$$\frac{1+p+q}{1-p} \tag{5}$$

is an extreme point of $\mathscr{P}$. Furthermore, every function in $\mathscr{F}(p)$ is a convex combination of at most $\langle k-1\rangle + 1$ extreme elements of $\mathscr{P}$ of the form (5), *where $\langle k-1\rangle$ is the real dimension of the space of polynomials of degree $\leq k-1$ in n complex variables.*

Proof. The existence of an extreme element of $\mathscr{P}$ of the form (5) follows from Theorem 3 and the remarks in the paragraph above. The "Furthermore" part of the corollary follows from a result of Caratheodory which asserts that every member of a compact convex subset K of real m-dimensional space can be written as a convex combination of at most $m+1$ extreme elements of K. See [5].

Examples

First we will establish some notation. S will denote the boundary of B and T^n will denote the n-dimensional torus

$$\left\{t \in C^n\colon |t_j| = 1 \quad \text{for } j = 1, 2, \ldots, n\right\}.$$

We observe that if $t \in T^n$ and if $z \in S$, then

$$tz = \left(t_1 z_1, t_2 z_2, \ldots, t_n z_n\right) \in S.$$

We will make use of the normalized Haar measure m on T^n. We will use lower case Greek letters without subscripts to indicate multi-indices. Thus, α denotes an n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where the α_j's are non-negative integers. We will write $\alpha \blacktriangleright \beta$ if $\alpha_j \geq \beta_j$ for $j = 1, 2, \ldots, n$ and $|\alpha| > |\beta|$, where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. It will be convenient to signify the n-tuple $(0, 0, \ldots, 0)$ by 0.

LEMMA 1. *Suppose that f is a member of $\mathscr{U}$ and has the Taylor expansion $f(z) = \Sigma_\alpha a_\alpha z^\alpha$ and the homogeneous expansion $f(z) = \Sigma_{k=0}^\infty f_k(z)$ for $z \in B$. Then*

$$\sum_\alpha |a_\alpha z^\alpha|^2 \leq 1 \tag{6}$$

and

$$\sum_{k=0}^{\infty} |f_k(z)|^2 \leq 1. \tag{7}$$

Proof. By Parseval's identity,

$$\begin{aligned}\sum_\alpha |a_\alpha z|^2 &= \int_{T^n} |f(tz)|^2 \, dm(t) \\ &\leq \|f\|^2 \\ &\leq 1.\end{aligned}$$

The inequality (7) follows by a similar argument.

LEMMA 2. *Let $g(z) = z_1^2 + z_2^2 + \cdots + z_n^2$. Then g is an extreme point of $\mathscr{U}$.*

Proof. We will show that, if $h \in H^\infty(B)$ and $g \pm h \in \mathscr{U}$, then $h = 0$. Let

$$h(z) = \sum_{k=0}^{\infty} h_k(z)$$

be the homogeneous expansion of h. By (7) we have

$$|g(z) \pm h_2(z)|^2 + \sum_{k \neq 2} |h_k(z)|^2 \leq 1.$$

Suppose that $x \in S \cap R^n$. Then $g(x) = 1$. Since $|g(x) \pm h_2(x)|^2 \leq 1$, it follows that $h_2(x) = 0$. Since h_2 is homogeneous of degree 2, it follows immediately that h_2 vanishes on $B \cap R^n$. Hence, h_2 must vanish on all of C^n. A similar argument shows that h_k vanishes for $k \neq 2$.

The following result was first obtained by Forelli in [1].

THEOREM 4. *Let g be as in Lemma 2. Then $f = (1 + g)/(1 - g)$ is an extreme element of $\mathscr{P}$.*

Proof. We note that the homogeneous expansion of f is

$$f(z) = 1 + 2 \sum_{k=1}^{\infty} (g(z))^k.$$

Hence, $f \in \mathscr{F}(g)$. Recall that, by Theorem 3, a function f_1 belongs to $\mathscr{F}(g)$ if and only if it is of the form

$$f_1(z) = \frac{1 + g(z) + q(z)}{1 + g(z)},$$

where q is a homogeneous polynomial of degree at most 1. But as the argument used by Rudin in his book [7, p. 412] shows, any degree one homogeneous polynomial q for which the function f_1 belongs to $\mathscr{P}$ must vanish. Thus the face $\mathscr{F}(g)$ contains only one element. It follows that f is extreme.

Next we consider monomials of the form $h(z) = cz^\alpha$, where c is chosen so that $\|h\| = 1$. For convenience sake we will assume that c is a positive real number. First we will develop necessary and sufficient conditions for h to be an extreme element of $\mathscr{U}$. We begin by observing that in the case where $n > 1$ and $|\alpha| = 1$, h is not an extreme point of $\mathscr{U}$. For if, say $h(z) = z_1$, then, since

$$|z_2|^2 \leq 1 - |z_1|^2 \leq 2(1 - |z_1|),$$

it follows that

$$|z_1 \pm 2^{-1}z_2^2| \leq |z_1| + |z_2^2/2| \leq 1.$$

Thus, h is not an extreme point of $\mathscr{U}$. The case where $|\alpha|$ and $n > 1$ is handled by the following:

LEMMA 3. *If $|\alpha|$ and $n > 1$, then h is an extreme point of $\mathscr{U}$ if and only if $\alpha_j > 0$ for $j = 1, 2, \ldots, n$.*

Proof. Suppose that $\alpha_j > 0$ for $j = 1, 2, \ldots, n$ and that v is a function in $H^\infty(B)$ with $\|h \pm v\| \leq 1$. We will show that $v = 0$. Denoting the Taylor series expansion of v by $v(z) = \Sigma_\beta v_\beta z^\beta$, from (6) we have

$$|cz^\alpha \pm v_\alpha z^\alpha|^2 + \sum_{\beta \neq \alpha} |v_\beta z^\beta|^2 \leq 1.$$

Since all of the α_j's are positive, there is a point w on S such that $cw^\alpha = 1$ and no w_j is zero. Thus,

$$|1 \pm v_\alpha w^\alpha|^2 + \sum_{\beta \neq \alpha} |v_\beta w^\beta|^2 \leq 1$$

leads to $v_\beta = 0$ for every β.

Next we assume that one of the α_j's, say α_1, is zero. Suppose that z is a point of B. Let

$$a = \left(1 - |z_1|^2\right)^{1/2} \quad \text{and} \quad w = (0, z_2/a, z_3/a, \ldots, z_n/a).$$

Since $w \in B$, it follows that $a^{-|\alpha|}|h(z)| = |h(w)| \leq 1$. Thus,

$$|h(z)| \leq \left(1 - |z_1|^2\right)^{|\alpha|/2} \leq 1 - |z_1|^2.$$

It follows that $|h(z) \pm z_1^2| < 1$, and, hence, that h is not extreme in $\mathscr{U}$.

From now on we will assume that α is a multi-index with $\alpha_j > 0$. We will characterize the face $\mathscr{F}(h)$. Recall that f_1 is a member of $\mathscr{F}(h)$ if and only if it is of the form

$$f_1(z) = \frac{1 + h(z) + q(z)}{1 - h(z)},$$

where q is a polynomial of the form

$$q(z) = \sum_{0<|\beta|<|\alpha|} q_\beta z^\beta,$$

and where

$$0 < 1 - |h(z)|^2 + \operatorname{Re}\left(\left(1 - \overline{h(z)}\right) q(z)\right) \tag{8}$$

for $z \in B$. We will see that (8) imposes strong restrictions on the coefficients of q.

LEMMA 4. *Suppose that* (8) *holds and that, for some* β, $q_\beta \neq 0$. *Then* $\alpha \blacktriangleright \beta$.

Proof. Let $H(z)$ denote the right hand side of (8). An easy calculation shows that

$$\int_{T^n} H(tz)\bar{t}^\beta \, dm(t) = \begin{cases} q_\beta z^\beta - \bar{q}_{\alpha-\beta}(\bar{z})^{\alpha-\beta} c z^\alpha & \text{if } \alpha \blacktriangleright \beta \\ q_\beta z^\beta & \text{otherwise.} \end{cases} \tag{9}$$

It follows that, if β fails to satisfy $\alpha \blacktriangleright \beta$, then

$$|q_\beta z^\beta| \leq \int_{T^n} H(tz) \, dm(t) = 1 - |h(z)|^2. \tag{10}$$

Hence, $|(h(z))^2 \pm q_\beta z^\beta| \leq 1$. By the extremality of $(h(z))^2$, we have $q_\beta = 0$.

LEMMA 5. *Suppose that* (8) *holds and that* y *is a point of* $S \cap R^n$ *such that* $cy^\alpha = 1$. *If* $\alpha \blacktriangleright \beta$, *then, either* $|q_\beta y^\beta| < 2$, *or* $\alpha = 2\beta$.

Proof. In the case where $\alpha \blacktriangleright \beta$ it follows from (9) that

$$|q_\beta z^\beta - \bar{q}_{\alpha-\beta}(\bar{z})^{\alpha-\beta} c z^\alpha| \leq 1 - |cz^\alpha|^2. \tag{11}$$

Let y be a point of $S \cap R^n$ with $cy^\alpha = 1$. Then it follows from (11) that

$$q_\beta y^\beta = \bar{q}_{\alpha-\beta} y^{\alpha-\beta}.$$

Let $z = (ay_1, y_2, \dots, y_n)$, where $0 < a < 1$. Then using (11) we obtain

$$|q_\beta y^\beta| |a^{\beta_1} - a^{2\alpha_1-\beta_1}| \le 1 - a^{2\alpha_1}.$$

Dividing both sides of this inequality by $1 - a$ and then letting a approach 1 we get

$$|q_\beta y^\beta| (\alpha_1 - \beta_1) \le \alpha_1.$$

Replacing β by $\alpha - \beta$ leads to

$$|q_\beta y^\beta| \beta_1 \le \alpha_1.$$

A similar argument shows that

$$|q_\beta y^\beta| (\alpha_j - \beta_j) \le \alpha_j \quad \text{and} \quad |q_\beta y^\beta| \beta_j \le \alpha_j$$

for $j = 2, 3, \dots, n$. If none of the inequalities above is strict, then it follows that $\alpha_j = 2\beta_j$ for $j = 1, 2, \dots, n$, i.e., $\alpha = 2\beta$. If at least one of the inequalities above is strict, it follows that, for some j, $|q_\beta y^\beta| \alpha_j < 2\alpha_j$. Thus, $|q_\beta y^\beta| < 2$. The lemma follows immediately from the two preceding inequalities.

LEMMA 6. *Let* (8) *hold. If* $q_\beta \neq 0$, *then there is a real number* A *such that* $\beta = A\alpha$.

Proof. It follows from Lemma 4 that $\alpha \blacktriangleright \beta$. Also, if $\alpha = 2\beta$, there is nothing to prove. We may, therefore, assume that $\alpha \neq 2\beta$.

Let $r_\beta = \operatorname{Re} q_\beta \neq 0$. Consider the function

$$K(x) = 1 - |x^\alpha|^2 + r_\beta x^\beta - r_{\alpha-\beta} c x^{2\alpha-\beta},$$

where x varies over $S \cap R^n$. It is clear that K is non-negative and that $K(y) = 0$ if $cy^\alpha = 1$. Using the operators $x_j \partial/\partial x_j$ together with the method of Lagrange multipliers, we obtain a real number C such that

$$-2\alpha + r_\beta y^\beta \beta - r_{\alpha-\beta} y^{\alpha-\beta}(2\alpha - \beta) = -2Cy^*,$$

where $y^* = (y_1^2, y_2^2, \dots, y_n^2)$. It follows from $r_\beta y^\beta = r_{\alpha-\beta} y^{\alpha-\beta}$ that

$$\left(1 + r_\beta y^\beta\right)\alpha - r_\beta y^\beta \beta = Cy^*.$$

Applying the same argument with β replaced by $\alpha - \beta$ and again using

$$r_\beta y^\beta = r_{\alpha-\beta} y^{\alpha-\beta},$$

we obtain a real number D such that

$$\alpha + r_\beta y^\beta \beta = D y^*.$$

Thus,

$$\left(-C + D + D r_\beta y^\beta\right)\alpha = (C + D) r_\beta y^\beta \beta.$$

To complete the argument in the case $r_\beta \neq 0$ we need only observe that, by Lemmas 4 and 5, $C + D = (2 + r_\beta y^\beta)|\alpha|$ does not vanish.

The case where $\operatorname{Re} q_\beta = 0$, but $\operatorname{Im} q_\beta \neq 0$ is handled in a similar fashion.

Suppose now that the greatest common divisor of the integers α_j, $j = 1, 2, \ldots, n$ is 1. Then it is not hard to show that the conditions $\beta = A\alpha$ and $\alpha \blacktriangleright \beta \blacktriangleright 0$ are incompatible. It follows that if $\gcd\{\alpha_j\} = 1$, then $q_\beta = 0$ for each β with $|\alpha| > |\beta|$. Hence, $\mathscr{F}(h)$ reduces to the single element $(1 + h)/(1 - h)$. Since $\mathscr{F}(h)$ is a face of $\mathscr{P}$ it follows that $(1 + h)/(1 - h)$ is an extreme point of $\mathscr{P}$.

Consider the case where $\gcd\{\alpha_j\} = k > 1$. Let $\theta = k^{-1}\alpha$. It is not hard to show that the conditions $\beta = A\alpha$ and $\alpha \blacktriangleright \beta \blacktriangleright 0$ imply that $\beta = m\theta$ for some integer m with $0 < m < k$. It follows that the polynomial q above takes the form

$$q(z) = \sum_{m=1}^{k-1} q_{m\theta} z^{m\theta}.$$

Let c_1 be a positive constant chosen so that the function $h_1(z) = c_1 z^\theta$ satisfies $\|h_1\| = 1$. It is easily seen that $h = (h_1)^k$. The polynomial q can be written in the form $q(z) = q^*(c_1 z^\theta)$, where q^* is the polynomial in one variable of degree $\leq k - 1$ defined by

$$q^*(u) = \sum_{m=1}^{k-1} q_{m\theta} c_1^{-m} u^m.$$

Using this notation, (8) can be written in the form

$$0 \leq 1 - |(c_1 z^\theta)^k|^2 + \operatorname{Re}\left(\left(1 - \left(c_1 \bar{z}^\theta\right)^k\right) q^*\left(c_1 z^\theta\right)\right)$$

or, equivalently,

$$0 < \operatorname{Re} \frac{1 + \left(c_1 z^\theta\right)^k + q^*\left(c_1 z^\theta\right)}{1 - \left(c_1 z^\theta\right)^k}.$$

Let $\mathscr{P}_{1,k}$ denote the set of functions of the form

$$f^*(u) = \frac{1 + u^k + q^*(u)}{1 - u^k}$$

where q^* is a polynomial in one variable of degree $\leq k - 1$ with $q^*(0) = 0$, and $f^*(u)$ has positive real part when $|u| < 1$.

The conclusions of the preceding discussion can be summarized by the following:

THEOREM 5. (a) *If* $\gcd(\alpha_j) = 1$, *then* $\mathscr{F}(h)$ *reduces to a single point and* $(1 + h)/(1 - h)$ *is an extreme point of* $\mathscr{P}$.

(b) *If* $\gcd(\alpha_j) = k > 1$, *then* $\mathscr{F}(h)$ *consists of all functions of the form* $f(z) = f^*(c_1 z^\theta)$, *where* $f^* \in \mathscr{P}_{1,k}$, $\theta = \alpha/k$, *and* $c_1 = c^{1/k}$.

Part (a) of Theorem 5 was proved by Forelli by other methods in [3].

It is a simple exercise to show that $\mathscr{P}_{1,2}$ consists of all functions of the form

$$f^*(u) = \frac{1 + u^2 + au}{1 - u^2},$$

where $-2 \leq a \leq 2$. Thus, in the notation of Theorem 5 the extreme elements of $\mathscr{F}((c_1 z^\theta)^2)$ are

$$\frac{1 + c_1 z^\theta}{1 - c_1 z^\theta} \quad \text{and} \quad \frac{1 - c_1 z^\theta}{1 + c_1 z^\theta}.$$

The collection $\mathscr{P}_{1,3}$ consists of functions of the form

$$f^*(u) = \frac{1 + u^3 + au + bu^2}{1 - u^3},$$

where $\operatorname{Re} f^*(u)$ is positive for $|u| < 1$. As a consequence of the condition $\operatorname{Re} f^*(u) > 0$ for $|u| < 1$, we have $b = \bar{a}$. Let

$$F(u) = |1 - \bar{u}^3| \operatorname{Re} f^*(u).$$

Then

$$r(1 - r^4)a = \frac{1}{2\pi}\int_0^{2\pi} e^{-it} F(re^{it})\, dt.$$

Thus, for $0 < r < 1$

$$r(1 - r^4)|a| \leq \frac{1}{2\pi}\int_0^{2\pi} F(re^{it})\, dt.$$

Hence,

$$|a| \leq \frac{1 - r^6}{r(1 - r^4)}.$$

A straightforward calculation shows that $|a| \leq 3/2$. Thus, the set of complex numbers

$$A = \left\{ a \colon (1 + u^3 + au + \bar{a}u^2)/(1 - u^3) \in \mathscr{P}_{1,3} \right\},$$

is a convex subset of the disk $\{a\colon |a| \leq 3/2\}$. It can be shown by a tedious argument that $3/2 \in A$. Hence, the function

$$f_1^*(u) = \frac{1 + u^3 + 1.5u + 1.5u^2}{1 - u^3}$$

is an extreme point of the set $\mathscr{P}_{1,3}$. It follows that the function

$$f_1(z) = \frac{1 + \left(c_1 z^\theta\right)^3 + 1.5 c_1 z^\theta + 1.5\left(c_1 z^\theta\right)^2}{1 - \left(c_1 z^\theta\right)^3}$$

is an extreme element of $\mathscr{F}((c_1 z^\theta)^3)$.

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Arizona State University
Tempe, Arizona