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COMPLEMENTED COPIES OF c_0 IN l^∞-SUMS OF BANACH SPACES

BY

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1. Introduction

By a result of W.B. Johnson [6], there exists a sequence (G_n) of finite dimensional Banach spaces such that for any separable Banach space G, the l^∞-sum of (G_n) contains an isometric 1-complemented copy of G'. From this we see that the l^∞-sum of a family $(E_\gamma)_{\gamma\in\Gamma}$ of Banach spaces may contain many very different complemented infinite dimensional subspaces even if none of the spaces E_γ does. However, the situation becomes quite different when we consider complemented subspaces isomorphic to c_0. In fact, our main result shows that if the cardinality of the index set Γ is smaller than the first real-valued measurable cardinal, then the l^∞-sum of $(E_\gamma)_{\gamma\in\Gamma}$ contains a complemented copy of c_0 if and only if one of the spaces E_γ does. From this we obtain that the l^∞-sum of certain families of Grothendieck spaces is again a Grothendieck space.

Our terminology is standard. However, we want to explain some frequently used terms and fix some notations. For a (real) Banach space E we denote by E' the dual and by E'' the bidual of E. On E', the weak* and weak topologies are the topologies $\sigma(E', E)$ and $\sigma(E', E'')$ respectively. A sequence (x_i) in E is called *weakly summable* if

$$\sum|\langle x_i, x'\rangle| < \infty \quad \text{for all } x' \in E'.$$

E is said to contain a complemented copy of c_0 if E has a complemented subspace isomorphic to c_0. Given a set Γ, we denote by $|\Gamma|$ the cardinality of Γ. If $(E_\gamma)_{\gamma\in\Gamma}$ is a family of Banach spaces and $1 \le p \le \infty$, the l^p-sum of $(E_\gamma)_{\gamma\in\Gamma}$ is the space $(\Sigma \oplus E_\gamma)_{l^p(\Gamma)}$ which consists of all $x = (x(\gamma))_{\gamma\in\Gamma}$ such that $x(\gamma) \in E_\gamma$ for all γ and

$$\sum_{\gamma\in\Gamma} \|x(\gamma)\|^p < \infty \quad \left(\sup_\gamma \|x(\gamma)\| < \infty \text{ if } p = \infty\right),$$

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endowed with the norm

$$\|(x(\gamma))\| = \Big(\sum_{\gamma\in\Gamma} \|x(\gamma)\|^p\Big)^{1/p} \quad \Big(\|(x(\gamma))\| = \sup_{\gamma} \|x(\gamma)\| \qquad \text{if } p = \infty\Big).$$

In case $E_\gamma = \mathbf{R}$ for all γ, we write $l^\infty(\Gamma)$ instead of $(\Sigma \oplus E_\gamma)_{l^\infty(\Gamma)}$. In the following, we will use the well known identification of $l^\infty(\Gamma)'$ as the space of all bounded finitely additive set functions on the power set of Γ [1, IV.5.3] without further comment.

2. Preliminary lemmas

In this section we state and prove several lemmas from which the main result follows easily. The first lemma is a well known result [2, Theorem 1] which provides a (necessary and) sufficient condition for a Banach space E to contain a complemented copy of c_0. Recall that a subset A of a Banach space E is a *limited set* if for every weak* null sequence (x_i') in E', $\lim_i \sup_{x\in A} |\langle x, x_i'\rangle| = 0$.

LEMMA 1. *Let (x_i) be a weakly summable sequence in a Banach space E. If (x_i) is not a limited set, then there is a subsequence (x_{i_k}) which spans a complemented subspace of E isomorphic to c_0.*

Let $(E_\gamma)_{\gamma\in\Gamma}$ be a family of Banach spaces and let $F \equiv (\Sigma \oplus E_\gamma)_{l^\infty(\Gamma)}$. For $x \in F$ and $A \subset \Gamma$, define $x\chi_A \in F$ by $x\chi_A(\gamma) = x(\gamma)$ if $\gamma \in A$ and $x\chi_A(\gamma) = 0$ otherwise. Furthermore, we will denote by P the projection from F' onto the subspace $(\Sigma \oplus E_\gamma')_{l^1(\Gamma)}$ given by

$$\langle x, Px'\rangle = \sum_{\gamma\in\Gamma} \langle x\chi_{\{\gamma\}}, x'\rangle$$

for all $x \in F$ and $x' \in F'$. We remark that P is $\sigma(F', F)$-sequentially continuous [10, Lemma 11.4]. The next lemma shows that the weak* null sequences in F' cannot have "coordinatewise disjoint supports".

LEMMA 2. *Let (x_i) be a bounded sequence in F and let (x_i') be a weak* null sequence in F'. Define finitely additive set functions (λ_i) on Γ by $\lambda_i(A) = \langle x_i\chi_A, x_i'\rangle$. Then (λ_i) is relatively weakly compact in $l^\infty(\Gamma)'$.*

Proof. If (λ_i) is not relatively weakly compact, then we may assume that there exist disjoint subsets (A_i) of Γ and $\varepsilon > 0$ such that $|\lambda_i(A_i)| > \varepsilon$ for all i. Let $y_i = x_i\chi_{A_i}$ for all i. Then (y_i) is a weakly summable sequence in F. Furthermore, (y_i) is not a limited subset of F since (x_i') is weak* null and

$|\langle y_i, x_i' \rangle| = |\lambda_i(A_i)| > \varepsilon$ for all i. According to Lemma 1, we may thus assume that (y_i) spans a subspace of F isomorphic to c_0 which is complemented by a projection S. On the other hand, for every $(a_i) \in L^\infty$, the coordinatewise sum $\Sigma a_i y_i$ exists trivially and T: $(a_i) \mapsto \Sigma a_i y_i$ is a bounded map from l^∞ into F. Clearly $S \circ T$ defines a map from l^∞ onto $[y_i]$, which is isomorphic to c_0; this is a known impossibility. ■

Before stating the next lemma, we recall some terminology regarding the measurability of cardinals.

DEFINITION. A cardinal $\mathbf{m}$ is said to be *measurable* (resp. *real-valued measurable*, resp. *atomlessly measurable*) if there exists a set Γ with cardinality $\mathbf{m}$ such that there is a non-zero $\{0,1\}$-valued (resp. $[0,1]$-valued, resp. atomless $[0,1]$-valued) $\mathbf{m}$-additive measure defined on the power set of Γ which vanishes on all subsets of Γ. The first real-valued measurable cardinal, it it exists, will be denoted by $\mathbf{m}_r$.

It is known that on the basis of axioms of Zermelo-Fraenkel and the axiom of choice the existence of real-valued measurable cardinals cannot be proved. If the continuum hypothesis or Martin's axiom is assumed to be true, then atomlessly measurable cardinals do not exist [5, 27.3, 27.7] and every real-valued measurable cardinal is measurable [5, 27.5, 27.7]. We will need the fact that if Γ is a set with $|\Gamma| < \mathbf{m}_r$, then any *countably* additive measure on the power set of Γ which vanishes on finite sets must be identically zero [5, 27.4 Cor.]. More on these cardinals can be found in [3, p. 972], [5], [12], and [13].

Under the appropriate measurability condition on the cardinality of Γ, we are now able to prove the following key lemma from which we can conclude that any projection map from F onto a subspace isomorphic to c_0 must originate from a weak* null sequence in $PF' = (\Sigma \oplus E_\gamma')_{l^1(\Gamma)}$.

LEMMA 3. *Let $(E_\gamma)_{\gamma \in \Gamma}$ be Banach spaces where Γ is an index set satisfying $|\Gamma| < \mathbf{m}_r$ and let $F = (\Sigma \oplus E_\gamma)_{l^\infty(\Gamma)}$. If (x_i) is a weakly summable sequence in F and (x_i') is a weak* null sequence in F' which lies in $(Id - P)F'$, then* $\lim_i \langle x_i, x_i' \rangle = 0$.

Proof. Define (λ_i) as in Lemma 2. Then (λ_i) is relatively weakly compact by the same lemma. We may thus assume that (λ_i) converges weakly to some $\lambda \in l^\infty(\Gamma)'$. We claim that λ is countably additive. Otherwise, there exist $\varepsilon > 0$ and a sequence (A_i) of subsets of Γ which decreases to $\varnothing$ while $|\lambda(A_i)| > \varepsilon$ for all i. Without loss of generality, we may then assume that $|\lambda_i(A_i)| > \varepsilon$ for all i. Let $y_i = x_i \chi_{A_i}$ for all i. Then (y_i) is a weakly summable sequence such that $|\langle y_i, x_i' \rangle| > \varepsilon$ for all i. Applying Lemma 1, we may assume that $[y_i]$ is a complemented subspace of F isomorphic to c_0. On the other hand, for every $\gamma \in \Gamma$, the sequence $(y_i(\gamma))$ has only finitely many non-zero terms since (A_i) decreases to $\varnothing$. Using this observation and the fact that (y_i) is weakly summable, we obtain that the coordinatewise sum $\Sigma a_i y_i$ exists and is in F for every $(a_i) \in l^\infty$, and that T: $(a_i) \mapsto \Sigma a_i y_i$ is a bounded map from l^∞ into F. We may then proceed as in the proof of Lemma 2 to obtain a contradiction. This establishes the claim that λ is countably additive. Now for all finite subsets A of Γ and for all i, $\lambda_i(A) = 0$ since $x_i' \in (Id - P)F'$; hence $\lambda(A) = 0$. By the measurability condition on the cardinality of Γ, we must have $\lambda = 0$. In particular,

$$\lim\langle x_i, x_i'\rangle = \lim \lambda_i(\Gamma) = \lambda(\Gamma) = 0,$$

as claimed. ■

Remark. The conclusion of Lemma 3 fails without the measurability assumption on Γ [11, Remark after Prop. 3.1].

The final lemma states that every weak* null sequence in

$$PF' = \left(\Sigma \oplus E_\gamma'\right)_{l^1(\Gamma)}$$

must essentially be supported by finitely many coordinates.

LEMMA 4. [8, Lemma 2.8]. *Let*

$$(x_i') \subset \left(\Sigma \oplus E_\gamma'\right)_{l^1(\Gamma)}$$

be $\sigma((\Sigma \oplus E_\gamma')_{l^1(\Gamma)}, F)$*-null,* $x_i' = (x_i'(\gamma))$. *Then for all* $\varepsilon > 0$, *there exists a cofinite subset* A *of* Γ *such that*

$$\sup_i \sum_{\gamma \in A} \|x_i'(\gamma)\| \leq \varepsilon.$$

3. The main result and its consequences

We are now in a position to prove our main result.

THEOREM. *Let* $(E_\gamma)_{\gamma\in\Gamma}$ *be Banach spaces where* Γ *is an index set with* $|\Gamma| < \mathbf{m}_r$. *Then* $F \equiv (\Sigma \oplus E_\gamma)_{l^\infty(\Gamma)}$ *contains a complemented copy of* c_0 *if and only if some* E_γ *does.*

Proof. Suppose F contains a complemented copy of c_0. There exist biorthogonal sequences (x_i) and (x_i') in F and F' respectively so that (x_i) is equivalent to the c_0 basis and (x_i') is weak* null. Since P is weak* sequentially continuous, $((Id - P)x_i')$ is weak* null. Thus

$$\langle x_i, (Id - P)x_i'\rangle \to 0$$

by Lemma 3. Let $y_i' = Px_i'$. Then $\langle x_i, y_i' \rangle \to 1$ and (y_i') is weak* null. Moreover,

$$y_i' \in \left(\sum \oplus E_\gamma'\right)_{l^1(\Gamma)}$$

and hence we may write $y_i' = (y_i'(\gamma))$ where $y_i'(\gamma) \in E_\gamma'$ for all γ. Now $(y_i'(\gamma))$ is $\sigma(E_\gamma', E_\gamma)$-null for every fixed γ and also

$$\sup_i \sum_{\gamma \in A} \|y_i'(\gamma)\| \le 1/2$$

for some cofinite subset A of Γ by Lemma 4. Hence there exists a finite subset B of Γ with $\limsup \langle x_i \chi_B, y_i' \rangle \ge 1/2$. By passing to a subsequence, we may assume that for some $\gamma \in B$, $\inf_i \langle x_i(\gamma), y_i'(\gamma) \rangle > 0$. Since $(x_i(\gamma))$ is a weakly summable sequence in E_γ and $(y_i'(\gamma))$ is a $\sigma(E_\gamma', E_\gamma)$-null sequence, Lemma 1 shows that E_γ has a complemented subspace isomorphic to c_0, as required. ∎

Before stating some corollaries, we recall that a Banach space E has *property* (V) if every operator from E into a Banach space G which maps weakly summable sequences onto summable sequences is weakly compact. The space E is a *Grothendieck space* if weak* null sequences in E' are weakly null. It is known that every L^1-predual (i.e., every Banach space whose dual is isometric to a space L^1) has property (V) [7] and a Banach space with property (V) is a Grothendieck space if and only if it contains no complemented copies of c_0 [10, Korollar 3.3].

COROLLARY 1. *Let $(E_\gamma)_{\gamma \in \Gamma}$ be Banach spaces where Γ is an index set with $|\Gamma| < \mathbf{m}_r$. Suppose $F \equiv (\sum \oplus E_\gamma)_{l^\infty(\Gamma)}$ has property (V) and no E_γ contains a complemented copy of c_0. Then F is a Grothendieck space.*

In particular, if E_γ is an L^1-predual for every $\gamma \in \Gamma$, then the hypotheses of Corollary 1 are fulfilled if and only if each E_γ is a Grothendieck space (cf. [11, Corollary 2.5]). We thus obtain the next result.

COROLLARY 2. *If $(E_\gamma)_{\gamma \in \Gamma}$ is a family of L^1-preduals where Γ satisfies $|\Gamma| < \mathbf{m}_r$, then $(\sum \oplus E_\gamma)_{\ell^\infty(\Gamma)}$ is a Grothendieck space if and only if each E_γ is a Grothendieck space.*

This generalizes results in [10, §11] on the l^∞-sum of Grothendieck spaces. We close this section with a question (cf. also the appendix) and a remark, phrased as a proposition.

Problem. Is the measurability condition necessary in the theorem?

It is known that the measurability condition may be removed for some special cases.

PROPOSITION. *Let Γ be an arbitrary index set and let $(E_\gamma)_{\gamma \in \Gamma}$ be a collection of Banach spaces satisfying one of the following properties:*

(a) *No E_γ contains a copy of c_0;*

(b) *There is a constant $k < \infty$ such that each E_γ is complemented in its bidual by a projection of norm $\leq k$.*

Then $F \equiv (\Sigma \oplus E_\gamma)_{l^\infty(\Gamma)}$ contains no complemented copies of c_0. In particular, the measurability condition in the theorem can be removed if each E_γ is separable.

Proof. Let (x_i) be a sequence in F equivalent to the standard c_0 basis so that there is a projection Q from F onto $[x_i]$. Regard F as a subspace of $H \equiv (\Sigma \oplus E_\gamma'')_{l^\infty(\Gamma)}$, which is the dual of $G \equiv (\Sigma \oplus E_\gamma')_{l^1(\Gamma)}$. Since (x_i) is weakly summable,

$$(a_i) \mapsto \sigma(H, G) - \Sigma a_i x_i$$

defines a bounded map T from l^∞ into H. A simple computation shows that

$$T(a_i)(\gamma) = \sigma\left(E_\gamma'', E_\gamma'\right) - \Sigma a_i x_i(\gamma) \quad \text{for all } \gamma.$$

If condition (a) holds, every weakly summable sequence in E_γ is summable [9, Proposition 2.e.4]; hence $T(a_i)(\gamma) \in E_\gamma$ for all γ. Thus $T(l^\infty) \subset F$. But then $Q \circ T$ maps l^∞ onto $[x_i]$, which is isomorphic to c_0; a known impossibility. If case (b) is valid, F is clearly complemented in H by some projection R. Thus $Q \circ R \circ T$ maps l^∞ onto $[x_i]$, yielding the same contradiction as in case (a). Finally, the last assertion follows immediately from case (a) due to the separable injectivity of c_0 [9, Theorem 2.f.5]. ■

Appendix

The authors wish to thank Professor Richard Haydon for pointing out some suggestions to remove the measurability condition in our main result. He conjectures the following.

CONJECTURE. *Let $(E_\gamma)_{\gamma \in \Gamma}$ be Banach spaces where $|\Gamma|$ is smaller than the first atomlessly measurable cardinal. Then $F \equiv (\Sigma \oplus E_\gamma)_{l^\infty(\Gamma)}$ contains a complemented copy of c_0 if and only if some E_γ does.*

Under the continuum hypothesis or Martin's axiom, a positive solution would imply that the main theorem is true for arbitrary index sets. However, all we know at this point is the following result of Haydon [4] which uses to some extent the methods of Lemma 3 together with facts on ultraproducts.

PROPOSITION. *Suppose that atomlessly measurable cardinals do not exist. Let* $(E_\gamma)_{\gamma\in\Gamma}$ *be Banach spaces and assume that* $\sup_\gamma |E_\gamma|$ *is smaller than the first measurable cardinal. Then* $F \equiv (\sum \oplus E_\gamma)_{l^\infty(\Gamma)}$ *contains a complemented copy of* c_0 *if and only if some* E_γ *does.*

This solves the conjecture in the affirmative provided there is a positive answer to the following question.

Problem. Is there a cardinal number $\mathbf{m}$, smaller than the first measurable cardinal, such that a given Banach space E and a non-complemented subspace G isomorphic to c_0, there is a subspace H with $G \subset H \subset E$, $|H| \leq \mathbf{m}$, and G is not complemented in H?

REFERENCES

1. N. DUNFORD and J.T. SCHWARTZ, *Linear operators, Part I*, Interscience, New York, 1958.
2. G. EMMANUELE, *A note on Banach spaces containing complemented copies of* c_0, preprint.
3. R.J. GARDNER and W.F. PFEFFER, "Borel measures," in *Handbook of set-theoretic topology*, ed. K. Kunen and J.E. Vaughan, North-Holland, Amsterdam, 1984, pp. 961–1043.
4. R. HAYDON, personal communication.
5. T. JECH, *Set theory*, Academic Press, San Diego, 1978.
6. W.B. JOHNSON, *A complementary universal conjugate Banach space and its relation to the approximation problem*, Israel J. Math., vol. 13 (1972), pp. 301–310.
7. W.B. JOHNSON and M. ZIPPIN, *Separable* L^1 *preduals are quotients of* $C(\Delta)$, Israel J. Math., vol. 16 (1973), pp. 198–202.
8. D. LEUNG, *Uniform convergence of operators and Grothendieck spaces with the Dunford-Pettis property*, Thesis, University of Illinois, 1987.
9. J. LINDENSTRAUSS and L. TZARFRIRI, *Classical Banach spaces II. Function spaces*, Springer-Verlag, New York, 1979.
10. F. RÄBIGER, "Beiträge zur Strukturtheorie der Grothendieck-Räume," in *Sitzungsber. Heidelberger Akad. Wiss. Math.-Naturwiss. Klasse*, Jg. 1985, 4. Abhandlung, Springer-Verlag, New York, 1985.
11. ______, *Complemented copies of* c_0 *in* $\mathcal{F}$*-products of Banach spaces*, preprint.
12. J. SILVER, *The consistence of the GCH with the existence of a measurable cardinal*, Proc. Symposia in Pure Mathematics, Vol. 13, Part 1, ed. D.S. Scott, American Math. Society, Providence, 1971, pp. 391–395.
13. R.M. SOLOVAY, *Real-valued measurable cardinals*, Proc. Symposia in Pure Mathematics, Vol. 13, Part 1, ed. D.S. Scott, American Math. Society, Providence, 1971, pp. 397–428.
14. S. ULAM, *Zur Maßtheorie in der allgemeinen Mengenlehre*, Fund. Math., vol. 16 (1930), pp. 140–150.

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