

## QUASI-HEREDITARY ALGEBRAS

BY

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**Dedicated to the memory of Irving Reiner**

In their work on highest weight categories arising in the representation theory of Lie algebras and algebraic groups, E. Cline, B. Parshall and L. Scott recently introduced the notion of a quasi-hereditary algebra (see [1] and [2]). They define a quasi-hereditary algebra recursively in terms of the existence of a particular idempotent ideal; finite-dimensional hereditary algebras are typical examples of quasi-hereditary algebras. On the other hand, they showed that every quasi-hereditary algebra has finite global dimension.

The purpose of this note is to establish the following three results. First, finite-dimensional hereditary algebras are characterized as those quasi-hereditary algebras which satisfy a certain refinement property on chains of their idempotent ideals (Theorem 1). Second, all finite-dimensional algebras of global dimension 2 are shown to be quasi-hereditary (Theorem 2). Third, the question of whether every finite-dimensional algebra of finite global dimension is quasi-hereditary is answered in the negative by providing an example of an (11-dimensional serial) algebra of global dimension 4 which is not quasi-hereditary. The same example illustrates that the class of quasi-hereditary algebras is not closed under tilting (in the sense of [4]).

In what follows, all rings are semiprimary rings. An associative ring  $A$  with 1 is called *semiprimary* if its Jacobson radical  $N$  is nilpotent and  $A/N$  is semisimple artinian. Recall that an ideal  $I$  of  $A$  is idempotent if and only if  $I = AeA$  for an idempotent  $e$  of  $A$ ; in particular,  $I$  is a minimal (non-zero) idempotent ideal provided that  $e$  is primitive. An ideal  $J$  of  $A$  is said to be a *heredity* ideal of  $A$  if  $J^2 = J$ ,  $JNJ = 0$  and  $J$ , considered as a right  $A$ -module  $J_A$ , is projective. In fact, this also implies that the left  $A$ -module  ${}_A J$  is projective (see [2] or [3]). A semiprimary ring  $A$  is called *quasi-hereditary* if there is a chain

$$0 = J_0 \subset J_1 \subset \cdots \subset J_{t-1} \subset J_t \subset \cdots \subset J_m = A$$

of ideals of  $A$  such that, for any  $1 \leq t \leq m$ ,  $J_t/J_{t-1}$  is a heredity ideal of  $A/J_{t-1}$ . Such a chain of idempotent ideals is called a *heredity chain*. Let us

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remark that  $A$  is quasi-hereditary if and only if  $A^{op}$  is quasi-hereditary (see [2] or [3]).

**THEOREM 1.** *Let  $A$  be a semiprimary ring. Then  $A$  is hereditary if and only if every chain of idempotent ideals of  $A$  can be refined to a heredity chain.*

*Proof.* First, let  $A$  be a hereditary semiprimary ring. Note that, for any idempotent ideal, the ring  $A/I$  is again hereditary. Thus, it is sufficient to show that any non-zero idempotent ideal  $I$  of  $A$  contains a non-zero heredity ideal. To this end, consider  $J = AeA$  with a primitive idempotent  $e$  in  $I$ . Since  $A$  is hereditary,  $J_A$  is projective. Moreover, since the left multiplication by any element  $x \in eNe$  defines a non-invertible map from the indecomposable module  $eA$  onto  $xA$  and  $xA$  is projective, necessarily  $x = 0$ . It follows that  $JNJ = AeNeA = 0$ .

Conversely, assume that every chain of idempotent ideals of  $A$  can be refined to a heredity chain. We want to show that  $A$  is hereditary. We shall proceed by induction; thus assume that  $A/I$  is hereditary for every non-zero idempotent ideal  $I$  of  $A$ .

Let us start by choosing a primitive idempotent  $f$  in  $A$  such that the Loewy length  $L(fA)$  of the module  $fA$  is maximal. Observe that  $I = AfA$  is a heredity ideal. Indeed,  $AfA$  is a minimal idempotent ideal of  $A$ . Thus  $fNf = 0$  and  $I_A$  is a projective  $A$ -module. We claim that  $Nf = 0$ . For, if  $Nf \neq 0$  then there is a primitive idempotent  $e$  such that  $eNf \neq 0$  and thus  $eNf = eAf$ . Furthermore,  $I = eI \oplus (1 - e)I$ , and thus  $(eI)_A$  is a projective  $A$ -module. Since  $eI = eAfA$  is generated by  $fA$ , it is a non-zero direct sum of copies of  $fA$  and therefore  $L(eI) = L(fA)$ . On the other hand,  $eI = eAfA = eNfA \subseteq eN$ , and consequently

$$L(eI) \leq (eN) = L(eA) - 1$$

which contradicts the maximality of  $L(fA)$ . Hence,  $Nf = 0$ .

Now, write  $1 = \sum_{i=1}^n e_i$  with primitive orthogonal idempotents  $e_i$ . Assume that  $f = e_1$  is equivalent to  $e_i$  if and only if  $1 \leq i \leq k$ . Thus  $e_j A e_i = 0$  for all  $i \leq k < j$ . If  $k = n$ , then  $A$  is simple artinian and thus hereditary. If  $k < n$ , we show that  $(AfN)_A$  is a projective  $A$ -module. For then, in conjunction with the fact that  $Nf = 0$  and that  $B = A/AfA$  is hereditary, we can conclude that  $A$  is hereditary, as required.

In order to prove that  $(AfN)_A$  is projective, consider the  $A$ -module  $X_A = AfN / (AfN \cap AgA)$ , where  $g$  is a primitive idempotent satisfying  $Ng \subseteq I (= AfA)$ . Such an idempotent  $g = e_j$  for some  $j > k$  exists, since  $B = A/I$  is

hereditary. Clearly,  $X$  is isomorphic to the ideal

$$(AfN + AgA)/AgA$$

of the ring  $C = A/AgA$  which is, by induction, hereditary. Consequently,  $X$  is a projective  $C$ -module. Since  $X$  is annihilated by  $A(f + g)A$ , it is a projective  $D$ -module for  $D = A/A(f + g)A$ . But, in view of  $Ng \subseteq I$ , every projective  $D$ -module is a projective  $B$ -module, and in view of  $Nf = 0$ , every projective  $B$ -module is a projective  $A$ -module. Hence,  $X_A$  is a projective  $A$ -module. As a result, the canonical  $A$ -homomorphism  $(AfN)_A \rightarrow X_A$  splits and thus  $(AfN)_A$  is isomorphic to the direct sum of the projective  $A$ -module  $X_A$  and  $(AfN \cap AgA)_A$ . However, the latter  $A$ -module is also projective. This follows from the fact that

$$(*) \quad AfN \cap AgA = \bigoplus_{i=1}^k e_i AgA,$$

which is trivially a direct summand of the projective  $A$ -module

$$AgA = \bigoplus_{i=1}^n e_i AgA.$$

In order to verify  $(*)$ , notice that, on the one hand,

$$e_i AgA \subseteq AgA \quad \text{and} \quad e_i AgA \subseteq (Ae_i A)(AgA) = AfNgA \subseteq AfN,$$

and that, on the other hand,

$$AfN \cap AgA \subseteq \left( \bigoplus_{i=1}^k e_i A \right) \cap AgA = \bigoplus_{i=1}^k e_i AgA.$$

This concludes the proof of Theorem 1.

**THEOREM 2.** *A semiprimary ring of global dimension 2 is quasi-hereditary.*

The proof is based on the following proposition.

**PROPOSITION.** *Let  $A$  be a semiprimary ring of global dimension at most 2. Let  $e$  be a primitive idempotent of  $A$  such that the Loewy length  $L(eA)$  is minimal. Then  $AeA$  is a heredity ideal.*

*Proof.* Write  $J = AeA$ . Clearly  $J^2 = J$ . First, we are going to show that  $eNe = 0$ ; this implies immediately that  $JNJ = 0$ . Assume that there is a non-zero element  $x \in eNe$ . The left multiplication by  $x$  yields an endomor-

phism of  $eA$  with a non-zero kernel  $K \subseteq eN$ . Since  $\text{gl dim } A = 2$ , kernels of maps between projective modules are projective. Thus  $K$  is projective and therefore the Loewy length  $L(K) \geq L(eA)$ . But

$$L(K) \leq L(eN) = L(eA) - 1,$$

in contradiction to the preceding inequality.

Second, in order to show that  $J_A$  is projective, consider the minimal projective cover  $p: P \rightarrow J_A$ . We claim that  $p$  is an isomorphism. Otherwise, there exists a finitely generated direct summand  $P'$  of  $P$  such that the restriction of  $p$  to  $P'$  is not a monomorphism. Now,  $P$  and therefore  $P'$  is a direct sum of copies of  $eA$ . We may assume that  $P' = P'' \oplus \bar{P}$  with  $\bar{P} \simeq eA$  and such that the restriction of  $p$  to  $P''$  is a monomorphism. Denoting by  $X$  the image of  $P''$  under  $p$ , consider the commutative diagram of  $A$ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & P'' & \longrightarrow & P' & \longrightarrow & \bar{P} \longrightarrow 0 \\ & & p'' \downarrow & & p' \downarrow & & \bar{p} \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & A & \longrightarrow & A/X \longrightarrow 0, \end{array}$$

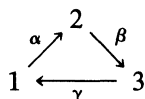
where the horizontal maps are the canonical inclusions and projections, and the vertical ones are induced by  $p$  and the embedding of  $J$  in  $A$ . Observe that  $p'$  is not a monomorphism and  $\bar{p} \neq 0$ . Forming the kernels, we obtain the exact sequence

$$0 = \text{Ker } p'' \rightarrow \text{Ker } p' \rightarrow \text{Ker } \bar{p}.$$

Now, since  $p'$  is a map between projective modules,  $\text{Ker } p'$  is a non-zero projective module, and thus its Loewy length  $L(\text{Ker } p') \geq L(eA)$ . But this contradicts the inequality  $L(\text{Ker } p') \leq L(\text{Ker } \bar{p}) < L(eA)$ . The proof of the proposition is completed.

*Proof of Theorem 2.* Choose a primitive idempotent  $e$  of  $A$  such that the Loewy length  $L(eA)$  is minimal. Since, by the proposition,  $J = AeA$  is a heredity ideal, it follows that  $\text{gl dim } A/J \leq \text{gl dim } A$  (see [2] or [3]). Therefore, by induction,  $A/J$  is quasi-hereditary, and thus  $A$  is quasi-hereditary.

*Example.* Let  $A$  be the path algebra (over a field) of the diagram



modulo the ideal generated by  $\gamma\alpha\beta$  and  $\alpha\beta\gamma\alpha$ . Denote by  $e_i$  the idempotent corresponding to the vertex  $i$ , and let  $P_i = e_iA$ . Thus,  $A$  is a serial algebra,

and the lengths of the indecomposable projective  $A$ -modules are as follows:  $P_1$  and  $P_2$  are of length 4 and  $P_3$  is of length 3. One can check easily that  $\text{gl dim } A = 4$  and that  $A$  has no (non-zero) heredity ideals; for,  $e_1 \text{ Rad } A e_1 \neq 0$ ,  $e_2 \text{ Rad } A e_2 \neq 0$  and  $(Ae_3A)_A$  is not projective.

Let  $S_2 = P_2/\text{Rad } P_2$  be the simple  $A$ -module of projective dimension 1. Since the modules  $P_1$  and  $P_2$  are both projective and injective, it follows that  $T = P_1 \oplus P_2 \oplus S_2$  is a tilting module [4]. Let  $B = \text{End } T_A$ . Then  $B$  is a quasi-hereditary algebra. Indeed, if  $J_1$  is the set of all endomorphisms of  $T$  which factor through  $S_2$  and  $J_2$  the set of all those which factor through  $S_2 \oplus P_2$ , then  $0 \subset J_1 \subset J_2 \subset A$  is a heredity chain. This may be verified without difficulty. In fact, one can see easily that  $B$  is the path algebra (over a field) of the diagram

$$1 \begin{matrix} \xleftarrow{\alpha} \\ \xrightarrow{\delta} \end{matrix} 2 \begin{matrix} \xleftarrow{\beta} \\ \xrightarrow{\gamma} \end{matrix} 3$$

modulo the ideal generated by  $\alpha\beta$ ,  $\alpha\delta$ ,  $\gamma\delta$  and  $\beta\gamma - \delta\alpha$ .

**Appendix: Heredity ideals**

For the convenience of the reader, we wish to collect here various results of [1] and [2]. In doing so, we aim to minimize assumptions and, at the same time, to strengthen conclusions (cf. examples at the end). Furthermore, we include some assertions from the general ring theory, and provide proofs which are direct and elementary.

In what follows, unless specified otherwise,  $A$  is an arbitrary associative ring with 1,  $J$  and ideal and  $B = A/J$ . The  $B$ -modules will always be viewed as those  $A$ -modules which are annihilated by  $J$ .

Let us remark that, under the assumption that  $A$  is semiprimary, every  $A$ -module  $X$  has finite Loewy length which will always be denoted by  $L(X)$ . Note that for semiprimary rings,  $L(A_A)$  and  $L({}_A A)$  are equal and will be simply denoted by  $L(A)$ . Always,  $L(X) \leq L(A)$ .

STATEMENT 1. *Let  $J_A$  be projective. If  $X$  is a  $B$ -module, then*

$$\text{proj dim } X_A \leq 1 + \text{proj dim } X_B.$$

*In particular,  $\text{proj dim } B_A \leq 1$ .*

*Proof.* Proceed by induction. The exact sequence  $0 \rightarrow J_A \rightarrow A_A \rightarrow B_A \rightarrow 0$  shows that

$$\text{proj dim } B_A \leq 1.$$

Therefore, for any projective  $B$ -module  $X_B$ ,  $\text{proj dim } X_A \leq 1$ . Now, assume that

$$\text{proj dim } X_B = d > 0,$$

and consider an exact sequence  $0 \rightarrow X'_B \rightarrow P_B \rightarrow X_B \rightarrow 0$  with a projective  $B$ -module  $P_B$ . Thus  $\text{proj dim } X'_B = d - 1$ . An application of  $\text{Hom}_A(-, Y_A)$  yields the exact sequence

$$\text{Ext}_A^{d+1}(X'_A, Y_A) \rightarrow \text{Ext}_A^{d+2}(X_A, Y_A) \rightarrow \text{Ext}_A^{d+2}(P_A, Y_A).$$

Since  $d \geq 1$  and  $\text{proj dim } P_A \leq 1$ , the last term is zero. By induction,

$$\text{proj dim } X'_A \leq d,$$

and consequently the first term is zero also. This yields

$$\text{Ext}_A^{d+2}(X_A, Y_A) = 0$$

for any  $Y_A$ , and thus  $\text{proj dim } X_A \leq d + 1$ .

**STATEMENT 2.**  $J^2 = J$  if and only if  $\text{Hom}_A(J_A, X_A) = 0$  for any  $B$ -module  $X$ . If  $J$  is projective, then  $J^2 = J$  if and only if  $\text{Hom}_A(J_A, B_A) = 0$ .

*Proof.* First, assume that  $J^2 = J$  and let  $\phi: J_A \rightarrow X_A$  be a homomorphism. Then  $\phi(J) = \phi(J^2) \subseteq XJ = 0$ , and thus  $\phi = 0$ . Conversely, let  $\text{Hom}_A(J_A, X_A) = 0$  for any  $B$ -module  $X$ . Write  $Y_A = J/J^2$ . Since  $YJ = 0$ ,  $Y$  can be viewed as a  $B$ -module. Hence,  $\text{Hom}_A(J_A, Y_A) = 0$ , and the canonical epimorphism  $J_A \rightarrow Y_A$  shows that  $Y = 0$ .

Finally, assume that  $J_A$  is projective and that  $\text{Hom}_A(J_A, B_A) = 0$ . Given a  $B$ -module  $X$ , let  $F_B$  be a free  $B$ -module with an epimorphism  $\pi: F_B \rightarrow X_B$ . Since  $J_A$  is projective, any map  $\phi: J_A \rightarrow X_A$  lifts to a map  $\phi': J_A \rightarrow F_A$  with  $\phi = \pi\phi'$ . But  $\text{Hom}_A(J_A, F_A) = 0$ , because  $F_A$  is a direct sum of copies of  $B_A$ .

**STATEMENT 3.** Let  $J^2 = J$  and  $J_A$  be projective. If  $X, Y$  are  $B$ -modules, then

$$\text{Ext}_B^i(X_B, Y_B) \simeq \text{Ext}_A^i(X_A, Y_A) \quad \text{for all } i \geq 0.$$

In particular,  $\text{Ext}_A^i(B_A, B_A) = 0$  for  $i \geq 1$ .

*Proof.* Clearly,

$$\text{Hom}_B(X_B, Y_B) = \text{Hom}_A(X_A, Y_A).$$

To prove the case  $i = 1$ , regard  $\text{Ext}_B^1(X_B, Y_B)$  as a subgroup of  $\text{Ext}_A^1(X_A, Y_A)$ . Then, given an exact sequence

$$0 \rightarrow Y_A \xrightarrow{\mu} Z_A \rightarrow X_A \rightarrow 0$$

of  $A$ -modules with an inclusion map  $\mu$ , it follows, in view of  $XJ = 0$ , that  $ZJ \subseteq \mu(Y)$ . Furthermore,

$$ZJ = ZJ^2 \subseteq \mu(Y)J = 0,$$

and thus  $Z$  is a  $B$ -module. This completes the proof for  $i = 1$ .

Finally, let  $0 \rightarrow X'_B \rightarrow P_B \rightarrow X_B \rightarrow 0$  be an exact sequence with a projective  $B$ -module  $P_B$ . An application of  $\text{Hom}_B(-, Y_B)$  yields, for all  $i \geq 0$ , the exact sequences

$$\text{Ext}_B^i(P_B, Y_B) \rightarrow \text{Ext}_B^i(X'_B, Y_B) \rightarrow \text{Ext}_B^{i+1}(X_B, Y_B) \rightarrow \text{Ext}_B^{i+1}(P_B, Y_B).$$

Here, since  $P_B$  is projective, the first and the last terms are zero for  $i \geq 1$ . Similarly, an application of  $\text{Hom}_A(-, Y_A)$  yields, for all  $i \geq 0$ , the exact sequences

$$\text{Ext}_A^i(P_A, Y_A) \rightarrow \text{Ext}_A^i(X'_A, Y_A) \rightarrow \text{Ext}_A^{i+1}(X_A, Y_A) \rightarrow \text{Ext}_A^{i+1}(P_A, Y_A).$$

Here, according to Statement 1,  $\text{proj dim } P_A \leq 1$ , and thus the last term is zero for all  $i \geq 1$ . Moreover,  $\text{Ext}_A^1(P_A, Y_A) \simeq \text{Ext}_B^1(P_B, Y_B) = 0$ , and therefore the first term is zero also for all  $i \geq 1$ . By induction, we may assume

$$\text{Ext}_B^i(X'_B, Y_B) \simeq \text{Ext}_A^i(X'_A, Y_A)$$

and

$$\text{Ext}_B^{i+1}(X_B, Y_B) \simeq \text{Ext}_A^{i+1}(X_A, Y_A),$$

as required.

STATEMENT 4. *Let  $J^2 = J$  and  $J_A$  be projective. Then*

$$\text{gl dim } B \leq \text{gl dim } A.$$

*Proof.* Let  $\text{gl dim } A = d < \infty$ . If  $X, Y$  are  $B$ -modules, then, in view of Statement 3,  $\text{Ext}_B^{d+1}(X_B, Y_B) \simeq \text{Ext}_A^{d+1}(X_A, Y_A) = 0$ . Hence,  $\text{gl dim } B \leq d$ .

STATEMENT 5. *Let  $A$  be a semiprimary ring with radical  $N$ . Let  $JNJ = 0$  and  $J_A$  be projective. Then*

$$\text{gl dim } A \leq \text{gl dim } B + 2.$$

*Proof.* Let  $\text{gl dim } B = d < \infty$ . Given an  $A$ -module  $X$ , first calculate the projective dimension of the  $A$ -module  $XJ$ . Let  $\pi: P \rightarrow XJ$  be a minimal projective cover. For every  $x \in X$ , let  $J_x = J_A$ , and let  $P' = \bigoplus_{x \in X} J_x$ ; finally, define  $\pi': P' \rightarrow XJ$  by sending  $y \in J_x$  to  $xy$ . Since  $P'$  is projective and  $\pi'$  is surjective, it follows that  $P$  can be identified with a direct summand of  $P'$ , and thus

$$\ker \pi \subseteq \text{rad } P = PN \subseteq P'N = \bigoplus_{x \in X} J_x N.$$

Consequently,  $(\ker \pi)J = 0$ , and thus  $\ker \pi$  is a  $B$ -module. Therefore, by Statement 1,  $\text{proj dim } (\ker \pi)_A \leq d + 1$ . Hence,  $\text{proj dim } (XJ)_A \leq d + 2$ .

On the other hand,  $X/XJ$  is a  $B$ -module, and thus, making use of Statement 1 again,  $\text{proj dim } (X/XJ)_A \leq d + 1$ . Now, since  $X$  is an extension of the  $A$ -modules  $XJ$  by  $X/XJ$  both of projective dimension  $\leq d + 2$ ,

$$\text{proj dim } X_A \leq d + 2.$$

**STATEMENT 6.** *If  $e$  is an idempotent of  $A$ , then  $(AeA)^2 = AeA$ . Conversely, if  $A$  is semiprimary and  $J$  is an ideal of  $A$  such that  $J^2 = J$ , then  $J = AeA$  for an idempotent  $e$  of  $A$ .*

*Proof.* The first assertion is trivial. So assume that  $A$  is semiprimary with radical  $N$  and that  $J^2 = J$ . Any ideal of  $A/N$  is generated by an idempotent, and any idempotent of  $A/N$  is of the form  $\bar{e} = e + N$  with an idempotent  $e$  in  $A$ . Thus  $J + N = AeA + N$  for some idempotent  $e$  of  $A$ . Now,  $J^2 = J$  implies  $(J + N)^i = J + N^i$  for all  $i \geq 1$ ; similarly,  $(AeA + N)^i = AeA + N^i$  for all  $i \geq 1$ . But for large  $i$ ,  $N^i = 0$ , and therefore  $J = AeA$ .

**STATEMENT 7.** *Let  $e$  be an idempotent of a ring  $A$ . If the right module  $(AeA)_A$  or the left module  ${}_A(AeA)$  is projective, then the multiplication map*

$$\mu: Ae \bigoplus_{eAe} eA \rightarrow AeA$$

*is bijective. Conversely, assume that  $A$  is semiprimary with radical  $N$  and that  $eNe = 0$ . Then, if  $\mu$  is bijective, both modules  $(AeA)_A$  and  ${}_A(AeA)$  are projective.*

*Proof.* For any  $A$ -module  $X_A$ , consider the multiplication map

$$\mu_X: X \bigotimes_A Ae \bigotimes_{eAe} eA \rightarrow X.$$



The map  $\mu_X$  is bijective for  $X_A = eA$ , and therefore for all direct summands of direct sums of the module  $eA$ . Now, if  $(AeA)_A$  is projective, there is a surjective map of the form  $\oplus eA \rightarrow AeA$ , where the direct sum is indexed by all elements of  $A$ . Since this epimorphism splits, it follows that  $\mu_{AeA}$  is bijective. But this means that  $\mu$  is bijective. The same argument applies in the case that  ${}_A(AeA)$  is projective.

Now, assume that  $A$  is semiprimary with radical  $N$  and that  $eNe = 0$ . Since  $eNe = 0$ , the ring  $eAe$  is simple artinian, and thus any  $eAe$ -module is projective. Since  $(Ae)_{eAe}$  and  $(eA)_A$  are projective,  $(Ae \otimes_{eAe} eA)_A$  is projective also. Thus, the bijectivity of  $\mu$  implies that  $(AeA)_A$  is projective. Similarly, it implies that  ${}_A(AeA)$  is projective.

STATEMENT 8. *Let  $A$  be a semiprimary ring with radical  $N$  and  $JNJ = 0$ . Then  $L(A) \leq 2L(B) + 1$ .*

*Proof.* Since  $J_A$  is an extension of  $J_A$  by  $B_A$ , we have

$$L(A) \leq L(J_A) + L(B).$$

Moreover,  $J_A$  is an extension of  $(JN)_A$  by  $(J/JN)_A$ . Since  $JNJ = 0$ ,  $JN$  is a  $B$ -module and thus  $L((JN)_A) \leq L(B)$ . On the other hand,  $J/JN$  is semisimple and thus  $L((J/JN)_A) \leq 1$ . Consequently,

$$L(J_A) \leq L((JN)_A) + L((J/JN)_A) \leq L(B) + 1.$$

Now, making a subsequent use of Statements 7, 5 and 8, one can easily derive the following statement for quasi-hereditary rings.

STATEMENT 9. *Let  $A$  be a semiprimary quasi-hereditary ring with a heredity chain*

$$0 = J_0 \subseteq J_1 \subseteq \dots \subseteq J_t \subseteq \dots \subseteq J_m = A.$$

*Then  $A^{\text{op}}$  is quasi-hereditary also, and*

$$0 = J_0^{\text{op}} \subseteq J_1^{\text{op}} \subseteq \dots \subseteq J_t^{\text{op}} \subseteq \dots \subseteq J_m^{\text{op}} = A^{\text{op}}$$

*is a heredity chain.*

*Moreover,  $\text{gl dim } A \leq 2m - 2$  and  $L(A) \leq 2^m - 1$ .*

The following assertion can also be easily obtained.

STATEMENT 10. *Let  $A$  be a semiprimary quasi-hereditary ring with a heredity chain*

$$0 = J_0 \subseteq J_1 \subseteq \dots \subseteq J_t \subseteq \dots \subseteq J_m = A.$$

Then, for each  $1 \leq t \leq m$ ,  $J_t = Ae_tA$  for an idempotent  $e_t$  of  $A$  and

$$0 = e_t J_0 e_t \subseteq e_t J_1 e_t \subseteq \cdots \subseteq e_t J_t e_t = e_t A e_t$$

is a heredity chain of  $e_t A e_t$ .

*Proof.* First,  $J_t = Ae_tA$  for some  $e_t$ ,  $e_t^2 = e_t$ , by Statement 6. Clearly,

$$e_t J_t e_t = e_t A e_t A e_t \supseteq e_t A e_t,$$

and thus  $e_t J_t e_t = e_t A e_t$ . Now, if we show that  $e_t J_1 e_t$  is a heredity ideal of  $e_t A e_t$ , then the statement follows easily by induction. Since  $J_1 = Ae_1A$  for an idempotent  $e_1$  and  $e_1 \in Ae_1A$ ,

$$e_t J_1 e_t \supseteq (e_t J_1 e_t)^2 = e_t A e_1 A e_t A e_1 A e_t \supseteq e_t A e_1 A e_t = e_t J_1 e_t.$$

Furthermore, if  $N$  is the radical of  $A$ , then  $e_t N e_t$  is the radical of  $e_t A e_t$  and

$$(e_t J_1 e_t)(e_t N e_t)(e_t J_1 e_t) \subseteq e_t J_1 N J_1 e_t = 0.$$

Finally, the  $A$ -module  $(J_1)_A$  can be written as the finite direct sum of projective  $A$ -modules

$$J_1 = \bigoplus_p a_p A \oplus P,$$

where  $P \subseteq (1 - e_t)A$ ,  $a_p A \simeq eA$  with  $e^2 = 1$ ,  $ee_1 = e_1e = e$  and  $a_p = e_t a_p = a_p e_1$ , i.e.,

$$a_p \in e_t A e_1 \subseteq e_t A e_t.$$

Hence,

$$e_t J_1 e_t = \bigoplus_p a_p (e_t A e_t)$$

is a decomposition into a direct sum of projective  $e_t A e_t$ -modules

$$a_p (e_t A e_t) \simeq e A e_t = (e_t e e_t)(e_t A e_t).$$

**Examples.** Now we exhibit some examples (mostly path algebras over a field) to illustrate the necessity of some assumptions in the previous statements and the optimality of some bounds.

- (i) The assumption that  $J_A$  is projective cannot be omitted in Statements 1, 2 and 5.

Let  $A$  be the path algebra of the diagram

$$\circ \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} \circ$$

modulo the ideal generated by  $\alpha\beta$  and  $\beta\alpha$ . Let  $J$  be a one-dimensional ideal. Then,  $\text{Hom}_A(J_A, B_A) = 0$ ,  $J^2 = 0 \neq J$ ,  $JNJ = 0$ ,

$$\text{gl dim } A = \infty \quad \text{and} \quad \text{gl dim } B = 1.$$

(ii) The assumption  $J^2 = J$  cannot be omitted in Statements 3 and 4.

Let  $A$  be the path algebra of  $\circ \rightarrow \circ \rightarrow \circ$  and  $J$  the unique one-dimensional ideal. Then  $J_A$  is projective,  $J^2 = 0 \neq J$ ,

$$\text{gl dim } A = 1 \quad \text{and} \quad \text{gl dim } B = 2.$$

(iii) The assumption that  $J_A$  is projective cannot be omitted in Statements 3 and 4.

Let  $A$  be the (basic) Auslander algebra of the algebra in (ii), i.e., the endomorphism algebra of the direct sum of all six indecomposable modules over the algebra in (ii). Let  $J$  be the set of all endomorphisms which factor through the largest indecomposable (projective and injective) module. Then  $J$  is an idempotent ideal,

$$\text{gl dim } A = 2 \quad \text{and} \quad \text{gl dim } B = 4.$$

(iv) The assumption  $JNJ = 0$  cannot be omitted in Statement 5.

Let  $A$  be the path algebra of the diagram

$$1 \xrightarrow{\alpha} 2 \supset \beta$$

modulo the ideal generated by  $\beta^2$ . Let  $J$  be the (idempotent) ideal  $AeA$ , where  $e$  is the idempotent corresponding to the vertex 2. Then  $J_A$  is projective,  $\text{gl dim } A = \infty$  and  $\text{gl dim } B = 0$ .

(v) The assumption  $eNe = 0$  cannot be omitted in Statement 7.

Let  $A$  be the path algebra of the diagram

$$1 \xrightarrow{\alpha} 2 \supset \beta$$

modulo the ideal generated by  $\alpha\beta$  and  $\beta^2$ . Let  $e$  be the idempotent corresponding to the vertex 2. Then the multiplication map

$$Ae \otimes_{eAe} eA \rightarrow AeA$$

is bijective,  ${}_A(AeA)$  is projective and  $(AeA)_A$  is not projective.

(vi) The bounds on global dimensions in Statements 5 and 9 are best possible.

Let  $A$  be the path algebra of the diagram

$$1 \begin{matrix} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{matrix} 2 \begin{matrix} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{matrix} 3 \cdots m-1 \begin{matrix} \xrightarrow{\alpha_{m-1}} \\ \xleftarrow{\beta_{m-1}} \end{matrix} m$$

modulo the ideal generated by all  $\alpha_t \alpha_{t+1}, \beta_{t+1} \beta_t, \beta_t \alpha_t - \alpha_{t+1} \beta_{t+1}$  for  $1 \leq t \leq m-2$  and  $\beta_{m-1} \alpha_{m-1}$ . Let  $e_t$  be the idempotent corresponding to the vertex  $t$ ,  $1 \leq t \leq m$ . Put  $J_t = A(e_m + e_{m-1} + \cdots + e_{m-t+1})A$ . Then  $0 = J_0 \subset J_1 \subset \cdots \subset J_t \subset \cdots \subset J_m = A$  is a heredity chain of the quasi-hereditary algebra  $A$  and  $\text{gl dim } A = 2m - 2$ . Let us remark that  $L(A) = 3$ .

(vii) The bounds on Loewy lengths in Statements 8 and 9 are best possible.

Consider the complete oriented graph (without loops) on  $m$  vertices; denote the arrow from the vertex  $r$  to the vertex  $s$  by  $\alpha_{rs}$ ,  $1 \leq r, s \leq m, r \neq s$ . Let  $A$  be the path algebra of this graph modulo the ideal  $I$  generated by all products  $\alpha_{t_1 t_2} \alpha_{t_2 t_3} \cdots \alpha_{t_k t_{k+1}}$ , where  $t_1 = t_{k+1}$  and all  $t_i < t_1$  for  $2 \leq i \leq k$ . Let  $e_t$  be the idempotent corresponding to the vertex  $t$ ,  $1 \leq t \leq m$ . Let

$$J_t = A(e_m + e_{m-1} + \cdots + e_{m-t+1})A.$$

Then

$$0 = J_0 \subset J_1 \subset \cdots \subset J_t \subset \cdots \subset J_m = A$$

is a heredity chain of the quasi-hereditary algebra  $A$ . Consider the element

$$a = \alpha_{t_1 t_2} \alpha_{t_2 t_3} \cdots \alpha_{t_k t_{k+1}} \cdots \alpha_{t_{2m-2} t_{2m-1}} \text{ modulo } I,$$

where  $t_k = r + 1$  such that  $k = 2^r(2s - 1)$ . It is not difficult to see that  $a$  is a non-zero element of the socle of  $A$ , and to deduce, by induction, that  $L(A) = 2^m - 1$ . Let us remark that  $\text{gl dim } A = 2$ .

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