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FITTING CLASSES OF $\mathscr{S}_1$-GROUPS, I

BY

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1. Introduction

Since they were first introduced by Fischer [1], Fitting classes and the associated injectors of a finite soluble group have provided one of the most fruitful topics in the theory of finite soluble groups; a useful introduction to the subject is contained in the essay by Hawkes [4]. However, there have been few attempts to obtain a similar theory in infinite groups, particularly in the case of groups which are not locally finite and in which, therefore, one has little hope of using any form of Sylow theory. One positive result that has been obtained is the existence and conjugacy of nilpotent (hypercentral) injectors in polycyclic groups ($\mathscr{S}_1$-groups) [8]. Our aim here is to give a general theory of Fitting classes of $\mathscr{S}_1$-groups and investigate the extent to which this will give rise to the existence and conjugacy of injectors.

An $\mathscr{S}_1$-group is one possessing a finite normal series in which the factors are abelian groups of finite rank whose torsion subgroups are Černikov groups. Our terminology is taken from Robinson [7, Part 2, p. 137]; $\mathscr{S}_1$-groups have also been called *soluble groups of type* A_3 by Mal'cev [6]. One of the properties of $\mathscr{S}_1$-groups that will be of crucial importance throughout is that they are nilpotent-by-abelian-by-finite [7, Theorem 3.25].

Throughout we shall work within a subclass $\mathscr{X}$ of $\mathscr{S}_1$ which contains $\mathscr{F}$, the class of all finite soluble groups, and which is $\{S, D_0\}$-closed; that is, if $G \in \mathscr{X}$ and $H \leq G$ then $H \in \mathscr{X}$ and if $N_1, N_2 \in \mathscr{X}$ then $N_1 \times N_2 \in \mathscr{X}$.

We shall occasionally refer to specific subclasses and one should note the following possibilities for $\mathscr{X}$:

(1) $\mathscr{X} = \mathscr{S}_1$,
(2) $\mathscr{X} = \mathscr{P}$, the class of polycyclic groups,
(3) $\mathscr{X} = \mathscr{F}$, the class of finite soluble groups
(4) $\mathscr{X} = \mathscr{C}$, the class of Černikov (or extremal) soluble groups,
(5) $\mathscr{X} = \mathscr{M}$, the class of soluble minimax groups.

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If $\mathscr{X}$ is a $\{S, D_0\}$-closed subclass of $\mathscr{S}_1$ containing $\mathscr{F}$ then a *Fitting class of $\mathscr{X}$-groups* (or a *$\mathscr{X}$-Fitting class*) is a subclass $\mathfrak{X}$ of $\mathscr{X}$ such that

(F1) if $G \in \mathfrak{X}$ and A is an ascendant subgroup of G, then $A \in \mathfrak{X}$,
(F2) if $G \in \mathscr{X}$ and G is generated by normal $\mathfrak{X}$-subgroups, then $G \in \mathfrak{X}$,
(F3) if $G \in \mathscr{X}$ and G is the union of an ascending chain of ascendant $\mathfrak{X}$-subgroups, then $G \in \mathfrak{X}$.

If $\mathscr{X} = \mathscr{F}$ or $\mathscr{P}$, then condition (F1) reduces to the usual S_n-closure and (F3) can be omitted. For $\mathscr{X} = \mathscr{F}$, the definition gives the usual definition of a Fitting class of finite soluble groups.

Condition (F2) ensures that the join of all normal $\mathfrak{X}$-subgroups of a $\mathscr{X}$-group G is also a normal $\mathfrak{X}$-subgroup of G. This unique maximal normal $\mathfrak{X}$-subgroup of G is called the $\mathfrak{X}$-radical of G and is denoted by $G_{\mathfrak{X}}$. We see from the following result that conditions (F2) and (F3) could be replaced by the single condition: if $G \in \mathscr{X}$ and G is generated by ascendant $\mathfrak{X}$-subgroups, then $G \in \mathfrak{X}$. Although this may be considered a more natural definition of a Fitting class it is usually easier to check separately that conditions (F2) and (F3) are satisfied.

LEMMA 1.1. *Let $\mathfrak{X}$ be a $\mathscr{X}$-Fitting class and $G \in \mathscr{X}$. Then:*
(i) *If A asc G and $A \in \mathfrak{X}$, then $A \leq G_{\mathfrak{X}}$,*
(ii) *$G_{\mathfrak{X}} = \langle A | A$ is an ascendant $\mathfrak{X}$-subgroup of $G \rangle$,*
(iii) *If B asc G, then $B_{\mathfrak{X}} = B \cap G_{\mathfrak{X}}$.*

Proof. Part (i) appears in [7, Lemma 1.31.] and (ii) is an immediate consequence.

(iii) Since $B \cap G_{\mathfrak{X}}$ asc $G_{\mathfrak{X}}$, we have $B \cap G_{\mathfrak{X}} \in \mathfrak{X}$. Also $B \cap G_{\mathfrak{X}} \triangleleft B$ and so $B \cap G_{\mathfrak{X}} \leq B_{\mathfrak{X}}$. Conversely $B_{\mathfrak{X}} \in \mathfrak{X}$ and $B_{\mathfrak{X}} \triangleleft B$ asc G so that $B_{\mathfrak{X}}$ asc G. Hence $B_{\mathfrak{X}} \leq G_{\mathfrak{X}}$ and we obtain $B_{\mathfrak{X}} = B \cap G_{\mathfrak{X}}$.

By considering the characteristic of a Fitting class in Section 2 we are able to show that a $\mathscr{X}$-Fitting class $\mathfrak{X}$ either contains all locally nilpotent $\mathscr{X}$-groups or consists entirely of π-groups for some set of primes π and, in this case, $\mathfrak{X}$ contains all locally nilpotent π-groups which are in the class $\mathscr{X}$.

These two situations give rise to rather different types of result and are largely considered separately. The more important case is when $\mathfrak{X}$ contains all locally nilpotent $\mathscr{X}$-groups so that the radical $G_{\mathfrak{X}}$ contains the Hirsch-Plotkin radical $G_{\mathscr{N}}$. Since $G/G_{\mathscr{N}}$ is a polycyclic abelian-by-finite group (Theorem 3.1) this means that we can usually restrict our discussion to groups in which $G/G_{\mathfrak{X}}$ is polycyclic and abelian-by-finite.

If $\mathfrak{X}$ is a Fitting class of $\mathscr{X}$-groups and $G \in \mathscr{X}$, then an *$\mathfrak{X}$-injector* of G is an $\mathfrak{X}$-subgroup V of G such that $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of A whenever A is an ascendant subgroup of G.

For the case in which $\mathfrak{X}$ consists entirely of periodic groups a sufficient condition for the existence and conjugacy of $\mathfrak{X}$-injectors is given in Theorem 4.7. A more important case is the following.

THEOREM 4.4. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class containing all locally nilpotent $\mathscr{K}$-groups and let $G \in \mathscr{K}$. Suppose that G has a normal subgroup M such that $M/G_{\mathfrak{X}}$ is finite and M contains all $\mathfrak{X}$-subgroups of G which contain $G_{\mathfrak{X}}$. Then G has $\mathfrak{X}$-injectors and any two such subgroups are conjugate in G. The $\mathfrak{X}$-injectors of G are the $\mathfrak{X}$-injectors of M.*

At first sight the conditions of Theorem 4.4 may appear somewhat artificial but we also prove a form of converse to this result showing that if $\mathfrak{X}$ is a Fitting class of $\mathscr{K}$-groups which gives rise to the existence and conjugacy of $\mathfrak{X}$-injectors, then every $\mathscr{K}$-group G must satisfy conditions similar to the hypothesis in Theorem 4.4. More precisely we prove the following theorem.

THEOREM 6.3. *Let $\mathscr{K} \supseteq \mathscr{P}$ and let $\mathfrak{X}$ be a Fitting class of $\mathscr{K}$-groups such that each $\mathscr{K}$-group G has a unique conjugacy class of $\mathfrak{X}$-injectors. Then each $\mathscr{K}$-group G has a normal subgroup M such that $M/G_{\mathfrak{X}}$ is finite and the $\mathfrak{X}$-injectors of G are the $\mathfrak{X}$-injectors of M.*

Although these could be considered as our main results, we are also able to prove that many of the properties of $\mathfrak{X}$-injectors of finite soluble groups (e.g., pronormality) can be extended to the infinite case; in some places these additional properties are, in fact, necessary as part of the main proofs. We have also included two sections described as examples. These sections do include examples to illustrate the theory (and also some of the difficulties involved in making natural generalizations from the finite case) but also include examples to show that there are limits to the results we can expect. These examples are necessary in reducing the type of Fitting class that we need to consider in Theorem 6.3.

2. Characteristic of a Fitting class

Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class. We define $\mathbf{C}_f(\mathfrak{X})$, the *finite characteristic* of $\mathfrak{X}$, to be the set of primes p such that $C_p \in \mathfrak{X}$, where C_p is the cyclic group of order p. If $\mathfrak{X}$ contains no such groups then we write $\mathbf{C}_f(\mathfrak{X}) = \varnothing$. The *infinite characteristic* $\mathbf{C}_i(\mathfrak{X})$ of $\mathfrak{X}$ is defined to be $\{\infty\}$ if $\mathfrak{X}$ contains the infinite cyclic group C_∞ and $\varnothing$ otherwise. The *characteristic* of $\mathfrak{X}$ is then $\mathbf{C}(\mathfrak{X}) = C_f(\mathfrak{X}) \cup \mathbf{C}_i(\mathfrak{X})$, a subset of $\mathbf{P} \cup \{\infty\}$, where $\mathbf{P}$ denotes the set of all primes. We shall see that either $\mathbf{C}(\mathfrak{X})$ is equal to the whole of $\mathbf{P} \cup \{\infty\}$ or $\mathbf{C}(\mathfrak{X})$ is a subset of $\mathbf{P}$ and every $\mathfrak{X}$-group is periodic.

LEMMA 2.1. *Let G be a hyperabelian group.*

(i) *If G contains an element x of prime order p, then there are subgroups $K \triangleleft H$ sn G such that $H/K \cong C_p$.*

(ii) *If G contains an element x of infinite order, then there are subgroups $K \triangleleft H$ sn G such that $H/K \cong C_\infty$.*

Proof. We suppose that G has an ascending normal series

$$1 = G_0 \triangleleft \cdots \triangleleft G_\alpha \triangleleft \cdots \triangleleft G_\rho = G$$

with $G_{\alpha+1}/G_\alpha$ abelian.

(i) Let α be minimal such that $x \in G_\alpha$; then $\alpha - 1$ exists. Clearly

$$\langle x, G_{\alpha-1}\rangle / G_{\alpha-1} \cong C_p \quad \text{and} \quad \langle x, G_{\alpha-1}\rangle \triangleleft G_\alpha \triangleleft G.$$

(ii) We use induction on ρ. Again let α be minimal such that $x \in G_\alpha$; then again $\alpha - 1$ exists and $\alpha - 1 < \rho$. If $xG_{\alpha-1}$ has infinite order then as in part (i) we can take the factor $\langle x, G_{\alpha-1}\rangle / G_{\alpha-1}$. If $x^n \in G_{\alpha-1}$, then x^n is an element of $G_{\alpha-1}$ of infinite order. By induction there are subgroups $K \triangleleft H$ sn $G_{\alpha-1}$ such that $H/K \cong C_\infty$. Since H sn G, this completes the proof.

By Lemma 4.2 and a generalization of a standard argument used in the finite case, we obtain the next result.

THEOREM 2.2. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class.*

(i) *If there is a group $G \in \mathfrak{X}$ which contains a p-element, then $C_p \in \mathfrak{X}$.*

(ii) *If there is a group $G \in \mathfrak{X}$ which contains an element of infinite order then $C_\infty \in \mathfrak{X}$.*

(iii) *If $C_\infty \in \mathfrak{X}$, then $\mathbf{C}(\mathfrak{X}) = \mathbf{P} \cup \{\infty\}$.*

A simple consequence of Theorem 2.2 is:

COROLLARY 2.3. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class. Then either*

(a) *$C(\mathfrak{X}) = \pi \subseteq \mathbf{P}$ and every $\mathfrak{X}$-group is a π-group, or*

(b) *$\mathbf{C}(\mathfrak{X}) = \mathbf{P} \cup \{\infty\}$.*

THEOREM 2.4. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class.*

(i) *If $\mathbf{C}(\mathfrak{X}) = \pi \subseteq \mathbf{P}$, then $\mathfrak{X}$ contains all locally nilpotent π-groups in $\mathscr{K}$.*

(ii) *If $\mathbf{C}(\mathfrak{X}) = \mathbf{P} \cup \{\infty\}$, then $\mathfrak{X}$ contains all locally nilpotent $\mathscr{K}$-groups.*

Proof. Since $\mathscr{K} \subseteq \mathscr{S}_1$, a locally nilpotent $\mathscr{K}$-group G is hypercentral [7, Part 2, p. 38, Corollary 1] and so every subgroup of G is ascendant. Now G is generated by its elements of infinite and prime-power order. It is therefore sufficient to note that if $C_p \in \mathfrak{X}$ then $C_{p^n} \in \mathfrak{X}$. This can be seen by considering the wreath product $C_{p^n} \wr C_p$. If $C_{p^n} = \langle a \rangle$ and $C_p = \langle b \rangle$ then $a^{-1}ba$ has order p and so $\langle b, a^{-1}ba \rangle$, being generated by subnormal $\mathfrak{X}$-subgroups, is an $\mathfrak{X}$-group. But

$$[b, a] \in \langle b, a^{-1}ba \rangle$$

and has order p^n. Thus

$$C_{p^n} \cong \langle [b, a] \rangle \in S_n \mathfrak{X} = \mathfrak{X}.$$

Many of our results will be concerned with the class $\mathscr{P}$ of polycyclic groups and it is worth noting the consequences of the above results for that case.

COROLLARY 2.5. *Let $\mathfrak{X}$ be a $\mathscr{P}$-Fitting class.*
(i) *If $C_\infty \notin \mathfrak{X}$, then $\mathfrak{X} \subseteq \mathscr{F}$.*
(ii) *If $C_\infty \in \mathfrak{X}$, then $\mathfrak{X}$ contains all finitely generated nilpotent groups.*

3. Examples of $\mathscr{S}_1$-Fitting classes

First note that if $\mathscr{K}$ is a $\{S, D_0\}$-closed subclass of $\mathscr{S}_1$ containing $\mathscr{F}$ and if $\mathfrak{X}$ is an $\mathscr{S}_1$-Fitting class then $\mathfrak{X} \cap \mathscr{K}$ is a $\mathscr{K}$-Fitting class. Thus our examples of $\mathscr{S}_1$-Fitting classes also lead to $\mathscr{K}$-Fitting classes for each $\mathscr{K}$.

Example A. Let $\mathscr{N}$ denote the class of locally nilpotent $\mathscr{S}_1$-groups. Then $\mathscr{N}$ is an $\mathscr{S}_1$-Fitting class and if $G \in \mathscr{S}_1$, then $G_{\mathscr{N}}$ is the Hirsch-Plotkin radical of G. (See [7, 2.3] for details.)

We saw in Section 2 that if $\mathfrak{X}$ is a Fitting class containing C_∞ then $\mathfrak{X}$ contains all locally nilpotent $\mathscr{K}$-groups and so $G_{\mathfrak{X}} \geq G_{\mathscr{N}}$. Our results for Fitting classes $\mathfrak{X}$ with $\mathbf{C}(\mathfrak{X}) = \mathbf{P} \cup \{\infty\}$ will be considerably simplified by the following result.

THEOREM 3.1. *If $G \in \mathscr{S}_1$, then $G/G_{\mathscr{N}}$ is a finitely generated abelian-by-finite group (and hence $G/G_{\mathscr{N}}$ is polycyclic).*

Proof. Since an $\mathscr{S}_1$-group is nilpotent-by-abelian-by-finite, we also have that $G/G_{\mathscr{N}}$ is abelian-by-finite and so only need prove that $G/G_{\mathscr{N}}$ is finitely generated.

By Theorem 10.3.3 of [7], G has a normal series

$$1 \lhd R \lhd F \lhd G$$

where R is the direct product of finitely many quasicyclic groups, F/R is nilpotent and G/F is finitely generated and abelian-by-finite.

Let P be the Sylow p-subgroup of R and let

$$P_n = \Omega_n(P) = \{x \in P | x^{p^n} = 1\}.$$

Then, for each $n \geq 1$, the mapping

$$\varphi: P_{n+1}/P_n \to P_n/P_{n-1}$$

is defined by making $(P_n x)\varphi = P_{n-1}x^p$ a G-isomorphism. Hence

$$C_F(P_1) = C_F(P_2/P_1) = C_F(P_3/P_2) = \cdots .$$

Writing C_P for $C_F(P_1)$, we have F/C_P finite since P_1 is finite. Let

$$C = \bigcap_{p \in \varpi(R)} C_p;$$

then F/C is finite and so G/C is finitely generated. Also C is a hypercentral group and so $C \leq G_{\mathscr{N}}$; hence $G/G_{\mathscr{N}}$ is finitely generated.

The class of $\mathscr{S}_1$-groups is not closed under homomorphic images but most of the difficulties which might arise from this are avoided by the following Corollary to Theorem 3.1.

COROLLARY 3.2. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class and $G \in \mathscr{K}$, then $G/G_{\mathfrak{X}}$ is an $\mathscr{S}_1$-group.*

Proof. If $\mathbf{C}(\mathfrak{X}) = \mathbf{P} \cup \{\infty\}$, then $\mathfrak{X}$ contains all $\mathscr{N} \cap \mathscr{K}$-groups and so $G_{\mathfrak{X}} \geq G_{\mathscr{N}}$ so that $G/G_{\mathfrak{X}}$ is polycyclic.

If $\mathbf{C}(\mathfrak{X}) = \pi \subseteq \mathbf{P}$, then $G_{\mathfrak{X}}$ is a normal Černikov subgroup of G and it is clear that $G/G_{\mathfrak{X}}$ is an $\mathscr{S}_1$-group.

Example B. Let $\mathscr{E}_p$ denote the class of Černikov p-groups. Then $\mathscr{E}_p$ is an $\mathscr{S}_1$-Fitting class and if $G \in \mathscr{S}_1$, we shall denote the $\mathscr{E}_p$-radical of G by $O_p(G)$.

Example C. Let p be a fixed prime. For $G \in \mathscr{S}_1$, the p-socle of G is

$$\mathrm{Soc}_p(G) = \langle M | M \text{ is a minimal normal } p\text{-subgroup of } G \rangle.$$

If G has no (minimal) normal p-subgroups then $\mathrm{Soc}_p(G) = 1$. Each minimal normal p-subgroup is elementary abelian. Thus $\mathrm{Soc}_p(G)$ is an elementary abelian p-group and is a Černikov group; hence $\mathrm{Soc}_p(G)$ is a finite elementary abelian characteristic p-subgroup of G.

We define $\mathscr{C}(p)$ to be the class of $\mathscr{S}_1$-groups G such that $\mathrm{Soc}_p(G) \leq Z(G)$ or, equivalently, each minimal normal p-subgroup is central.

THEOREM 3.3. *$\mathscr{C}(p)$ is an $\mathscr{S}_1$-Fitting class and if $G \in \mathscr{S}_1$, then*

$$G_{\mathscr{C}(p)} = C_G(\mathrm{Soc}_p(G)).$$

Proof. (I) *An ascendant subgroup of a $\mathscr{C}(p)$-group is a $\mathscr{C}(p)$-group.*

We consider an ascendant subgroup A of an $\mathscr{S}_1$-group G and show that if $A \notin \mathscr{C}(p)$ then $G \notin \mathscr{C}(p)$. In fact, we shall prove that if A has a non-central minimal normal p-subgroup M_0 then G has a non-central minimal normal p-subgroup M contained in M_0^G. Our proof is by induction on α the length of an ascending series

$$A = A_0 \triangleleft \cdots \triangleleft A_\beta \triangleleft A_{\beta+1} \triangleleft \cdots \triangleleft A_\alpha = G$$

from A to G. We assume therefore that if $H \triangleleft^\beta K$, $\beta < \alpha$, and H has a non-central minimal normal p-subgroup N_0, then K has a non-central minimal normal p-subgroup $M \leq N_0^K \leq N_0^G$.

Case (a). $\alpha = \beta + 1$.

By the induction hypothesis A_β has a non-central minimal normal p-subgroup $M_\beta \leq M_0^G$. Now M_β is contained in the finite characteristic subgroup $\mathrm{Soc}_p(A_\beta)$ of A_β and so M_β^G is finite. Thus there are elements $g_1, \dots, g_n$ of G such that

$$M_\beta^G = M_\beta^{g_1} \times \cdots \times M_\beta^{g_n}$$

and M_β^G is a direct product of non-central minimal normal p-subgroups of A_β. Now M_β^G contains a minimal normal p-subgroup M of G and M contains a minimal normal subgroup N of A_β. Since N is A_β-isomorphic to some $M_\beta^{g_i}$, N is non-central in A_β. Hence M is not centralized by A_β and so it is a non-central minimal normal p-subgroup of G and $M \leq M_\beta^G \leq M_0^G$, as required.

Case (b). α is a limit ordinal.

Note in this case that all finite ordinals are less than α and so it follows from our induction hypothesis that if H sn K and H has a non-central minimal normal p-subgroup N_0 then K has a non-central minimal normal p-subgroup $M \leq N_0^K \leq N_0^G$.

Since M_0 is an ascendant p-subgroup of G we have $M_0 \leq O_p(G) = P$, say. Now P is a Černikov group and so contains a characteristic divisible abelian subgroup R of finite index. If we let R_i be the subgroup of R consisting of those elements of order dividing p^i, then $R = \bigcup_{i=1}^\infty R_i$ and R_i, being characteristic in R, is normal in G. Since $R = \bigcup_{i=1}^{\infty} R_i$,

$$C_P(R) = \bigcap_{i=1}^{\infty} C_P(R_i) \geq R.$$

Since P/R is finite, there is an integer k such that $C_P(R) = C_P(R_k)$.

The subgroup AR_k is ascendant in G and, since $|AR_k : A|$ is finite, A is subnormal in AR_k and hence AR_k has a non-central minimal normal p-subgroup $N \leq M_0^G \leq P$. If $N \nleq R_k$, then $N \cap R_k = 1$ and so $N \leq C_P(R_k) = C_P(R)$. But $C_P(R)$ is a centre-by-finite group and so is finite-by-abelian [7, Theorem 4.12]. Hence $C_P(R)$ is a finite-by-(divisible abelian) p-group. Then N^G, being generated by elements of order p and contained in the normal subgroup $C_P(R)$, is clearly finite. If $N \leq R_k$, then $N^G \leq R_k$ is again finite.

We now have an ascendant subgroup $AR_k \lhd^{\alpha} G$ and a non-central minimal normal subgroup N of AR_k such that N^G is finite. Suppose that

$$AR_k = H_0 \lhd \cdots \lhd H_\beta \lhd \cdots \lhd H_\alpha = G.$$

Then, by induction, we may assume that each H_β contains a non-central minimal normal p-subgroup $N_\beta \leq N^G \leq M_0^G$.

Since N^G is finite there is a cofinal subset $I \subseteq \{\beta | \beta < \alpha\}$ such that, for all $\gamma \in I$, $N_\gamma = L$, say. We now have $L \lhd H_\gamma$, for all $\gamma \in I$, and so

$$L \lhd \bigcup_{\gamma \in I} H_\gamma = G$$

and L is a non-central minimal normal p-subgroup of G contained in M_0^G.

(II) *An $\mathscr{S}_1$-group generated by normal $\mathscr{C}(p)$-groups is a $\mathscr{C}(p)$-group.*

Let $G = \langle N_i | i \in I \rangle$ be an $\mathscr{S}_1$-group with each N_i being a normal $\mathscr{C}(p)$-subgroup of G. We let M be a minimal normal p-subgroup of G and show that $M \leq Z(G)$. For each $i \in I$, $M \cap N_i \lhd G$ and so either $M \cap N_i = 1$ or $M \leq N_i$. If $M \cap N_i = 1$, then clearly $[M, N_i] = 1$. If $M \leq N_i$, then by Clifford's Theorem, M is a direct product of minimal normal subgroups of N_i so that $M \leq \mathrm{Soc}_p(N_i) \leq Z(N_i)$. In both cases N_i centralizes M and hence M is centralized by $\langle N_i | i \in I \rangle = G$.

(III) *An $\mathscr{S}_1$-group which is the union of an ascending chain of ascendant $\mathscr{C}(p)$-subgroups is a $\mathscr{C}(p)$-group.*

Let $G = \bigcup_{\alpha < \rho} A_\alpha$ where each A_α is an ascendant $\mathscr{C}(p)$-subgroup of G. Let M be a minimal normal p-subgroup of G; then M is finite and so contains only finitely many proper non-trivial subgroups $N_1, \ldots, N_k$, say. For each

$i = 1, \ldots, k$, there is an $\alpha(i)$ such that N_i is not normalized by $A_{\alpha(i)}$. For

$$\beta \geq \max\{\alpha(1), \ldots, \alpha(k)\},$$

M is a minimal normal p-subgroup of A_β and hence $[M, A_\beta] = 1$. It follows that $M \leq Z(G)$ and so $G \in \mathscr{C}(p)$.

(IV) $G_{\mathscr{C}(p)} = C_G(\mathrm{Soc}_p(G))$.

Let $C = C_G(\mathrm{Soc}_p(G))$ and let M be a minimal normal p-subgroup of C. Then $M^G \leq \mathrm{Soc}_p(G)$ is finite and so there are elements $g_1, \ldots, g_n$ of G such that

$$M^G = M^{g_1} \times \cdots \times M^{g_n}$$

is a direct product of minimal normal p-subgroups of C. Let N be a minimal normal subgroup of G contained in M^G; then N is centralized by C. There is a minimal normal subgroup L of C contained in N and L is C-isomorphic to M^{g_i} for some i. Since $L \leq N$ it is centralized by C and hence M^{g_i} is centralized by C. Therefore $M \leq g_i Z(C) g_i^{-1} = Z(C)$ and it follows that $C \in \mathscr{C}(p)$.

Conversely, let H be a normal $\mathscr{C}(p)$-subgroup of G and let T be a minimal normal p-subgroup of G. Then either $T \cap H = 1$ or $T \leq H$. If $T \cap H = 1$ then $[T, H] = 1$ and so $H \leq C_G(T)$. If $T \leq H$, then it follows from Clifford's Theorem that T is a direct product of minimal normal p-subgroups of H. Since $H \in \mathscr{C}(p)$, each of these subgroups is central in H and we again have $[T, H] = 1$. Thus H centralizes each minimal normal p-subgroup of G and so $H \leq C$.

It should be noted for future applications that since $\mathrm{Soc}_p(G)$ is finite, the final part of the above theorem shows that $G/G_{\mathscr{C}(p)}$ is finite.

Example D. Let $G \in \mathscr{S}_1$; we define G to be a $\mathscr{T}(p)$-group if and only if $G/C_G(O_p(G))$ is a p-group, p a prime number. Let $\mathscr{F}(p) = \mathscr{T}(p) \cap \mathscr{P}$.

Some of the properties of the class $\mathscr{F}(p)$ are presented without proof in Theorem 3.4.

THEOREM 3.4. *$\mathscr{F}(p)$ is a $\mathscr{P}$-Fitting class and if $G \in \mathscr{P}$, then $O_p(G)$ is finite,*

$$G_{\mathscr{F}(p)} = W \quad \textit{where } W/C_G\big(O_p(G)\big) = O_p\big(G/C_G\big(O_p(G)\big)\big),$$

and $G/G_{\mathscr{F}(p)}$ is finite.

This example does not give a Fitting class of $\mathscr{S}_1$-groups. For, let $A \cong C_{p^\infty}$ and form the split extension G of A by the infinite cyclic group $\langle x \rangle$ in which

$x^{-1}ax = a^{p+1}$, for all $a \in A$. Then G is the union of the ascendant $\mathscr{F}(p)$-subgroups $A_n = \Omega_n(A)\langle x\rangle$ but $O_p(G) = A$ and $G/C_G(A)$ is infinite cyclic.

Example E. The class $\mathscr{C}$ of soluble Černikov groups is an $\mathscr{S}_1$-Fitting class. Furthermore, if $\mathfrak{X}$ is an $\mathscr{C}$-Fitting class then $\mathfrak{X}$ is also an $\mathscr{S}_1$-Fitting class. Similarly any Fitting class $\mathfrak{Y}$ of finite soluble groups is a Fitting class of polycyclic groups.

Example F. Let $\mathfrak{X}, \mathfrak{Y}$ be Fitting classes of $\mathscr{K}$-groups and let $\mathfrak{X}\mathfrak{Y}$ denote the class of $\mathscr{K}$-groups G with $G/G_{\mathfrak{X}} \in \mathfrak{Y}$. Some difficulties could arise with this definition if $\mathscr{K}$ is not Q-closed but for most classes $\mathscr{K}$, these difficulties are avoided by using Theorem 3.1.

THEOREM 3.5. *Let $\mathscr{K}$ be an $\{S, D_0\}$-closed subclass of $\mathscr{S}_1$ containing $\mathscr{F}$ and satisfying either* (a) *$\mathscr{K}$ contains $\mathscr{P}$ or* (b) *$\mathscr{K}$ is Q-closed.*
Then for any $\mathscr{K}$-Fitting classes $\mathfrak{X}$ and $\mathfrak{Y}$ with $\mathscr{N} \cap \mathscr{K} \subseteq \mathfrak{X}$, the class $\mathfrak{X}\mathfrak{Y}$ is also a Fitting class of $\mathscr{K}$-groups and $G_{\mathfrak{X}\mathfrak{Y}}/G_{\mathfrak{X}} = (G/G_{\mathfrak{X}})_{\mathfrak{Y}}$.

Proof. It is straightforward to prove that an ascendant subgroup of an $\mathfrak{X}\mathfrak{Y}$-group is an $\mathfrak{X}\mathfrak{Y}$-group. The hypotheses on $\mathscr{K}$ are required to show that a $\mathscr{K}$-group G which is the join of ascendant $\mathfrak{X}\mathfrak{Y}$-subgroups A_i, $i \in I$, is also an $\mathfrak{X}\mathfrak{Y}$-group. Since $\mathfrak{X}$ is a $\mathscr{K}$-Fitting class, G has an $\mathfrak{X}$-radical $G_{\mathfrak{X}}$. By Theorem 3.1 or the Q-closure of $\mathscr{K}$, $G/G_{\mathfrak{X}} \in \mathscr{K}$. Also $G/G_{\mathfrak{X}}$ is generated by the ascendant subgroups $A_iG_{\mathfrak{X}}/G_{\mathfrak{X}}$, $i \in I$, and

$$A_iG_{\mathfrak{X}}/G_{\mathfrak{X}} \cong A_i/A_i \cap G_{\mathfrak{X}} = A_i/(A_i)_{\mathfrak{X}} \in \mathfrak{Y}.$$

Thus $G/G_{\mathfrak{X}} \in \mathfrak{Y}$ and $G \in \mathfrak{X}\mathfrak{Y}$.

Taking $\mathfrak{X} = \mathfrak{Y} = \mathscr{N}$, we obtain the Fitting class $\mathscr{N}^2$ of $\mathscr{S}_1$-groups in which $G/G_{\mathscr{N}}$ is locally nilpotent. Since $\mathscr{S}_1$-groups are nilpotent-by-abelian-by-finite we always have $G/G_{\mathscr{N}^2}$ finite.

Let $\mathfrak{X}$ be a Fitting class of $\mathscr{S}_1$-group and let $G \in \mathscr{S}_1$. Then $G/G_{\mathscr{N}} \in \mathscr{P}$ by Theorem 3.1 and $G/G_{\mathfrak{X}} \in \mathscr{S}_1$, by Corollary 3.2. Thus we can therefore obtain the following variations of the above construction.

THEOREM 3.6. *Let $\mathfrak{X}$ be an $\mathscr{S}_1$-Fitting class. Then*:
(i) *If $\mathscr{N} \subseteq \mathfrak{X}$ and $\mathfrak{Y}$ is a $\mathscr{P}$-Fitting class, then $\mathfrak{X}\mathfrak{Y}$ is an $\mathscr{S}_1$-Fitting class.*
(ii) *If $\mathfrak{Y}$ is an $\mathscr{S}_1$-Fitting class, then $\mathfrak{X}\mathfrak{Y}$ is an $\mathscr{S}_1$-Fitting class.*
In particular $\mathscr{N}\mathscr{F}(p)$ is an $\mathscr{S}_1$-Fitting class even though $\mathscr{F}(p)$ is not.

4. Injectors

Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class and let $G \in \mathscr{K}$. A subgroup of X of G is called an *$\mathfrak{X}$-injector* of G if $X \cap A$ is a maximal $\mathfrak{X}$-subgroup of A for each ascendant subgroup A of G. We shall denote the set of $\mathfrak{X}$-injectors of the $\mathscr{K}$-group G by $\mathrm{Inj}_{\mathfrak{X}}(G)$.

The methods used to prove the existence and conjugacy of $\mathfrak{X}$-injectors of finite soluble groups in [2] can be adapted to deal with infinite groups in which $G/G_{\mathfrak{X}}$ is a finite soluble group. The basic result concerning this situation is given in Theorem 4.3; we then consider separately the cases in which

$$\mathbf{C}(\mathfrak{X}) = \mathbf{P} \cup \{\infty\} \quad \text{and} \quad \mathbf{C}(\mathfrak{X}) = \pi \subseteq \mathbf{P}.$$

The proof of the following generalization of Hartley's Lemma is similar to the finite case and hence is omitted.

LEMMA 4.1. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class and let G be a $\mathscr{K}$-group with a normal $\mathfrak{X}$-subgroup R such that G/R is finite. Let $R \leq N \lhd G$ with G/N nilpotent and let W be a maximal $\mathfrak{X}$-subgroup of N containing R. If V_1 and V_2 are maximal $\mathfrak{X}$-subgroups of G containing W, then V_1 and V_2 are conjugate in G.*

The next lemma, whose proof can be found on p. 193 of [8], will be used in the proof of Theorem 4.3 and elsewhere in this paper.

LEMMA 4.2. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class and let $G \in \mathscr{K}$. If W is an $\mathfrak{X}$-subgroup of the ascendant subgroup A and $W \geq A_{\mathfrak{X}}$, then $WG_{\mathfrak{X}} \in \mathfrak{X}$.*

THEOREM 4.3. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class and let $G \in \mathscr{K}$. If $G/G_{\mathfrak{X}}$ is finite, then G has $\mathfrak{X}$-injectors and any two $\mathfrak{X}$-injectors of G are conjugate.*

Proof. We use induction on $|G/G_{\mathfrak{X}}|$. If $G = G_{\mathfrak{X}}$, then G is itself an $\mathfrak{X}$-injector so we may assume that there is a maximal normal subgroup $M/G_{\mathfrak{X}}$ of $G/G_{\mathfrak{X}}$. By induction, M has an $\mathfrak{X}$-injector U. Let V be a maximal $\mathfrak{X}$-subgroup of G containing U; we show that V is an $\mathfrak{X}$-injector of G.

Let A be an ascendant subgroup of G. If $AG_{\mathfrak{X}} = G$, let W be a maximal $\mathfrak{X}$-subgroup of A containing $V \cap A$. Then by Lemma 4.2, $WG_{\mathfrak{X}} \in \mathfrak{X}$. But

$$WG_{\mathfrak{X}} \geq (V \cap A)G_{\mathfrak{X}} = V$$

and so, by the maximality of V, $WG_{\mathfrak{X}} = V$. In particular $W \leq V$ and so $W = V \cap A$, as required.

We may suppose therefore that $AG_{\mathfrak{X}} < G$. Then $AG_{\mathfrak{X}}$ is a subnormal subgroup having finite index in G. A simple induction argument allows us to assume that $AG_{\mathfrak{X}}$ is a maximal normal subgroup of G. By induction $AG_{\mathfrak{X}}$ has an $\mathfrak{X}$-injector X, say, and so $M \cap X$ is an $\mathfrak{X}$-injector of $M \cap AG_{\mathfrak{X}}$. Since U is an $\mathfrak{X}$-injector of M we also see that $U \cap AG_{\mathfrak{X}}$ is an $\mathfrak{X}$-injector of $M \cap AG_{\mathfrak{X}}$. The induction hypothesis shows that $M \cap X$ is conjugate to $U \cap AG_{\mathfrak{X}}$ and we may assume that $X \geq U \cap AG_{\mathfrak{X}}$. Now X is contained in a maximal $\mathfrak{X}$-subgroup Y of G and V is also a maximal $\mathfrak{X}$-subgroup of G with V and Y both containing the $\mathfrak{X}$-injector $U \cap AG_{\mathfrak{X}}$ of $M \cap AG_{\mathfrak{X}}$. Since $G/M \cap AG_{\mathfrak{X}}$ is nilpotent we may apply Lemma 4.1 to see that V and Y are conjugate. Thus $V \cap AG_{\mathfrak{X}}$ is conjugate to $Y \cap AG_{\mathfrak{X}} = X$ and so $V \cap AG_{\mathfrak{X}}$ is an $\mathfrak{X}$-injector of $AG_{\mathfrak{X}}$. Hence $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of A.

Now let V_1, V_2 be two $\mathfrak{X}$-injectors of G. Then $V_1 \cap M$ and $V_2 \cap M$ are $\mathfrak{X}$-injectors of M and so, by induction, are conjugate. We may assume that $V_1 \cap M = V_2 \cap M$ so that V_1 and V_2 are maximal $\mathfrak{X}$-subgroups of G containing the $\mathfrak{X}$-injector $V_1 \cap M$ of M. By Lemma 4.1, V_1 and V_2 are conjugate in G.

There are many examples of Fitting classes which give rise to conjugate injectors even though $G/G_{\mathfrak{X}}$ is not always finite. We therefore need to extend the above result and the generalizations required are different depending on whether or not $\mathfrak{X}$ contains the infinite cyclic group.

THEOREM 4.4. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class with $\mathbf{C}(\mathfrak{X}) = \mathbf{P} \cup \{\infty\}$ and let $G \in \mathscr{K}$. Suppose that G has a normal subgroup M such that $M/G_{\mathfrak{X}}$ is finite and M contains all $\mathfrak{X}$-subgroups of G which contain $G_{\mathfrak{X}}$. Then*:

(i) *G has $\mathfrak{X}$-injectors and any two such subgroups are conjugate in G.*

(ii) $\mathrm{Inj}_{\mathfrak{X}}(G) = \mathrm{Inj}_{\mathfrak{X}}(M)$.

(iii) *If $X \in \mathrm{Inj}_{\mathfrak{X}}(G)$, then $X^G/G_{\mathfrak{X}}$ is finite.*

Proof. By Theorem 4.3, M has a unique conjugacy class of $\mathfrak{X}$-injectors. It is clearly sufficient to show that each $\mathfrak{X}$-injector of M is an $\mathfrak{X}$-injector of G and, conversely, that each $\mathfrak{X}$-injector of G is an $\mathfrak{X}$-injector of M.

Let X be an $\mathfrak{X}$-injector of M and let A be an ascendant subgroup of G. If W is an $\mathfrak{X}$-subgroup of A containing $X \cap A$, then $W \geq X \cap A \geq G_{\mathfrak{X}} \cap A = A_{\mathfrak{X}}$ and so, by Lemma 4.2, $WG_{\mathfrak{X}} \in \mathfrak{X}$. By the hypotheses of the theorem, $WG_{\mathfrak{X}} \leq M$ and, in particular, $W \leq M \cap A$. But $M \cap A$ is an ascendant subgroup of M and so $X \cap A$ is a maximal $\mathfrak{X}$-subgroup of $M \cap A$. Hence $W = X \cap A$ and so $X \cap A$ is a maximal $\mathfrak{X}$-subgroup of A. Thus X is an $\mathfrak{X}$-injector of G.

Conversely, let Y be an $\mathfrak{X}$-injector of G. Then $Y \geq G_{\mathfrak{X}}$ and so $Y \leq M$. Thus Y is an $\mathfrak{X}$-injector of M.

The case in which $\mathfrak{X}$ consists entirely of periodic groups essentially reduces to considering injectors of Černikov groups. The proof presented here is based on Theorem 4.3 but requires one further lemma.

LEMMA 4.5. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class and let G be a $\mathscr{K}$-group such that $G/G_{\mathfrak{X}}$ is finite. Let*

$$(*) \qquad G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = B \rhd 1$$

be a series for G with $B \in \mathfrak{X}$ and G_{i-1}/G_i a finite nilpotent group for each $i = 1, \dots, n$. Then V is an $\mathfrak{X}$-injector of G if and only if $G_i \cap V$ is a maximal $\mathfrak{X}$-subgroup of G_i, for each $i = 0, \dots, n$.

Proof. If V is an $\mathfrak{X}$-injector of G then since G_i sn G it is clear that $V \cap G_i$ is a maximal $\mathfrak{X}$-subgroup of G_i, for each i.

We prove the converse by induction on n. If $n = 0$, then $G = B \in \mathfrak{X}$ and so $V = G$ is clearly an $\mathfrak{X}$-injector of G. If $n \geq 1$, then by induction $V \cap G_1$ is an $\mathfrak{X}$-injector of G_1. By Theorem 4.3, G has an $\mathfrak{X}$-injector X, say. Then $X \cap G_1$ is an $\mathfrak{X}$-injector of G_1 and so is conjugate to $V \cap G_1$. We may assume that $V \cap G_1 = X \cap G_1$. Then V and X are maximal $\mathfrak{X}$-subgroups containing the $\mathfrak{X}$-injector $X \cap G_1$ of G_1. By Lemma 4.1, V is conjugate to X and so is an $\mathfrak{X}$-injector of G.

An easy consequence of Lemma 4.5 is the following corollary.

COROLLARY 4.6. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class and let G be a $\mathscr{K}$-group such that $G/G_{\mathfrak{X}}$ is finite. If $V \in \operatorname{Inj}_{\mathfrak{X}}(G)$ and $V \leq L \leq G$, then $V \in \operatorname{Inj}_{\mathfrak{X}}(L)$.*

THEOREM 4.7. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class with $\mathbf{C}(\mathfrak{X}) = \pi \subset \mathbf{P}$ and let $G \in \mathscr{K}$. Suppose that G has a normal Černikov subgroup M such that M contains all $\mathfrak{X}$-subgroups of G which contain $G_{\mathfrak{X}}$. Then:*

(i) *G has $\mathfrak{X}$-injectors and any two such subgroups are conjugate in G.*

(ii) $\operatorname{Inj}_{\mathfrak{X}}(G) = \operatorname{Inj}_{\mathfrak{X}}(M)$.

Proof. Let S be a Sylow π-subgroup of M. Then $S/G_{\mathfrak{X}}$ is finite and so, by Theorem 4.3, S has an $\mathfrak{X}$-injector X. We show that X is an $\mathfrak{X}$-injector of M.

Let A be an ascendant subgroup of M and let V be a maximal $\mathfrak{X}$-subgroup of A containing $X \cap A$. Then V is contained in a Sylow π-subgroup T of A. Now consider

$$H = \langle T, S \cap A \rangle;$$

since $T/A_{\mathfrak{X}}$ and $S \cap A/A_{\mathfrak{X}}$ are finite, $H/A_{\mathfrak{X}}$ and hence $H/H_{\mathfrak{X}}$ must be finite. By Theorem 4.3, H has an $\mathfrak{X}$-injector Y. By Corollary 4.6, Y is an $\mathfrak{X}$-injector of some Sylow π-subgroup of H and so some conjugate of Y is an $\mathfrak{X}$-injector of $S \cap A$. But $X \cap A$ is an $\mathfrak{X}$-injector of $S \cap A$ and so $X \cap A$ is conjugate to Y. In particular, $X \cap A$ is a maximal $\mathfrak{X}$-subgroup of H and so $X \cap A = V$, as required.

It is now clear from Corollary 4.6 that the $\mathfrak{X}$-injectors of M are just the $\mathfrak{X}$-injectors of the Sylow π-subgroups of M and hence are conjugate in M.

The proof that the $\mathfrak{X}$-injectors of M are the $\mathfrak{X}$-injectors of G is exactly as in the proof of Theorem 4.4.

It is necessary to give a form of this result for Fitting classes of Černikov groups which is rather different from Theorem 4.4. For, if G is the split extension of C_{3^∞} by the automorphism α which inverts each element and if $X = \langle \alpha \rangle$ then X is an $\mathfrak{X}$-injector of G with $\mathfrak{X}$ the class of 2-groups. But $G_{\mathfrak{X}} = 1$ and $X^G = G$ so that $X^G/G_{\mathfrak{X}}$ may be an infinite group.

The hypotheses of Theorems 4.4 and 4.7 can be combined to say that $M/G_{\mathfrak{X}}$ is a Černikov group. Note that if $C_\infty \notin \mathfrak{X}$, then M is Černikov. We use Theorem 3.1 in this formulation of the following improved version of Lemma 4.5.

THEOREM 4.8. *Let $\mathfrak{X}$ be a $\mathscr{X}$-Fitting class and let $G \in \mathscr{X}$. Suppose that G has a normal subgroup M such that $M/G_{\mathfrak{X}}$ is a Černikov group and M contains all $\mathfrak{X}$-subgroups of G which contain $G_{\mathfrak{X}}$. Let*

$$M = M_0 \rhd M_1 \rhd \cdots \rhd M_n = B \rhd 1$$

be a series for M with M_{i-1}/M_i nilpotent, for each $i = 1, \ldots, n$, and $B \in \mathfrak{X}$. Then:

(i) *V is an $\mathfrak{X}$-injector of G if and only if V is a maximal $\mathfrak{X}$-subgroup of M_i, for each $i = 0, \ldots, n$.*
(ii) *If $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$ and $V \le L \le G$, then $V \in \mathrm{Inj}_{\mathfrak{X}}(L)$.*
(iii) *The $\mathfrak{X}$-injectors of G are pronormal in G.*
(iv) *If $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$ and $N \lhd G$, then $G = NN_G(V \cap N)$.*
(v) *If $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$ and H/K is a chief factor of G, then V either covers or avoids H/K.*

Proof. (i) By Theorem 4.7 and an argument similar to the proof of Lemma 4.5 shows that $V \in \mathrm{Inj}_{\mathfrak{X}}(M)$ if and only if V is a maximal $\mathfrak{X}$-subgroup of M_i, for each $i = 0, \ldots, n$. We have seen in Theorems 4.4 and 4.7 that $\mathrm{Inj}_{\mathfrak{X}}(M) = \mathrm{Inj}_{\mathfrak{X}}(G)$ so the result follows.

(ii) Since $G_{\mathfrak{X}} \le V \le L$ we have $G_{\mathfrak{X}} \le L_{\mathfrak{X}}$. If X is an $\mathfrak{X}$-subgroup of L containing $L_{\mathfrak{X}}$ then $X \ge G_{\mathfrak{X}}$ and so $X \le M$. Thus every $\mathfrak{X}$-subgroup of L containing $L_{\mathfrak{X}}$ is contained in $L \cap M$ and $L \cap M/L_{\mathfrak{X}}$, being a section of $M/G_{\mathfrak{X}}$, is a Černikov group. Now $L \cap M$ has a series

$$L \cap M = L \cap M_0 \rhd \cdots \rhd L \cap M_n = L \cap B \rhd 1$$

with $L \cap M_{i-1}/L \cap M_i$ nilpotent, for each $i = 1, \ldots, n$, and $B \le G_{\mathfrak{X}} \le L$ so that $L \cap B = B \in \mathfrak{X}$. Since $V \le L$ and $V \cap M_i$ is a maximal $\mathfrak{X}$-subgroup of M_i it is clear that $V \cap M_i$ is a maximal $\mathfrak{X}$-subgroup of $L \cap M_i$. It follows from (i) that $V \in \mathrm{Inj}_{\mathfrak{X}}(L)$.

(iii) Let V be an $\mathfrak{X}$-injector of G and let $x \in G$. Then, by (ii), V and V^x are $\mathfrak{X}$-injectors of $\langle V, V^x \rangle$ and so are conjugate in $\langle V, V^x \rangle$.

(iv) Let $x \in G$; then $(V \cap N)^x = V^x \cap N$ is an $\mathfrak{X}$-injector of N as also is $V \cap N$. Therefore

$$(V \cap N)^x = (V \cap N)^n$$

for some $n \in N$ and $xn^{-1} \in N_G(V \cap N)$. Hence $x \in NN_G(V \cap N)$.

(v) By (iv), $G = NH_G(V \cap H)$. Since H/K is abelian, $K(V \cap H)$ is normalized by H and it is also normalized by $N_G(V \cap H)$. Hence $K(V \cap H) \lhd G$ and so

$$K(V \cap H) = H \text{ or } K,$$

as required.

5. Examples of injectors

5.1. $\mathscr{N}$-injectors.

It was shown in [8] that every $\mathscr{S}_1$-group has a unique conjugacy class of $\mathscr{N}$-injectors. Although the approach used there was rather different from that used here, an important point in the proof was the introduction of a normal subgroup N such that $N/G_{\mathscr{N}}$ is finite and each hypercentral subgroup U containing $G_{\mathscr{N}}$ is contained in N. Thus the subgroup N takes the place of M in Theorem 4.4.

We describe the construction of N briefly. The radical $G_{\mathscr{N}}$ of the $\mathscr{S}_1$-group G has a finite series

$$1 = R_0 \leq \cdots \leq R_n = G_{\mathscr{N}}$$

of normal subgroups of G such that each factor R_i/R_{i-1} is either torsion-free abelian and rationally irreducible or is a Černikov group. The torsion-free factors are central in $G_{\mathscr{N}}$ and we define

$$N = \bigcap \{ C_G(R_i/R_{i-1}) \mid R_i/R_{i-1} \text{ is torsion-free} \}.$$

To show that each hypercentral subgroup U containing $G_{\mathscr{N}}$ is contained in N the key observation is that $G/G_{\mathscr{N}}$ is abelian-by-finite and so $U/G_{\mathscr{N}}$ is finite. It then follows that U centralizes each torsion-free R_i/R_{i-1} and so is contained in N.

5.2. Fitting classes of $\mathscr{P}$-groups and $\mathscr{S}_1$-groups.

THEOREM 5.1. *Let $\mathfrak{X}$ be a Fitting class of $\mathscr{K}$-groups, where $\mathscr{K} \supseteq \mathscr{P}$, and suppose that $\mathbf{C}(\mathfrak{X}) = \pi$, a nonempty set of primes. Then there is a polycyclic group G which does not have $\mathfrak{X}$-injectors.*

Proof. There is a prime p such that $C_p \in \mathfrak{X}$. Let

$$A = \langle a_1 \rangle \times \cdots \times \langle a_{p-1} \rangle$$

be a free abelian group of rank $p - 1$ and let $X = \langle x \rangle \cong C_p$. Form the split extension G of A by X in which

$$x^{-1} a_i x = a_{i+1} \quad (i = 1, \ldots, p-2),$$
$$x^{-1} a_{p-1} x = a_1^{-1} \cdots a_{p-1}^{-1}.$$

(The group G is, in fact, isomorphic to $C_\infty \wr C_p$ with the centre factored out.) Then, for $p \neq 2$,

$$G' = [A, x] = \langle a_1^{-1} a_2, \ldots, a_{p-2}^{-1} a_{p-1}, a_1^{-1} \ldots a_{p-1}^{-2} \rangle$$
$$= \langle a_1^{-1} a_2, \ldots, a_{p-2}^{-1} a_{p-1}, a_1^{p} \rangle$$

so that $G/G' \cong C_p \times C_p$. (If $p = 2$, then $G = \langle a_1, x \rangle$ is the infinite dihedral group and $G' = \langle a_1^2 \rangle$ so that again $G/G' \cong C_2 \times C_2$.) Thus

$$\langle G', x \rangle \quad \text{and} \quad \langle G', a_1 x \rangle$$

are normal subgroups of G containing elements of order p. But the maximal p-subgroups of G all have order p. Thus an $\mathfrak{X}$-injector of G must be a subgroup of order p contained in $\langle G', x \rangle \cap \langle G', a_1 x \rangle$. But

$$\langle G', x \rangle \cap \langle G', a_1 x \rangle = G'$$

is torsion-free and so G has no $\mathfrak{X}$-injectors.

This means that for any class $\mathscr{K}$ containing the class of polycyclic groups we can reduce our investigation to Fitting classes $\mathfrak{X}$ which contain all locally nilpotent $\mathscr{K}$-groups. For these classes we have the following elementary but important result.

LEMMA 5.2. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class such that $\mathscr{K} \cap \mathscr{N} \subset \mathfrak{X}$. If $G \in \mathscr{K}$, then*:

(i) *$G/G_{\mathfrak{X}}$ has an abelian normal subgroup $D/G_{\mathfrak{X}}$ of finite index n, say.*
(ii) *Each $\mathfrak{X}$-subgroup U of G containing $G_{\mathfrak{X}}$ satisfies $|U/G_{\mathfrak{X}}| \leq n$.*
(iii) *Each $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$ is contained in a maximal $\mathfrak{X}$-subgroup.*

Proof. (i) This follows from the fact that $\mathscr{S}_1$-groups are nilpotent-by-abelian-by-finite and $G_{\mathfrak{X}} \geq G_{\mathscr{N}}$.

(ii) Since $U \cap D \lhd U$ we have $U \cap D \in \mathfrak{X}$. Also $U \cap D \lhd D \lhd G$ and so $U \cap D \leq G_{\mathfrak{X}}$. Hence $|U/G_{\mathfrak{X}}| \leq |G/D| = n$.

(iii) Follows immediately from (ii).

This lemma seems to give considerable control over the maximal $\mathfrak{X}$-subgroups containing $G_{\mathfrak{X}}$ and might give hope that we would always have $\mathfrak{X}$-injectors in this case. However the construction of Theorem 5.1 can be easily extended to provide examples of a Fitting class $\mathfrak{X}$ such that $\mathfrak{X} \supseteq \mathscr{K} \cap \mathscr{N}$ but there are $\mathscr{K}$-groups not having $\mathfrak{X}$-injectors.

THEOREM 5.3. *Let $\mathscr{K} \supseteq \mathscr{P}$ and let $\mathfrak{X}$ be a Fitting class of $\mathscr{K}$-groups such that* $\mathbf{C}(\mathfrak{X}) = \pi$, *a non-empty set of primes. Then there is a polycyclic group H which does not have $\mathscr{N}\mathfrak{X}$-injectors.*

Proof. Let $G = AX$ be the group constructed in Theorem 5.1. The infinite cyclic group can be made to act $\mathbf{Z}$-irreducibly on $\mathbf{Z} \oplus \mathbf{Z}$; for example the generator may act on the basis elements as the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

By letting each $\langle a_i \rangle$ act on a copy of $\mathbf{Z} \oplus \mathbf{Z}$ in this way we obtain an action of A on $M = \mathbf{Z}^{2(p-1)}$. Now form the induced representation of G; in this representation G acts on $M \otimes_{\mathbf{Z}A} \mathbf{Z}G$ as follows. For each $g \in G$, and each $k = 0, \ldots, p-1$, there is an $a \in A$ such that $x^k g = a x^l$, for some $l = 0, \ldots, p-1$, and g acts on

$$M \otimes_{\mathbf{Z}A} \mathbf{Z}G = (M \otimes 1) \oplus (M \otimes x) \oplus \cdots \oplus (M \otimes x^{p-1})$$

according to the rule

$$(m \otimes x^k) g = ma \otimes x^l.$$

Form the split extension H of $M_1 = M \otimes_{\mathbf{Z}A} \mathbf{Z}G$ by G. Then $H_{\mathscr{N}} = M_1$ and the $\mathscr{N}\mathfrak{X}$-subgroups of H containing $H_{\mathscr{N}}$ are just extensions of M_1 by groups of order p. As in Theorem 5.1, we can show that these are not $\mathscr{N}\mathfrak{X}$-injectors of H.

It is clear that an $\mathfrak{X}$-injector of a group G is a maximal $\mathfrak{X}$-subgroup containing $G_{\mathfrak{X}}$. The existence and conjugacy of $\mathscr{N}$-injectors is proved in [8] by showing that the maximal $\mathscr{N}$-subgroups containing $G_{\mathscr{N}}$ form a single conjugacy class and then showing that these are the $\mathscr{N}$-injectors. As it often happens in finite soluble groups that the $\mathfrak{X}$-injectors are precisely the maximal $\mathfrak{X}$-subgroups containing $G_{\mathfrak{X}}$ it seems worthwhile extending this part of the argument used in [8].

THEOREM 5.4. *Let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class containing all locally nilpotent $\mathscr{K}$-groups. Suppose that, in every $\mathscr{K}$-group G, the maximal $\mathfrak{X}$-subgroups containing $G_{\mathfrak{X}}$ form a single conjugacy class. Then these subgroups are the $\mathfrak{X}$-injectors of G.*

Proof. Let V be a maximal $\mathfrak{X}$-subgroup of G containing $G_{\mathfrak{X}}$; then by Lemma 5.2, $|V/G_{\mathfrak{X}}|$ is finite. Let A be an ascendant subgroup of G. We show that $V \cap A$ is a maximal $\mathfrak{X}$-subgroup of A.

Since $V \cap A$ is ascendant in V, we have $V \cap A \in \mathfrak{X}$. Consideration of the subgroups $A \leq AG_{\mathfrak{X}} \leq G$ shows that we may first suppose that $A \geq G_{\mathfrak{X}}$ to show that $V \cap AG_{\mathfrak{X}}$ is a maximal $\mathfrak{X}$-subgroup of $AG_{\mathfrak{X}}$ containing $G_{\mathfrak{X}} = (AG_{\mathfrak{X}})_{\mathfrak{X}}$ and then we need only consider the case in which $AG_{\mathfrak{X}} = G$.

(I) $A \geq G_{\mathfrak{X}}$.

We have an ascending series

$$A = A_0 \lhd \cdots \lhd A_\alpha \lhd \cdots \lhd A_\rho = G.$$

If $V \cap A$ is not a maximal $\mathfrak{X}$-subgroup of A then we take α to be minimal such that $V \cap A_\alpha$ is a maximal $\mathfrak{X}$-subgroup of A_α. If α were a limit ordinal then $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$ and, since $V/G_{\mathfrak{X}}$ is finite there would be some $\gamma < \alpha$ such that $V \cap A_\alpha = V \cap A_\gamma$. Since $V \cap A_\alpha$ is a maximal $\mathfrak{X}$-subgroup of A_α it would follow that $V \cap A_\alpha$ is also a maximal $\mathfrak{X}$-subgroup of A_γ, contrary to the minimality of α. Therefore $\alpha - 1$ exists and $V \cap A_{\alpha-1}$ is not a maximal $\mathfrak{X}$-subgroup of $A_{\alpha-1}$.

Let W be a maximal $\mathfrak{X}$-subgroup of $A_{\alpha-1}$ containing $V \cap A_{\alpha-1}$ and let U be a maximal $\mathfrak{X}$-subgroup of A_α containing W so that $W = U \cap A_{\alpha-1}$. By the hypothesis of the theorem, there is an $x \in A_\alpha$ such that

$$U = (V \cap A_\alpha)^x = V^x \cap A_\alpha.$$

Then

$$W = U \cap A_{\alpha-1} = V^x \cap A_{\alpha-1} = (V \cap A_{\alpha-1})^x.$$

Since x induces an automorphism of $A_{\alpha-1}$ this is contrary to $V \cap A_{\alpha-1}$ not being a maximal $\mathfrak{X}$-subgroup of $A_{\alpha-1}$.

(II) $AG_{\mathfrak{X}} = G$.

If W is an $\mathfrak{X}$-subgroup of A containing $V \cap A$, then by Lemma 4.2, $WG_{\mathfrak{X}}$ is an $\mathfrak{X}$-subgroup and $WG_{\mathfrak{X}} \geq (V \cap A)G_{\mathfrak{X}} = V$. By the maximality of V, $WG_{\mathfrak{X}} = V$. But then

$$WG_{\mathfrak{X}} \cap A \geq W \geq V \cap A = WG_{\mathfrak{X}} \cap A$$

shows that $W = V \cap A$.

5.3. $\mathscr{C}(p)$-injectors.

We saw in Section 3 that $G/G_{\mathscr{C}(p)}$ is finite for any $\mathscr{S}_1$-group G and so, by Theorem 4.3, every $\mathscr{S}_1$-group G has $\mathscr{C}(p)$-injectors and any two such subgroups are conjugate in G. To identify the $\mathscr{C}(p)$-injectors of an $\mathscr{S}_1$-group G we actually describe the maximal $\mathscr{C}(p)$-subgroups of G containing $G_{\mathscr{C}(p)}$. We shall see that these subgroups are conjugate and so are the $\mathscr{C}(p)$-injectors of G. This can be considered as a consequence of Theorem 5.4 or we can use our knowledge that the $\mathscr{C}(p)$-injectors are conjugate to deduce that the two sets of subgroups coincide. Most of the ideas used in the proofs of the following three lemmas may be found in [5] and hence will be omitted.

LEMMA 5.5. *Let G be an $\mathscr{S}_1$-group and let $C = G_{\mathscr{C}(p)} = C_G(\mathrm{Soc}_p(G))$. Then*:
(i) $C = C_G(P)$, *where* $P = \mathrm{Soc}_p(C)$.
(ii) *If* $C \leq H \leq G$, *then* $\mathrm{Soc}_P(H) \leq P$,
(iii) *If* $C \leq H \leq G$ *and* $H \in \mathscr{C}(p)$, *then* $\mathrm{Soc}_p(H) = C_P(H)$.

LEMMA 5.6. *Let $G_{\mathscr{C}(p)} = C \leq H \leq G \in \mathscr{S}_1$, $H \in \mathscr{C}(p)$ and let T/C be a Sylow p-subgroup of the finite group H/C. Then* $\mathrm{Soc}_p(H) = C_P(H) = C_P(T)$.

LEMMA 5.7. *Let $G_{\mathscr{C}(p)} = C \leq H \leq G \in \mathscr{S}_1$ and $H \in \mathscr{C}(p)$. Then* $C_G(\mathrm{Soc}_p(H)) \in \mathscr{C}(p)$.

THEOREM 5.8. *Let $G \in \mathscr{S}_1$ and $C = G_{\mathscr{C}(p)} = C_G(\mathrm{Soc}_p(G))$.*
(i) *The maximal $\mathscr{C}(p)$-subgroups of G containing C are the subgroups $C_G(C_P(S))$, where S/C is a Sylow p-subgroup of G/C.*
(ii) *The maximal $\mathscr{C}(p)$-subgroups of G containing C are the $\mathscr{C}(p)$-injectors of G.*

Proof. (i) By Lemma 5.5(i), $C = C_G(P)$ and so $C_G(C_P(S)) \geq C$. We show that $C_G(C_P(S))$ is a $\mathscr{C}(p)$-group. By Lemma 5.5(ii), $\mathrm{Soc}_p(S) \leq P$ and so C centralizes $\mathrm{Soc}_p(S)$. Now S/C is a p-group and so S induces a p-group of automorphisms on $\mathrm{Soc}_p(S)$ and hence $\mathrm{Soc}_p(S)$ is central in S; that is, $S \in \mathscr{C}(p)$. It follows from Lemma 5.5(iii) that $\mathrm{Soc}_p(S) = C_P(S)$ and now Lemma 5.7 shows that $C_G(C_P(S)) \in \mathscr{C}(p)$.

Conversely, let H be a $\mathscr{C}(p)$-subgroup containing C. Let T/C be a Sylow p-subgroup of H/C and let S/C be a Sylow p-subgroup of G/C containing T/C. Then

$$\begin{aligned} H &\leq C_G\big(\mathrm{Soc}_p(H)\big) && \text{since } H \in \mathscr{C}(p) \\ &= C_G\big(C_P(T)\big) && \text{by Lemma 5.6} \\ &\leq C_G\big(C_P(S)\big) && \text{since } C_P(S) \leq C_P(T). \end{aligned}$$

(ii) By Sylow's Theorem it is clear that the maximal $\mathscr{C}(p)$-subgroups containing $G_{\mathscr{C}(p)}$ are always conjugate and so we may apply Theorem 5.4 to deduce that these subgroups are the $\mathscr{C}(p)$-injectors of G.

5.4. $\mathscr{F}(p)$-injectors.

We saw in Example D of Section 3 that if $G \in \mathscr{P}$ then

$$G_{\mathscr{F}(p)} = W \quad \text{where } W/C_G(O_p(G)) = O_p(G/C_G(O_p(G))).$$

In this case we are again able to identify the maximal $\mathscr{F}(p)$-subgroups containing $G_{\mathscr{F}(p)}$ and so obtain a characterization of the $\mathscr{F}(p)$-injectors by using Theorem 5.4. These facts are contained in the next theorem whose proof we omit.

THEOREM 5.9. *Let G be a polycyclic group.*
(i) *A subgroup T containing $G_{\mathscr{F}(p)}$ is an $\mathscr{F}(p)$-group if and only if $T/G_{\mathscr{F}(p)}$ is a p-group.*
(ii) *G has a unique conjugacy class of $\mathscr{F}(p)$-injectors. They are subgroups V where $V/C_G(O_p(G))$ is a Sylow p-subgroup of $G/C_G(O_p(G))$.*

5.5. $\mathfrak{X}\mathfrak{Y}$-injectors.

THEOREM 5.10. *Let $\mathscr{K}$ be an $\{S, D_0\}$-closed subclass of $\mathscr{S}_1$ containing $\mathscr{F}$ and satisfying either* (a) *$\mathscr{K}$ contains $\mathscr{P}$ or* (b) *$\mathscr{K}$ is Q-closed.*
Let $\mathfrak{X}$ and $\mathfrak{Y}$ be $\mathscr{K}$-Fitting classes containing $\mathscr{N} \cap \mathscr{K}$ and let $G \in \mathscr{K}$. Then the subgroup V of G is an $\mathfrak{X}\mathfrak{Y}$-injector of G if and only if $V \geq G_{\mathfrak{X}}$ and $V/G_{\mathfrak{X}}$ is a $\mathfrak{Y}$-injector of $G/G_{\mathfrak{X}}$.

Proof. Let $K = G_{\mathfrak{X}\mathscr{N}}$ so that $K/G_{\mathfrak{X}} = (G/G_{\mathfrak{X}})_{\mathscr{N}}$ and let $H \geq K$.
(I) $H_{\mathfrak{X}} = G_{\mathfrak{X}}$.

Since $K \lhd G$, $[K, H_{\mathfrak{X}}] \leq K \cap H_{\mathfrak{X}} \in s_n\mathfrak{X} = \mathfrak{X}$. Also $G_{\mathfrak{X}} \leq K \leq H$ and so $G_{\mathfrak{X}} \leq H_{\mathfrak{X}}$. Thus $G_{\mathfrak{X}} \leq K \cap H_{\mathfrak{X}}$. But $K/G_{\mathfrak{X}}$ is a hypercentral group and so $K \cap H_{\mathfrak{X}}$ asc G. Therefore $K \cap H_{\mathfrak{X}} \leq G_{\mathfrak{X}}$ and we have $K \cap H_{\mathfrak{X}} = G_{\mathfrak{X}}$.
Hence $[K, H_{\mathfrak{X}}] \leq G_{\mathfrak{X}}$ and so $H_{\mathfrak{X}} \leq C_G(K/G_{\mathfrak{X}}) \leq K$. But this means that

$$H_{\mathfrak{X}} = K \cap H_{\mathfrak{X}} = G_{\mathfrak{X}}.$$

(II) *$H \in \mathfrak{X}\mathfrak{Y}$ if and only if $H/G_{\mathfrak{X}} \in \mathfrak{Y}$.*

This follows immediately from (I).

(III) *If $V/G_{\mathfrak{X}}$ is a $\mathfrak{Y}$-injector of $G/G_{\mathfrak{X}}$, then V is an $\mathfrak{X}\mathfrak{Y}$-injector of G.*

It follows from (II) that $V \in \mathfrak{X}\mathfrak{Y}$. Let A be an ascendant subgroup of G. Then

$$AG_{\mathfrak{X}}/G_{\mathfrak{X}} \text{ asc } G/G_{\mathfrak{X}}$$

and so $V \cap AG_{\mathfrak{X}}/G_{\mathfrak{X}}$ is a $\mathfrak{Y}$-injector of $AG_{\mathfrak{X}}/G_{\mathfrak{X}}$. Since $A_{\mathfrak{X}} = A \cap G_{\mathfrak{X}}$ we have

$$AG_{\mathfrak{X}}/G_{\mathfrak{X}} \cong A/A_{\mathfrak{X}}$$

and $V \cap A/A_{\mathfrak{X}}$ is a $\mathfrak{Y}$-injector of $A/A_{\mathfrak{X}}$. Since $\mathscr{N} \cap \mathscr{K} \subset \mathfrak{Y}$, $(G/G_{\mathfrak{X}})_{\mathscr{N}} \leq V/G_{\mathfrak{X}}$ and so $K \leq V$ and $A_{\mathfrak{X}\mathscr{N}} = K \cap A \leq V \cap A$. If $V \cap A$ is contained in an $\mathfrak{X}\mathfrak{Y}$-subgroup U of A, then by (II), $U/A_{\mathfrak{X}} \in \mathfrak{Y}$ and, since $V \cap A/A_{\mathfrak{X}}$ is a $\mathfrak{Y}$-injector of $A/A_{\mathfrak{X}}$, we have $V \cap A = U$ is a maximal $\mathfrak{X}\mathfrak{Y}$-subgroup of A.

(IV) *If V is an $\mathfrak{X}\mathfrak{Y}$-injector of G, then $V/G_{\mathfrak{X}}$ is a $\mathfrak{Y}$-injector of $G/G_{\mathfrak{X}}$.*

Certainly $V \geq G_{\mathfrak{X}\mathfrak{Y}} \geq G_{\mathfrak{X}\mathscr{N}}$ and so it follows from (II) that $V/G_{\mathfrak{X}} \in \mathfrak{Y}$. Let $A/G_{\mathfrak{X}}$ be an ascendant subgroup of $G/G_{\mathfrak{X}}$. Then A asc G and so $V \cap A$ is a maximal $\mathfrak{X}\mathfrak{Y}$-subgroup of A and $V \cap A \geq K \cap A = A_{\mathfrak{X}\mathscr{N}}$. If $Y/G_{\mathfrak{X}}$ is a $\mathfrak{Y}$-subgroup of $A/G_{\mathfrak{X}}$ containing $V \cap A/G_{\mathfrak{X}}$, then by (II), Y is an $\mathfrak{X}\mathfrak{Y}$-subgroup of A containing $V \cap A$ and so $Y = V \cap A$. Thus $V \cap A/G_{\mathfrak{X}}$ is a maximal $\mathfrak{Y}$-subgroup of $A/G_{\mathfrak{X}}$.

Let $\mathscr{K}$ be an $\{S, D_0\}$-closed subclass of $\mathscr{S}_1$ containing $\mathscr{F}$ and satisfying $\mathscr{K} \supseteq \mathscr{P}$ or $\mathscr{K}$ is Q-closed. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be $\mathscr{K}$-Fitting classes containing $\mathscr{N} \cap \mathscr{K}$. Theorem 5.10 gives a correspondence between $\mathfrak{X}\mathfrak{Y}$-injectors of G, $G \in \mathscr{K}$, and $\mathfrak{Y}$-injectors of $G/G_{\mathfrak{X}}$. In particular, G will have conjugate $\mathfrak{X}\mathfrak{Y}$-injectors whenever $G/G_{\mathfrak{X}}$ has conjugate $\mathfrak{Y}$-injectors. This correspondence also enables us to give simple characterizations of some injectors. For example, using Theorem 5.10 and Theorem 3 of [8] it is easy to see that the $\mathscr{N}^2$-injectors of an $\mathscr{S}_1$-group G are the maximal $\mathscr{N}^2$-subgroups of G containing $G_{\mathscr{N}^2}$.

Using Theorem 3.6 we can obtain the following variation of Theorem 5.10.

THEOREM 5.11. *Let $\mathfrak{X}$ be an $\mathscr{S}_1$-Fitting class.*

(i) *Let $\mathscr{N} \subset \mathfrak{X}$ and let $\mathfrak{Y}$ be a $\mathscr{P}$-Fitting class such that every $\mathscr{P}$-group has a unique conjugacy class of $\mathfrak{Y}$-injectors. Then every $\mathscr{S}_1$-group has a unique conjugacy class of $\mathfrak{X}\mathfrak{Y}$-injectors.*

(ii) *Let $\mathfrak{Y}$ be an $\mathscr{S}_1$-Fitting class such that every $\mathscr{S}_1$-group has a unique conjugacy class of $\mathfrak{Y}$-injectors. Then every $\mathscr{S}_1$-group has a unique conjugacy class $\mathfrak{X}\mathfrak{Y}$-injectors.*

The interesting point in the first part of Theorem 5.11, of course, is that $\mathfrak{Y}$ need not even be a Fitting class of $\mathscr{S}_1$-groups. For example, we could take $\mathfrak{Y} = \mathscr{F}(p)$ to obtain $\mathscr{N}\mathscr{F}(p)$-injectors of any $\mathscr{S}_1$-group. In the second part of

Theorem 5.11, no assumption is required about the existence of $\mathfrak{X}$-injectors. For example, it shows that every $\mathscr{S}_1$-group has conjugate $\mathscr{E}\mathscr{N}$-injectors although $\mathscr{E}$-injectors need not exist by 5.1.

6. Fitting classes of $\mathscr{P}$-groups and $\mathscr{S}_1$-groups

The condition in Theorem 4.4 requiring the existence of a finite normal subgroup $M/G_{\mathfrak{X}}$ of $G/G_{\mathfrak{X}}$ such that M contains every $\mathfrak{X}$-subgroup of G which contains $G_{\mathfrak{X}}$ seems to be a very strong hypothesis. Our main result in this section is that a condition of this type is in fact a consequence of the existence and conjugacy of $\mathfrak{X}$-injectors.

We consider only classes $\mathscr{K}$ which contain $\mathscr{P}$. We have already made one important reduction in Theorem 5.1 which says that if every $\mathscr{K}$-group has $\mathfrak{X}$-injectors then $\mathfrak{X} \supseteq \mathscr{K} \cap \mathscr{N}$. It follows that $G_{\mathfrak{X}} \geq G_{\mathscr{N}}$ so that, by Theorem 3.1, $G/G_{\mathfrak{X}}$ is a finitely generated abelian-by-finite group and each $\mathfrak{X}$-subgroup containing $G_{\mathfrak{X}}$ is a finite extension of $G_{\mathfrak{X}}$ (Lemma 5.2). We begin by showing that if $\mathfrak{X}$ is a $\mathscr{K}$-Fitting class which leads to existence and conjugacy of $\mathfrak{X}$-injectors then we can characterize the $\mathfrak{X}$-injectors in a similar way to Lemma 4.5. In that result we had $G/G_{\mathfrak{X}}$ a finite group and could take a finite series of $G/G_{\mathfrak{X}}$ with nilpotent factors. In the situation considered here $G/G_{\mathfrak{X}}$ is (free abelian)-by-finite. Thus any $\mathscr{K}$-group G has a series

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = A \rhd A_{\mathfrak{X}} \rhd 1 \qquad (*)$$

such that G_{i-1}/G_i is cyclic of prime order, for each $i = 1, \ldots, n$, and $A/A_{\mathfrak{X}}$ is free abelian of finite rank. The natural way to construct the series $(*)$ is to take a free abelian subgroup $A/G_{\mathfrak{X}}$ of finite index in $G/G_{\mathfrak{X}}$ but for the purposes of our proof we require the slight variation given above.

LEMMA 6.1. *Let $\mathscr{K}$ be an $\{S, D_0\}$-closed subclass of $\mathscr{S}_1$ containing $\mathscr{P}$ and let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class such that every $\mathscr{K}$-group has a unique conjugacy class of $\mathfrak{X}$-injectors. Then a subgroup V of the $\mathscr{K}$-group G is an $\mathfrak{X}$-injector of G if and only if G has a series $(*)$ such that $V \geq A_{\mathfrak{X}}$ and $V \cap G_i$ is a maximal $\mathfrak{X}$-subgroup of G_i, for each $i = 0, \ldots, n$.*

Proof. We use induction on n and the rank of $A/A_{\mathfrak{X}}$.

If V is an $\mathfrak{X}$-injector of G then it is clear that $V \geq G_{\mathfrak{X}} \geq A_{\mathfrak{X}}$ and that $V \cap G_i$ is a maximal $\mathfrak{X}$-subgroup of each subnormal subgroup G_i for any series $(*)$.

Conversely, suppose that G has a series $(*)$ with $V \geq A_{\mathfrak{X}}$ and $V \cap G_i$ a maximal $\mathfrak{X}$-subgroup of G_i for each i.

If $n = 0$, then $G = A$ so that $V = G_{\mathfrak{X}} = A_{\mathfrak{X}}$ is an $\mathscr{X}$-injector. So we may assume that $n \geq 1$. By induction on n, $V \cap G_1$ is an $\mathscr{X}$-injector of G_1. Let X be an $\mathscr{X}$-injector of G; then $X \cap G_1$ is an $\mathscr{X}$-injector of G_1 and so is conjugate to $V \cap G_1$. We may assume that $X \cap G_1 = V \cap G_1$. Now $V \cap A$ is an $\mathscr{X}$-injector of A and $V \cap A \lhd A$. Hence $V \cap A = A_{\mathfrak{X}}$ and so $V/A_{\mathfrak{X}}$ is finite. Similarly $X \cap A = A_{\mathfrak{X}}$ and $X/A_{\mathfrak{X}}$ is finite.

Now $V \cap G_1 \lhd V$ and $V \cap G_1 = X \cap G_1 \lhd X$ so that $V,\ X \leq N_G(V \cap G_1)$. The subgroup $N = N_G(V \cap G_1)$ has a series

$$(**) \qquad N = N \cap G_0 \rhd \cdots \rhd N \cap G_n = N \cap A \rhd A_{\mathfrak{X}} \rhd 1$$

with

$$(N \cap G_{i-1})/(N \cap G_i) \cong (N \cap G_{i-1})G_i/G_i$$

cyclic of prime order or trivial,

$$(N \cap A)/A_{\mathfrak{X}} \leq A/A_{\mathfrak{X}}$$

free abelian and $A_{\mathfrak{X}} = (N \cap A)_{\mathfrak{X}}$. Since $V \cap G_i$ is a maximal $\mathscr{X}$-subgroup of G_i and $V \cap G_i \leq N \cap G_i$ it follows that $V \cap G_i$ is a maximal $\mathscr{X}$-subgroup of $N \cap G_i$, for each $i = 0, \ldots, n$. Similarly $X \cap G_i$ is a maximal $\mathscr{X}$-subgroup of $N \cap G_i$, for each $i = 0, \ldots, n$. Also $V \geq (N \cap A)_{\mathfrak{X}}$ and $X \geq (N \cap A)_{\mathfrak{X}}$.

Suppose that $AN < G$; then $|N/(N \cap G_n)| = |NA/A| < |G/G_n|$ and so there is some collapse in the series $(**)$. By induction on n, X and V are $\mathscr{X}$-injectors of N. Thus V and X are conjugate (in N) and so V is an $\mathscr{X}$-injector of G. We can therefore assume that $AN = G$.

Suppose that $A/(A \cap N)$ is infinite; then

$$r((A \cap N)/(A \cap N)_{\mathfrak{X}}) < r(A/A_{\mathfrak{X}})$$

and so, by induction on r applied to the series $(**)$, V and X are again $\mathscr{X}$-injectors of N and, as above, we can deduce that $V \in \mathrm{Inj}_{\mathfrak{X}}(G)$. Therefore we may assume that $AN = G$ and $A/(A \cap N)$ is finite.

Since $(V \cap G_1) \cap A = A_{\mathfrak{X}}$, we have

$$N_{A/A_{\mathfrak{X}}}(V \cap G_1/A_{\mathfrak{X}}) = C_{A/A_{\mathfrak{X}}}(V \cap G_1/A_{\mathfrak{X}})$$

and so

$$\begin{aligned}(A \cap N)/A_{\mathfrak{X}} = N_A(V \cap G_1)/A_{\mathfrak{X}} &= C_{A/A_{\mathfrak{X}}}(V \cap G_1/A_{\mathfrak{X}}) \\ &\leq Z(A(V \cap G_1)/A_{\mathfrak{X}}).\end{aligned}$$

Since $A/(A \cap N)$ is finite, this means that $A(V \cap G_1)/A_{\mathfrak{X}}$ is central-by-finite and hence the set of elements of finite order in $A(V \cap G_1)/A_{\mathfrak{X}}$ forms a finite normal subgroup $T/A_{\mathfrak{X}}$. Since $A/A_{\mathfrak{X}}$ is torsion-free we must have $T \cap A = A_{\mathfrak{X}}$ and hence $T = V \cap G_1$. In particular, $V \cap G_1 \lhd A(V \cap G_1)$ so that $N \geq A$ and therefore, since $AN = G$, we have $N = G$. This means that $V \cap G_1 = X \cap G_1$ is a normal $\mathfrak{X}$-subgroup of G and $V \cap G_1 = (G_1)_{\mathfrak{X}}$.

Write $H = (G_1)_{\mathfrak{X}}$ so that $V \cap G_1 = X \cap G_1 = H$ is the unique $\mathfrak{X}$-injector of G_1. If $X \leq G_1$ then H is a maximal $\mathfrak{X}$-subgroup of G and $V = X$ is an $\mathfrak{X}$-injector of G. So we may assume that V and X properly contain H. Then

(1) $$G/G_1 \cong V/H \cong X/H \cong C_p \text{ for some prime } p.$$

Now let S be any ascendant subgroup of G; we have to show that $V \cap S$ is a maximal $\mathfrak{X}$-subgroup of S. Since H is the unique $\mathfrak{X}$-injector of G_1,

$$S \cap H = S \cap V \cap G_1$$

is the unique $\mathfrak{X}$-injector of $S \cap G_1$. If $S \leq G_1$, then $V \cap S$ is an $\mathfrak{X}$-injector of S and so we may assume that $SG_1 = G$. By (1), $S/(S \cap G_1) \cong C_p$. Let W be a maximal $\mathfrak{X}$-subgroup of S containing $S \cap V$; then $W \cap G_1 = S \cap H$ and so

$$|W/(S \cap H)| = p \text{ or } 1.$$

If $|(S \cap V)/(S \cap H)| = p$, then $S \cap V = W$ is a maximal $\mathfrak{X}$-subgroup of S. We may therefore assume that

(2) $$S \cap V = S \cap H \text{ and so } SH \cap V = H(S \cap V) = H.$$

By (1), $|X/H| = p$, and hence either $X \leq SH$ or $SH \cap X = H$. If $SH \cap X = H$, then $S \cap X = S \cap H$ is an $\mathfrak{X}$-injector of S. By (2), $S \cap V = S \cap H$ is a maximal $\mathfrak{X}$-subgroup of S. Hence we may assume that

(3) $$X \leq SH.$$

Since $G_1/H = G_1/(G_1)_{\mathfrak{X}}$ is polycyclic by Theorem 3.1 and $G/G_1 = C_p$, G/H is polycyclic and hence SH is subnormal in G. Thus we have a finite series

$$SH = S_m \lhd \cdots \lhd S_1 = G.$$

By (1), $|V/H| = p$ and so there is an integer k such that $V \leq S_{k-1}$ but $S_k \cap V = H$. Since $S_k \geq H$, S_{k-1}/S_k is polycyclic and hence residually finite [7, Corollary 2 to Theorem 9.31]. Therefore there is a normal subgroup M/S_k of finite index in S_{k-1}/S_k maximal with respect to $M \cap VS_k = S_k$ and hence $M \cap V = H$. By (3), $X \leq SH \leq S_k \leq M$ and so we have:

(4) $X \leq M \lhd S_{k-1}$, S_{k-1}/M is finite, $M \cap V = H$ and if $M < N \lhd S_{k-1}$ then $V \leq N$.

Since $M \geq X$, we have $MG_1 = G$ and so

$$M(S_{k-1} \cap G_1) = S_{k-1},\ M \cap G_1 < S_{k-1}$$

and

$$S_{k-1}/(M \cap G_1) = (M/M \cap G_1) \times (S_{k-1} \cap G_1/M \cap G_1).$$

Let K/M be a minimal normal subgroup of S_{k-1}/M. By (4), $V \leq K$. Since $H \leq V$, it follows from (1) that K/M is an elementary abelian p-group. Thus

$$K/(M \cap G_1) = (M/M \cap G_1) \times (K \cap G_1/M \cap G_1)$$

is an elementary abelian p-group and

$$V(M \cap G_1) \lhd K \lhd S_{k-1}.$$

Since X is an $\mathfrak{X}$-injector of S_{k-1}, $X \cap V(M \cap G_1)$ is a maximal $\mathfrak{X}$-subgroup of $V(M \cap G_1)$. But

$$\begin{aligned} X \cap V(M \cap G_1) &= X \cap M \cap V(M \cap G_1) \\ &= X \cap (M \cap V)(M \cap G_1) \\ &= X \cap M \cap G_1 = X \cap G_1 = H \end{aligned}$$

and $H < V \leq V(M \cap G_1)$. This final contradiction completes the proof of the lemma.

COROLLARY 6.2. *Let $\mathscr{K}$ be an $\{S, D_0\}$-closed subclass of $\mathscr{S}_1$ containing $\mathscr{P}$ and let $\mathfrak{X}$ be a $\mathscr{K}$-Fitting class such that every $\mathscr{K}$-group has a unique conjugacy class of $\mathfrak{X}$-injectors. If V is an $\mathfrak{X}$-injector of the $\mathscr{K}$-group G and $V \leq L \leq G$, then $V \in \mathrm{Inj}_{\mathfrak{X}}(L)$.*

Proof. Intersecting the series $(*)$ with L, we obtain the series

$$L = L \cap G_0 \rhd L \cap G_1 \rhd \cdots \rhd L \cap G_n = L \cap A \rhd A_{\mathfrak{X}} \rhd 1.$$

For each $i = 1, \ldots, n$,

$$(L \cap G_i)/(L \cap G_{i-1})$$

is either trivial or cyclic of prime order,

$$(L \cap A)/A_{\mathfrak{X}} \leq A/A_{\mathfrak{X}}$$

is free abelian and since $L \cap A \lhd A$, we have $A_{\mathfrak{X}} = (L \cap A)_{\mathfrak{X}}$. Since $V \cap G_i$ is a maximal $\mathfrak{X}$-subgroup of G_i and $V \leq L$ it is clear that $V \cap (L \cap G_i) = V \cap G_i$ is a maximal $\mathfrak{X}$-subgroup of $L \cap G_i$, for each $i = 0, \dots, n$, and so, by Lemma 6.1, V is an $\mathfrak{X}$-injector of L.

THEOREM 6.3. *Let $\mathscr{X}$ be an $\{S, D_0\}$-closed subclass of $\mathscr{S}_1$ containing $\mathscr{P}$ and let $\mathfrak{X}$ be a $\mathscr{X}$-Fitting class such that every $\mathscr{X}$-group has a unique conjugacy class of $\mathfrak{X}$-injectors. If X is an $\mathfrak{X}$-injector of the $\mathscr{X}$-group G, then*
 (i) *$X^G/G_{\mathfrak{X}}$ is finite,*
 (ii) $\mathrm{Inj}_{\mathfrak{X}}(G) = \mathrm{Inj}_{\mathfrak{X}}(X^G)$.

Proof. Let $Y = X^G$. Since $Y \lhd G$, X is an $\mathfrak{X}$-injector of Y. Further, $\mathrm{Inj}_{\mathfrak{X}}(Y)$ is a unique conjugacy class of subgroups and hence $Y = X^Y$.

Since $G/G_{\mathfrak{X}}$ is abelian-by-finite, $Y/G_{\mathfrak{X}}$ has a free abelian normal subgroup $A/G_{\mathfrak{X}}$ of finite index. We use induction on $|Y/A|$ and $|X/G_{\mathfrak{X}}|$. If either $|Y/A|$ or $|X/G_{\mathfrak{X}}|$ is 1, then $G_{\mathfrak{X}} = X = Y$.

If $AX < Y$, then by Corollary 6.2, X is an $\mathfrak{X}$-injector of AX. Since $|AX/A| < |Y/A|$, the induction hypothesis shows that $|X^{AX}: X|$ is finite. Let $F = X^{AX}$; then $F/G_{\mathfrak{X}}$ is finite and so $F \cap A = G_{\mathfrak{X}}$. Therefore $F = X$ and so $X \lhd AX$. Therefore $C_Y(A/G_{\mathfrak{X}}) \geq X$. But $A/G_{\mathfrak{X}} \lhd Y/G_{\mathfrak{X}}$ and so $C_Y(A/G_{\mathfrak{X}}) \geq X^Y = Y$. So $A/G_{\mathfrak{X}}$ is contained in the centre of $Y/G_{\mathfrak{X}}$. In the central-by-finite group $Y/G_{\mathfrak{X}}$, the elements of finite order form a finite normal subgroup $T/G_{\mathfrak{X}}$. The finite subgroup $X/G_{\mathfrak{X}}$ is contained in $T/G_{\mathfrak{X}}$ and hence $Y = X^Y = T$ so that $Y/G_{\mathfrak{X}}$ is finite.

So we may assume that $AX = Y$. The finite soluble group $X/G_{\mathfrak{X}}$ contains a maximal normal subgroup $M/G_{\mathfrak{X}}$ such that $X/M \cong C_p$, for some prime p. Then $AM \lhd Y$ and so $X \cap AM = M$ is an $\mathfrak{X}$-injector of AM. Since $AM \lhd Y \lhd G$, we have $(AM)_{\mathfrak{X}} = G_{\mathfrak{X}}$. Since

$$|M/(AM)_{\mathfrak{X}}| < |X/G_{\mathfrak{X}}|,$$

$M^{AM}/G_{\mathfrak{X}}$ is finite. Since $A/G_{\mathfrak{X}}$ is torsion-free, $M^{AM} \cap A = G_{\mathfrak{X}}$ and so $M^{AM} = M$. It follows that $M \lhd AM \lhd Y \lhd G$ so that M is a subnormal $\mathfrak{X}$-subgroup. Thus $M = G_{\mathfrak{X}}$ and $AM = A$ so that $Y/A \cong X/G_{\mathfrak{X}} \cong C_p$.

Suppose now that the free abelian normal subgroup $A/G_{\mathfrak{X}}$ of $Y/G_{\mathfrak{X}}$ is non-trivial and let $B/G_{\mathfrak{X}} = (A/G_{\mathfrak{X}})^p \lhd Y/G_{\mathfrak{X}}$. Then A/B is non-trivial and Y/B is a finite p-group of order greater than p. Therefore $XB/B \cong C_p$ is a proper subgroup of Y/B and so there is a normal subgroup K of Y such that $XB \leq K < Y$, contrary to $X^Y = Y$. This contradiction shows that $A/G_{\mathfrak{X}}$ is trivial and so $Y/G_{\mathfrak{X}}$ is finite.

The main deficiency in this result is that we are not able to obtain information about all the maximal $\mathfrak{X}$-subgroups containing $G_{\mathfrak{X}}$ so that it cannot be considered as a complete converse to our Theorem 4.4.

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