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MINIMAL FREE RESOLUTIONS OF LATTICES OVER FINITE GROUPS

BY

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In Memoriam, Irving Reiner

Let G be a finite group and let M be a finitely generated FG-module, where F is a field. In representation theory, the most useful type of projective resolution of M is obtained by the following method: take a projective cover of M, then a projective cover of the kernel, and so on. Such a projective resolution is minimal and two minimal resolutions are isomorphic as augmented complexes. Now replace M by a $\mathbf{Z}G$-lattice A. Here there are no projective covers to work with but nevertheless the exact analogue of the above result is true: two minimal projective resolutions of A are in the same genus (i.e., locally isomorphic) as augmented complexes. By a minimal projective resolution we mean a projective resolution in which no kernel contains a non-zero projective summand. (Over a field this is equivalent to the projective cover condition.) This was proved in [2].

We shall address here the question of what happens for minimal *free* resolutions of A. Things can go wrong: we give an example where two minimal free resolutions have different rank sequences; in this case A does not satisfy the Eichler condition. The main fact to be proved here is that *in the presence of the Eichler condition* all is well: *two minimal free resolutions of A lie in the same genus.*

The difference between a minimal projective resolution and a minimal free resolution may be considered as concentrated in a finite interval. We prove that if A is non-periodic, then every minimal free resolution of A is minimal projective beyond some finite dimension; while if A is periodic then there exists a minimal free resolution that is periodic beyond some finite dimension. Furthermore in the latter case, if A satisfies the Eichler condition and $\mathbf{Z}G$ is not a direct summand of A, we show that A has a periodic minimal free resolution.

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This paper is dedicated to the memory of Irving Reiner. The first author's understanding of integral representation theory owes much to Irving's interest and help during a close friendship of more than twenty years.

1. Notation and terminology

If R is an integral domain and L is an RG-module, we write $d_{RG}(L)$, or $d_G(L)$, for the minimum number of elements needed to RG-generate L, and L^G for the G-invariant elements of L. *All modules are assumed to be finitely generated*. The direct sum of t copies of L is written $L^{(t)}$. In particular, $RG^{(t)}$ is the free module of RG-rank $t = d_G(RG^{(t)})$.

The order of G is $|G|$, and the set of prime divisors of $|G|$ is $\pi(G)$. If p is a prime, $\mathbf{Z}_{(p)}$ denotes the local ring at p and we set

$$\mathbf{Z}_{(G)} = \bigcap_{p \in \pi(G)} \mathbf{Z}_{(p)}.$$

If L is a $\mathbf{Z}G$-lattice, $L_{(G)}$ means $L \otimes_{\mathbf{Z}} \mathbf{Z}_{(G)}$ and $\mathbf{Q}L$ means $L \otimes_{\mathbf{Z}} \mathbf{Q}$.

A *projective excision* of the $\mathbf{Z}G$-lattice L is a decomposition $L = L' \oplus P$, where P is projective and L' has no non-zero projective direct summand. We call L' an *L-core*. Projective excisions are only unique to within genus. If $P_{(G)}$ is free of $\mathbf{Z}_{(G)}G$-rank t, we call t the projective rank of L and write $t = \operatorname{pr} L$.

A projective representation of L is a short exact sequence

$$0 \to K \to P \to L \to 0,$$

where P is projective. It is *minimal* if K is its own core: this is not the general definition of minimality proposed in [2] but is equivalent to it when the coefficient ring is $\mathbf{Z}$ (cf. [2], §2). The presentation is free if P is $\mathbf{Z}G$-free and it is a *minimal free presentation* if $d_G(P) = d_G(L)$.

A $\mathbf{Z}G$-projective resolution of the $\mathbf{Z}G$-lattice A,

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to A \to 0$$

in which the image of P_i in P_{i-1} is C_i, will be abbreviated as (P, C), or just (P). We set $C_0 = A$. If $P_k \to C_k$ is minimal, we say that (P) is *minimal in* *dimension* k (or *at* k) and if (P) is minimal in all non-negative dimensions, then (P) is a minimal projective resolution. A similar terminology is used for free resolutions.

An elementary property that we use repeatedly is that *a short exact sequence of* **Z***G-lattices splits if the left hand term is projective*. We record some further facts for later use.

If (P, C) is a projective resolution of A, then

$$\chi_n(P) = \sum_{i=0}^{n} (-1)^{n-i} \mathrm{rk}(P_i),$$

where $\mathrm{rk}(P_i)$ is the **Z**-rank of P_i. The infimum of the set $\{\chi_n(P)\colon \text{all } (P)\}$ is $\chi_n(A)$, called the n-th *partial projective Euler characteristic* of A. The resolution (P) is minimal if, and only if, $\chi_n(P) = \chi_n(A)$ for all $n \geq 0$ [2, (3.4)].

(1.1) *If* (P, C) *and* (P', C') *are projective resolutions of* A, *then* $\chi_n(P) = \chi_n(P')$ *if, and only if,* C_{n+1} *and* C'_{n+1} *belong to the same genus.*

For if we tensor the resolutions with $\mathbf{Z}_{(G)}$, they become free resolutions of $A_{(G)}$ and then the result is an immediate consequence of Schanuel's lemma and the cancellation property over $\mathbf{Z}_{(G)}$ (cf. [2], the proof of (3.3), for details).

(1.2) *If the projective resolutions* (P, C) *and* (P', C') *of* A *have the same rank sequences* (*equivalently, if* $\chi_n(P) = \chi_n(P')$ *for all* $n \geq 0$), *then the resolutions belong to the same genus* (*as augmented complexes*).

This is really Theorem (3.5) in [2]: the proof given there does not use the hypothesis that the resolutions are minimal but only its consequence that the rank sequences are the same.

A genus Γ is said to *allow cancellation* if $M \oplus L \cong N \oplus L$ implies that $M \cong N$, whenever $L, M, N \in \Gamma$. An equivalent formulation is that for M and N in Γ, $M \oplus A \cong N \oplus A$ implies $M \cong N$ provided $A_{(G)}$ is a direct summand of $M_{(G)}^{(n)}$ for some $n \geq 1$. If L is a lattice in a genus that allows cancellation, we shall usually say that L allows cancellation. A sufficient (but not necessary) condition for L to allow cancellation is that L be an *Eichler lattice*. This means that the semisimple rational algebra $\mathrm{End}_{\mathbf{Q}G}(\mathbf{Q}L)$ has no Wedderburn component isomorphic to a totally definite quaternion algebra. Hence L is certainly an Eichler lattice if, for each non-trivial simple $\mathbf{Q}G$-module W occurring in $\mathbf{Q}L$, $W \oplus W$ is a direct summand of $\mathbf{Q}L$.

It is clear that if $\mathbf{Z}G$ is an Eichler lattice and L is any lattice, then $L \oplus \mathbf{Z}G$ is also an Eichler lattice. We shall need the following more general result; it is proved by a slight extension of arguments in Chapter 9 of [6].

(1.3) PROPOSITION. *If $\mathbf{Z}G$ allows cancellation and L is a $\mathbf{Z}G$-lattice, then $L \oplus \mathbf{Z}G$ also allows cancellation.*

Proof. The lattice A allows cancellation if, and only if, A satisfies the following condition:

($*$) If S is a simple F_pG-module with $p \notin \pi(G)$, then any two epimorphisms α_1, α_2: $A \to S$ have isomorphic kernels.

The "only if" part is easy and only depends on the observation that the pull-back to (α_1, α_2) is expressible as $A \oplus \operatorname{Ker} \alpha_1$ and also as $A \oplus \operatorname{Ker} \alpha_2$; the "if" part is (9.4) in [6].

We shall verify condition ($*$) for $A = L \oplus \mathbf{Z}G$. Choose S and α_1, α_2 as in ($*$) and then choose a $\mathbf{Z}G$-lattice C having S as a homomorphic image and with $\mathbf{Q}C$ a simple $\mathbf{Q}G$-module. This determines $\mathbf{Q}C$ up to isomorphism.

Suppose first that $\mathbf{Q}C$ is not a direct summand of $\mathbf{Q}L$. Then L is contained in $\operatorname{Ker} \alpha_i$ and so

$$\operatorname{Ker} \alpha_i = L \oplus P_i, \quad \text{where } P_i = \operatorname{Ker} \alpha_i \cap \mathbf{Z}G \ (i = 1, 2).$$

Since $\mathbf{Z}G$ allows cancellation, $P_1 \cong P_2$ by ($*$) and hence $\operatorname{Ker} \alpha_1 \cong \operatorname{Ker} \alpha_2$.

It remains to consider the case when $\mathbf{Q}C$ is a summand of $\mathbf{Q}L$. Then $\mathbf{Q}C \oplus \mathbf{Q}C$ is a summand of $\mathbf{Q}A$ and our proof of ($*$) will be complete once we have established the following result.

(1.4) LEMMA. *Let A be a $\mathbf{Z}G$-lattice and α_1, α_2: $A \to S$ be epimorphisms to a simple F_pG-module S with p prime to $|G|$. If C is a $\mathbf{Z}G$-lattice such that $\mathbf{Q}C$ is simple and S is an image of C, assume that $\mathbf{Q}C \oplus \mathbf{Q}C$ is a summand of $\mathbf{Q}A$. Then there exists an automorphism θ of A so that $\alpha_2 = \theta\alpha_1$ (whence* $\operatorname{Ker} \alpha_1 \cong \operatorname{Ker} \alpha_2$).

Proof. Let Λ be a maximal order containing $\mathbf{Z}G$ and set $M = A \otimes_{\mathbf{Z}G} \Lambda$, so that $\mathbf{Q}M \cong \mathbf{Q}A$. Hence the Λ-lattice M decomposes as $M = M_1 \oplus M_2$, where each of $\mathbf{Q}M_1$ and $\mathbf{Q}M_2$ contains a copy of $\mathbf{Q}C$ and so there are epimorphisms γ_i: $M_i \to S$ $(i = 1, 2)$. Extend α_i to an epimorphism μ_i: $M \to S$. We now claim that there exists an automorphism θ of M such that $\mu_2 = \theta\mu_1$

and for all x in M, $x\theta \equiv x \pmod{|G|M}$. To see this, proceed as in the *proof* of (9.5) [6], but replace Swan's map ϕ by $k|G|\phi$, where the integer k is chosen so that $k|G| \equiv 1 \pmod{p}$. This ensures that $\theta \equiv 1 \pmod{|G|}$. Since $|G|M \subseteq A$, it follows that our Λ-automorphism θ of M induces a $\mathbf{Z}G$-automorphism on A and $\alpha_2 = \theta\alpha_1$, as required.

2. Swan modules

We recall first the definition of Swan modules. If A is a $\mathbf{Z}G$-lattice, then

$$(2.1) \qquad d_G(A_{(G)}) \le d_G(A) \le d_G(A_{(G)}) + 1.$$

The first inequality is obvious, while the second is due to Swan [5]. As in [1], the lattice A is called a *Swan* module if the first of these inequalities is an equality, i.e., if $d_G(A_{(G)}) = d_G(A)$.[1] This is not a genus property. Indeed, a projective module is a Swan module if, and only if, it is free.

We introduce a genus version of the Swan condition. A $\mathbf{Z}G$-lattice A satisfies condition (S) if all lattices in the genus of A are Swan modules. Swan discovered [5] a very useful condition for A to satisfy (S). If

$$0 \to K \to \mathbf{Z}_{(G)}G^{(d)} \to A_{(G)} \to 0$$

is a minimal free presentation of $A_{(G)}$ and $\mathbf{Q}K$ is divisible by (has as a direct summand) every non-trivial simple $\mathbf{Q}G$-module, then A satisfies (S). We shall use this in (3.5).

(2.2) LEMMA. *Let* $0 \to K \to E \to A \to 0$ *be a minimal free presentation. Then*:

(i) *A is a Swan module if, and only if,* $\operatorname{pr} K = 0$;
(ii) *A is not a Swan module if, and only if,* $\operatorname{pr} K = 1$;
(iii) *if $\mathbf{Z}G$ allows cancellation, $\mathbf{Z}G$ does not divide K.*

Proof. Let $0 \to C \to P \to A \to 0$ be a minimal projective presentation. So $K \oplus P \cong C \oplus E$, whence $\operatorname{pr} K + \operatorname{pr} P = \operatorname{pr} E$. Now A is a Swan module precisely when $\operatorname{pr} P = \operatorname{pr} E$. Also $\operatorname{pr} E \le \operatorname{pr} P + 1$, by (2.1), so (i) and (ii) are proved.

For (iii), if $K = \mathbf{Z}G \oplus L$, then $E \cong \mathbf{Z}G \oplus Q$ and the surjection of E on A induces one of Q on A. Since $\mathbf{Z}G$ allows cancellation, Q is also free, so $d_G(A) < \operatorname{pr} E$, which is a contradiction.

[1] Note that some authors have used a different definition of Swan module.

We shall deduce our main result, Theorem (2.5), from the following lemma.

(2.3) LEMMA. *Suppose* $\mathbf{Z}G$ *allows cancellation. Then* L *is a Swan module if, and only if,* $L \oplus \mathbf{Z}G$ *is a Swan module.*

Proof. First note that

$$d_G\big(L_{(G)} \oplus \mathbf{Z}_{(G)}G\big) = d_G\big(L_{(G)}\big) + 1. \tag{i}$$

Next we claim that

$$d_G(L \oplus \mathbf{Z}G) = d_G(L) + 1. \tag{ii}$$

Clearly (i) and (ii) will establish the lemma. Since $d_G(L \oplus \mathbf{Z}G) \leq d_G(L) + 1$, we need to show $d_G(L \oplus \mathbf{Z}G) > d_G(L)$. Write $d = d_G(L)$ and suppose there exists a short exact sequence

$$0 \to K \to \mathbf{Z}G^{(d)} \xrightarrow{\theta} L \oplus \mathbf{Z}G \to 0.$$

If $E = L\theta^{-1}$, then we have short exact sequences

$$0 \to K \to E \to L \to 0 \tag{iii}$$

and

$$0 \to E \to \mathbf{Z}G^{(d)} \to \mathbf{Z}G \to 0. \tag{iv}$$

Since $\mathbf{Z}G$ allows cancellation, (iv) shows that $E \cong \mathbf{Z}G^{(d-1)}$ and then it follows from (iii) that $d_G(L) \leq d - 1$, a contradiction. This establishes (ii).

(2.4) *Remark.* (2.3) fails in general. If G is the generalized quaternion group of order 32, Swan [4] showed that there exists a non-free projective $\mathbf{Z}G$-module P such that $P \oplus \mathbf{Z}G \cong \mathbf{Z}G \oplus \mathbf{Z}G$. Moreover, and we need this later, $d_G(P/P^G) = 2$.

(2.5) THEOREM. *Assume that* $\mathbf{Z}G$ *allows cancellation and let* A *be a* $\mathbf{Z}G$-*lattice. Then all minimal free resolutions of* A *belong to one genus class.*

Proof. In view of (1.2) it will suffice to show that two minimal free resolutions have the same rank sequences. So let $(E, K), (F, L)$ be minimal free resolutions of A and write e_i for the $\mathbf{Z}G$-rank of E_i, f_i of F_i. We shall prove $e_i = f_i$ by an induction on i. Clearly e_0 equals f_0 because each is $d_G(A)$. Assume that $e_i = f_i$ for all $i \leq n$. By Schanuel's lemma,

$$K_{n+1} \oplus \mathbf{Z}G^{(f_n + e_{n-1} + \cdots)} \cong L_{n+1} \oplus \mathbf{Z}G^{(e_n + f_{n-1} + \cdots)};$$

and, by hypothesis,

$$f_n + e_{n+1} + \cdots = e_n + f_{n-1} + \cdots .$$

Hence $d_G((K_{n+1})_{(G)}) = d_G((L_{n+1})_{(G)})$. But by (2.3) K_{n+1} is a Swan module if, and only if, L_{n+1} is a Swan module. Consequently

$$d_G(K_{n+1}) = d_G(L_{n+1}), \quad \text{i.e., } e_{n+1} = f_{n+1}.$$

So the induction is complete and hence the proof.

(2.6) To obtain an example of the failure of (2.5) when there is no hypothesis on $\mathbf{Z}G$, it is obviously sufficient to construct two minimal free presentations of a lattice where the kernel of one is a Swan module and that of the other is not. We use the following construction based on Swan's example (2.4) above. Let $\mathfrak{g}^*$ be the $\mathbf{Z}$-dual $\mathrm{Hom}_{\mathbf{Z}}(\mathfrak{g}, \mathbf{Z})$ of the augmentation ideal $\mathfrak{g}$, and write $A = P/P^G$. Since $d_G(A) = 2$ we see that A is not a Swan module. Now $\mathfrak{g}^* \oplus \mathbf{Z}G$ allows cancellation since each simple $\mathbf{Q}G$-module other than $\mathbf{Q}$ occurs at least twice in $\mathbf{Q}\mathfrak{g}^* \oplus \mathbf{Q}G$. Factoring out the G-invariant elements in the relation $P \oplus \mathbf{Z}G \cong \mathbf{Z}G \oplus \mathbf{Z}G$ gives

$$A \oplus \mathfrak{g}^* \cong \mathfrak{g}^* \oplus \mathfrak{g}^*. \tag{i}$$

So if we add $\mathbf{Z}G$ to both sides of (i), we may cancel and obtain

$$A \oplus \mathbf{Z}G \cong \mathfrak{g}^* \oplus \mathbf{Z}G. \tag{ii}$$

Next, take the relation sequence from a minimal free presentation of G [1, p. 7] and dualise it to obtain

$$0 \to \mathfrak{g}^* \to \mathbf{Z}G^{(2)} \to \overline{R}^* \to 0. \tag{iii}$$

Adding $\mathbf{Z}G$ to the middle and left hand terms and using (ii) gives

$$0 \to A \oplus \mathbf{Z}G \to \mathbf{Z}G^{(3)} \to \overline{R}^* \to 0$$

and hence

$$0 \to A \to Q \to \overline{R}^* \to 0,$$

where $Q \oplus \mathbf{Z}G \cong \mathbf{Z}G^{(3)}$. Again we may cancel and so $Q \cong \mathbf{Z}G^{(2)}$, whence

$$0 \to A \to \mathbf{Z}G^{(2)} \to \overline{R}^* \to 0. \tag{iv}$$

The sequences (iii) and (iv) are the required free presentations.

3. Comparisons

The $\mathbf{Z}G$-lattice A is called *periodic* if there exists a positive integer q such that the functors $\operatorname{Ext}^n_{\mathbf{Z}G}(A, \ )$ and $\operatorname{Ext}^{n+q}_{\mathbf{Z}G}(A, \ )$ are naturally equivalent for all $n \geq 1$. The minimum such q is the (projective) period of A. It is well known that the period of $\mathbf{Z}$ (assuming it exists) can only be even. However it should be said that for each positive integer q, there exist G and A so that A has period q.

Suppose A has period q and (P, C) is a minimal projective resolution of A. By dimension shifting, $\operatorname{Ext}^{n+q}_{\mathbf{Z}G}(A, \ )$ is naturally equivalent to $\operatorname{Ext}^n_{\mathbf{Z}G}(C_q, \ )$ and hence $\operatorname{Ext}^1_{\mathbf{Z}G}(A, \ )$ is naturally equivalent to $\operatorname{Ext}^1_{\mathbf{Z}G}(C_q, \ )$. By a result of Hilton and Rees [3], there exist projective modules Q', Q'' so that $C_q \oplus Q' \cong A \oplus Q''$, whence, by [2, (4.1)], C_q (which is its own core) belongs to the genus of A-cores. Let A' be an A-core and let θ: $C_q \to A'$ be an embedding with cokernel finite and prime to the order of G. Then the push-out to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_q & \longrightarrow & P_{q-1} & \longrightarrow & C_{q-1} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\theta} & & & & & & \\ & & A' & & & & & & \end{array}$$

is a projective presentation

$$0 \to A' \to P'_{q-1} \to C_{q-1} \to 0$$

which is minimal because P'_{q-1} is in the genus of P_{q-1}.

Let $A = A' \oplus Q$ be a projective excision of A. Then the minimal projective presentation $P_0 \to A$ breaks up as

$$0 \to C_1 \to P'_0 \to A' \to 0, \quad P_0 \cong P'_0 \oplus Q.$$

Repeating the segment

$$0 \to A' \to P'_{q-1} \to P_{q-2} \to \cdots \to P'_0 \to A' \to 0$$

yields a periodic minimal projective resolution (P') of A'. Thus we have established

(3.1) PROPOSITION. *The lattice A has projective period q if, and only if, there exists a minimal projective resolution of an A-core having period q. Consequently*

$$\chi_{2rq+i}(A') = \chi_i(A') \text{ for all } i, r \geq 0.$$

Now suppose (E, K) is a minimal free resolution of the periodic lattice A. Let (P, C) be a minimal projective resolution contained in (E, K) as direct summand [2, (3.2)]. Thus $K_n \cong C_n \oplus D_n$ is a projective excision. By (2.2), $\operatorname{pr} D_n \leq 1$ and by (3.1), $\operatorname{rk} C_n < N$ for some positive integer N independent of n. Therefore $\operatorname{rk} K_n < N + |G|$ and it follows by the Jordan-Zassenhaus theorem that there exist $m < n$ so that $K_m \cong K_n$. Replacing (E, K) above dimension n by repeats of the segment from m to n gives a minimal free resolution that is periodic above m.

(3.2) PROPOSITION. *If A is periodic, then there exists a minimal free resolution that is periodic above some finite dimension.*

A more precise result is true. If q is the period of A, then there exists a minimal free resolution (E, K) of A and some $i \geq 1$ so that (E) becomes periodic at dimension iq, with period a multiple of q, and K_{iq} belongs to the genus of A' or of $A' \oplus \mathbf{Z}G$, where A' is a core of A. If $\mathbf{Z}G$ allows cancellation, the case $A' \oplus \mathbf{Z}G$ does not arise. To prove this requires a somewhat different circle of ideas.

When $\mathbf{Z}G$ allows cancellation we can improve (3.2):

(3.3) PROPOSITION. *Assume that $\mathbf{Z}G$ allows cancellation. If A is periodic and $\mathbf{Z}G$ is not a direct summand of A, then A has a periodic minimal free resolution.*

The hypothesis that $\mathbf{Z}G$ does not divide A is necessary, in view of (2.2) (iii), if we wish A to have a periodic minimal free resolution.

Proof. By (3.2) any minimal free resolution (E, K) of A contains a segment

$$0 \to K \to E_l \to \cdots \to E_m \to K \to 0$$

(where we have written $K = K_m$). We claim that there exists a minimal free segment

$$0 \to L \to E'_{l-1} \to E_{l-2} \to \cdots \to E_{m-1} \to L \to 0,$$

where $L = K_{m-1}$. A repeated application of this fact proves the proposition.

To establish the claim we apply the dual form of Schanuel's lemma to

$$0 \to K \to E_l \to K_l \to 0, \quad 0 \to K \to E_{m-1} \to L \to 0$$

giving $L \oplus E_l \cong K_l \oplus E_{m-1}$.

If E_l and E_{m-1} have unequal ranks, say $\operatorname{pr} E_l < \operatorname{pr} E_{m-1}$, then $\operatorname{pr} L > \operatorname{pr} K_l$ and so (by (2.2)) $\operatorname{pr} K_l = 0$ and $\operatorname{pr} L = 1$. Therefore $\operatorname{pr} E_{m-1} = 1 + \operatorname{pr} E_l$ and consequently

$$L \oplus \mathbf{Z}G^{(e)} \cong K_l \oplus \mathbf{Z}G \oplus \mathbf{Z}G^{(e)},$$

where $e = \operatorname{pr} E_l$. By (1.3), $K_l \oplus \mathbf{Z}G$ allows cancellation and so $L \cong K_l \oplus \mathbf{Z}G$, which contradicts the fact that $\mathbf{Z}G$ is known not to be a summand of L (by (2.2) (iii) if $m > 1$ and by hypothesis if $m = 1$). Hence $\operatorname{pr} E_l = \operatorname{pr} E_{m-1}$. Now

$$0 \to K_l \oplus E_{m-1} \to E_{l-1} \oplus E_{m-1} \to K_{l-1} \to 0$$

becomes

$$0 \to L \oplus E_l \to E_{l-1} \oplus E_{m-1} \to K_{l-1} \to 0$$

and gives the projective presentation

$$0 \to L \to E' \to K_{l-1} \to 0,$$

where $E' \oplus E_l \cong E_{l-1} \oplus E_{m-1}$. By cancellation E' is free and $\operatorname{pr} E' = \operatorname{pr} E_{l-1}$. So this projective presentation of K_{l-1} is a minimal free presentation and our claim is established.

On the other hand, (3.3) fails if $\mathbf{Z}G$ does not allow cancellation. To show this we need:

(3.4) LEMMA. *Let P be a non-free $\mathbf{Z}G$-module such that*

$$P \oplus \mathbf{Z}G \cong \mathbf{Z}G \oplus \mathbf{Z}G.$$

Then $P \oplus \mathbf{Z}^{(2)}$ does not have a periodic minimal free resolution.

Proof. If $P \oplus \mathbf{Z}^{(2)}$ has a periodic minimal free resolution, then there exists a short exact sequence of $\mathbf{Z}G$-lattices

$$0 \to P \oplus \mathbf{Z}^{(2)} \to \mathbf{Z}G^{(n)} \to X \to 0 \tag{i}$$

where $n = d_G(X)$. Note that $n \geq 3$. Since we have a short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z}G \to \mathfrak{g}^* \to 0,$$

there exists a short exact sequence

$$0 \to P \oplus \mathbf{Z}^{(2)} \to \mathbf{Z}G^{(4)} \to \mathbf{Z}G \oplus \mathfrak{g}^{*(2)} \to 0, \tag{ii}$$

so by the dual of Schanuel's lemma applied to (i) and (ii),

$$\mathfrak{g}^{*(2)} \oplus \mathbf{Z}G^{(n+1)} \cong X \oplus \mathbf{Z}G^{(4)}.$$

We now have two cases to consider.

Case 1. $n > 3$. By cancellation $\mathfrak{g}^{*(2)} \oplus \mathbf{Z}G^{(n-3)} \cong X$, so $d_G(X) = n - 1$, a contradiction.

Case 2. $n = 3$. Factoring out the G-invariant elements yields

$$\mathfrak{g}^{*(6)} \cong X \oplus \mathfrak{g}^{*(4)},$$

so by cancellation $X \cong \mathfrak{g}^{*(2)}$. Thus $d_G(X) = 2$, another contradiction, and the proof is complete.

Now take P to be the projective module in Swan's example (2.4). If $A = P \oplus \mathbf{Z}^{(2)}$, then by (3.4) we will have the required counterexample to (3.3) provided we can show that $\mathbf{Z}G$ is not a direct summand of A. Suppose on the contrary that $A = \mathbf{Z}G \oplus Y$ for some $\mathbf{Z}G$-module Y. Then factoring out the G-invariant elements would yield

$$P/P^G \cong A/A^G \cong \mathfrak{g}^* \oplus Y/Y^G. \tag{i}$$

Also by factoring out the G-invariant elements in $P \oplus \mathbf{Z}G \cong \mathbf{Z}G \oplus \mathbf{Z}G$, we see that

$$P/P^G \oplus \mathfrak{g}^* \cong \mathfrak{g}^* \oplus \mathfrak{g}^*, \tag{ii}$$

and it follows from (i) and (ii) that $Y/Y^G = 0$. Therefore $d_G(P/P^G) = 1$, which is not the case.

We turn now to non-periodic lattices and prove:

(3.5) THEOREM. *If A is a non-periodic $\mathbf{Z}G$-lattice, then every minimal free resolution is minimal projective beyond some dimension.*

We begin the proof with:

(3.6) LEMMA. *The lattice A is non-periodic if, and only if,*

$$\lim_{n\to\infty} \chi_n(A) = +\infty.$$

Proof. When A is periodic, then by (3.1) the function $\chi(A)$ is bounded. We now assume that $\chi(A)$ does not have limit $+\infty$ as $n \to \infty$ and shall show that A is periodic.

Choose a minimal projective resolution (P, C) of A. For each $n \geq 0$,

$$\operatorname{rk} C_{n+1} - \chi_n(A) + (-1)^n \operatorname{rk} A = 0 \tag{i}$$

and therefore the function $\chi(A)$ is bounded below. Since $\chi(A)$ does not have $+\infty$ as limit, there exists $t > 0$ so that $\chi_n(A) \leq t$ for infinitely many n. Hence there is an integer $k \leq t$ so that $\chi_n(A) = k$ for infinitely many n. Among these choose an infinite set J so that if s is the smallest element in J, then $n - s$ is even, for all n in J. It follows, using (i), that $\operatorname{rk} C_{n+1} = \operatorname{rk} C_{s+1}$ for all n in J. By the Jordan-Zassenhaus theorem there exist $m < n$ in J so that $C_m \cong C_n$. We conclude that C_m is periodic and so, by dimension shifting, is A.

Proof of (3.5). We mimic an argument of Swan [5, p. 202] to show that, if (P, C) is a minimal projective resolution of A, then ultimately all C_n have the genus property (S). Let us write $s_n = \chi_n(P)/|G|$. For each $n \geq 0$, we obtain a formal equation (cf. (i) in (3.6) above)

$$\mathbf{Q}C_{n+1} - s_n\mathbf{Q}G + (-1)^n\mathbf{Q}A = 0$$

(which we could regard as an actual equation in the ring of rational characters of G). Since $s_n \to +\infty$ as $n \to \infty$ by (3.6), we can find n_0 so that $n \geq n_0$ implies that C_n has the property (S).

We may now replace the part of our resolution (P) above dimension n_0 by a free resolution that is still minimal projective using the following stepwise construction. If we have done it up to dimension $n - 1$, say

$$\begin{array}{ccccccccccccc} 0 \to K_n \to E_{n-1} \to \cdots \to & E_{n_0} & & \longrightarrow & & P_{n_0-1} & \to \cdots \to A \to 0, \\ & & \searrow & & \nearrow & & \\ & & & C_{n_0} & & & \end{array}$$

then K_n belongs to the genus of C_n. Hence $K_n \in (S)$ and so any minimal free presentation

$$0 \to K_{n+1} \to E_n \to K_n \to 0$$

is also minimal projective. We shall denote by (E, K) the new minimal projective resolution of A constructed in this way (so $E_i = P_i$, $K_i = C_i$ for $i < n_0$).

Suppose (E', K') is a given minimal free resolution of A. We need to show (E') and (E) are ultimately in the same genus. It will suffice to prove K_n' is in the genus of K_n for all $n > n_0 + 1$ (use (1.1) and (1.2)).

By [2, (3.2)], (E', K') contains a minimal projective resolution as direct summand and so each K_i is in the genus of K_i'-cores. Since $\mathbf{Q}G$ divides $\mathbf{Q}K_m$ if $m > n_0$, so K_m is faithful and therefore so is every core of K_m'. By Roiter's replacement theorem [1, 5.9] we may choose a projective excision of the form

$$K_m' = B_m \oplus \mathbf{Z}G^{(r)}.$$

We know $K_m \in (S)$ and B_m is in the genus of K_m, so $B_m \in (S)$, whence also $K_m' \in (S)$. If $e_m = d_G(E_m)$, then $e_m = d_G(K_m)$ and so, if $e_m' = d_G(E_m')$, then comparing the free presentations of K_m', K_m by Schanuel's lemma and using $e_m' = e_m + r$, yields

$$(K_{m+1}')_{(G)} \cong (K_{m+1})_{(G)},$$

as was required. This completes the proof of (3.5).

All the results in this paper remain true if $\mathbf{Z}$ is replaced by any Dedekind domain of characteristic 0 in which no prime number dividing the order of G is invertible and whose field of fractions is a global field.

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