MIXING ACTIONS OF GROUPS[1]

BY

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0. Given a measure-preserving transformation T of a probability space (X, β, m), T is weakly mixing if and only if for all F_1, $F_2 \in L_2(X, \beta, m)$,

$$\lim_{N \to \infty} (1/N) \sum_{n=1}^{N} \left| \int F_1 T^n F_2 \, dm - \int F_1 \, dm \int F_2 \, dm \right| = 0. \qquad (1)$$

This special extreme is important for recurrence theorems and can be characterized in a number of interesting ways, see Furstenberg, Katznelson, and Ornstein [8]. This concept was extended to amenable groups by Dye [5] and is closely related to properties of the unitary representations of G, see Schmidt [16]. By using the invariant mean on the weakly almost periodic functions, weakly mixing unitary representations on a Hilbert space H can be defined for all groups in a manner that directly extends (1). Moreover, by using the standard methods of harmonic analysis in von Neumann [13] and Godement [9], all the characterizations of weakly mixing actions hold here. This gives new and different proofs of these theorems in the cases studied in [5], [8], [16].

In Section 1, the general definition is discussed and many alternative characterizing properties of weakly mixing unitary representations are given. In Section 2, a category result shows that on the unitary level weakly mixing actions are residual for amenable groups. In Section 3, examples of special groups and properties of their actions due to the representations of the group are discussed. In Section 4, the previous abstract theory is summarized for actions induced on $L_2(X)$ by groups of measure-preserving transformations on X.

1. Let G be a σ-compact locally compact Hausdorff group. Such a group will be called a locally compact group. Let $\lambda = \lambda_G$ be a fixed left-invariant Haar measure on G. Let H be a separable infinite-dimensional Hilbert space and let τ be a continuous unitary representation of G on H. Let $CB(G)$ denote the continuous bounded functions on G, let $WAP(G)$ denote the weakly almost periodic functions on G, let $B(G)$ denote the Fourier-Stieltjes

algebra of G, and let $AP(G)$ denote the almost periodic functions on G. Here $B(G)$ is the linear span of coefficient functions of continuous unitary representations of G. Hence, $AP(G) \subset \overline{B(G)} \subset WAP(G)$. See Burckel [1] and Chou [2], [3], [4] for background facts. The first inclusion is von Neumann's theorem, see [13], and the second is easy because coefficient functions are in $WAP(G)$. See Eberlein [6] and Chou [3], [4]. There is a unique G-invariant mean on $WAP(G)$, $B(G)$, or $AP(G)$. See Chou [4] for the references. Denote this invariant mean by M. We often write $\langle M(g), f(g) \rangle$ for the mean value $M(f)$.

The concepts discussed in this section were motivated by the corresponding ones for measure-preserving transformations T, especially as treated in Furstenberg, Katznelson, and Ornstein [8], or for amenable groups, see Dye [5]. Since all the methods are Hilbert space methods, it is natural to take this as the context now and to specialize later in Section 4. To appreciate the terminology in this section, think of the Hilbert space H as $L_2^0(X) = \{ f \in L_2(X): \int f \, dm = 0 \}$.

1.1 DEFINITION. The representation τ is *weakly mixing* if for all F_1, $F_2 \in H$,

$$\left\langle M(g), \left| \langle \tau(g) F_1, F_2 \rangle \right| \right\rangle = 0.$$

The representation τ is *strongly mixing* if for all $F_1, F_2 \in H$, $g \to \langle \tau(g) F_1, F_2 \rangle$ is in $C_0(G)$, the continuous functions on G vanishing at infinity.

This says that τ is strongly mixing if and only if $B_\tau(G) \subset C_0(G)$ where

$$B_\tau(G) = \text{span}\{ \gamma : \gamma(g) = \langle \tau(g) F_1, F_2 \rangle \text{ for some } F_1, F_2 \in H \}.$$

It is easy to see that $B_\tau(G)$ is also the span of γ of the form $\gamma(g) = \langle \tau(g) F, F \rangle$ for some F in H, and this gives the definition of strongly mixing corresponding to the one in Schmidt [16]. But $C_0(G) \subset \overline{B(G)}$ as is observed in Eymard [7]. Also, for non-compact groups G, $M(f) = 0$ if $f \in C_0(G)$. So it is clear that a strongly mixing representation of a non-compact group is weakly mixing.

By the same argument that shows $B(G) \subset WAP(G)$ in [6], it is easy to show that if $F_1, F_2 \in H$ and $p, 1 \le p < \infty$, $\gamma_p(G) = |\langle \tau(g) F_1, F_2 \rangle|^p$ is in $WAP(G)$. Fix $p, 1 \le p < \infty$.

1.2 PROPOSITION. *The representation τ is weakly mixing if and only if*

$$\left\langle M(g), \left| \langle \tau(g) F_1, F_2 \rangle \right|^p \right\rangle = 0 \quad \text{for all } F_1, F_2 \in H.$$

Proof. Since $|\langle \tau(g)F_1, F_2 \rangle| \leq \|F_1\| \, \|F_2\|$, $|\langle \tau(g)F_1, F_2 \rangle|^p \leq C|\langle \tau(g)F_1, F_2 \rangle|$ where $C = \|F_1\|^{p-1}\|F_2\|^{p-1}$, a constant. Hence, if τ is weakly mixing, the means above are always zero. Conversely, assume the above property for some fixed p. The construction of M on $WAP(G)$ shows that $M(\gamma_p)$ is the unique constant in the sup-closed convex hull of $\{ {}_g\gamma_p : g \in G\}$. Hence, for all $F_1, F_2 \in H$, $\varepsilon > 0$, there exists $g_1, \ldots, g_n \in G$, and $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$, such that

$$\sum_{i=1}^n \lambda_i \gamma_p(g_i g) < \varepsilon^p \quad \text{for all } g \in G.$$

The inequality $(\sum_{i=1}^n \lambda_i \varepsilon_i)^p \leq \sum_{i=1}^n \lambda_i \varepsilon_i^p$ holds for all $\varepsilon_i \geq 0$, $i = 1, \ldots, n$. Hence,

$$\sum_{i=1}^n \lambda_i \gamma_1(g_i g) \leq \left(\sum_{i=1}^n \lambda_i \gamma_p(g_i g) \right)^{1/p} \leq \varepsilon.$$

But then 0 is in the closure of the convex hull of $\{ {}_g\gamma_1 : g \in G\}$. So $M(\gamma_1) = 0$ and τ is weakly mixing. $\square$

1.3 THEOREM. *The representation τ is weakly mixing if and only if for all $\varepsilon > 0$, F_1, $F_2 \in H$, and $S \subset G$, S finite, there exists $g \in G$ such that $|\langle \tau(gs)F_1, F_2 \rangle| < \varepsilon$ for all $s \in S$.*

Proof. Assume first that τ is weakly mixing. Then for all F_1, $F_2 \in H$,

$$\langle M(g), |\langle \tau(g)F_1, F_2 \rangle| \rangle = 0.$$

Hence, if $S \subset G$ is a finite set, $\langle M(g), \sum_{s \in S}|\langle \tau(gs)F_1, F_2 \rangle| \rangle = 0$. But then $\inf_{g \in G} \sum_{s \in S}|\langle \tau(gs)F_1, F_2 \rangle| = 0$, and for any $\varepsilon > 0$, there exists $g \in G$ such that $\sum_{s \in S}|\langle \tau(gs)F_1, F_2 \rangle| < \varepsilon$. Thus $|\langle \tau(gs)F_1, F_2 \rangle| < \varepsilon$ for $s \in S$.

Conversely, assume that the property holds. Consider the function

$$\gamma(g) = |\langle \tau(g)F_1, F_2 \rangle|.$$

Fix $\varepsilon > 0$ and choose $g_1, \ldots, g_n \in G$, $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$, such that

$$\left| \sum_{i=1}^n \lambda_i \gamma(g_i^{-1} g) - M(\gamma) \right| < \varepsilon$$

for all $g \in G$. Notice that

$$\gamma(g_i^{-1} g) = |\langle F_1, \tau(g^{-1} g_i)F_2 \rangle| = |\langle \tau(g^{-1} g_i)F_2, F_1 \rangle|.$$

But then there exists $g \in G$ such that $\gamma(g_i^{-1}g) < \varepsilon$ for $i = 1, \ldots, n$. So

$$|M(\gamma)| \le \left| M(\gamma) - \sum_{i=1}^{n} \lambda_i \gamma(g_i^{-1}g) \right| + \left| \sum_{i=1}^{n} \lambda_i \gamma(g_i^{-1}g) \right|$$

$$\le \varepsilon + \sum_{i=1}^{n} \lambda_i \varepsilon$$

$$= 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary $M(\gamma) = 0$. Since $F_1, F_2 \in H$ are arbitrary, τ is weakly mixing. $\square$

Remarks. This proof shows that the weak almost periodicity of $\gamma(g) = \langle \tau(g)F_1, F_2 \rangle$ is a very useful piece of information here. Indeed, suppose that $f \in CB(G)$ and that for all $\varepsilon > 0$, $S \subset G$, S finite, there exists $g \in G$ such that $|f(g^{-1}s)| < \varepsilon$ for $s \in S$. If G is amenable, there would exist a G-invariant mean $\mathcal{M}$ on $CB(G)$ such that $\mathcal{M}(|f|) = 0$, but it would not necessarily be the case that $|f|$ is almost convergent to 0. Also since it is not assumed above that G is amenable, there may not in general be a G-invariant subspace of $CB(G)$ on which there exists an invariant mean $\mathcal{M}$ with $\mathcal{M}(|f|) = 0$.

1.4 DEFINITION. A set $L \subset G$ is *syndetic* if there exists $S \subset G$, S finite with $SL = G$. A set L is *permanently syndetic* if for all $g_1, \ldots, g_n \in G$, $\bigcap_{i=1}^{n} g_i L$ is syndetic.

The argument above shows this.

1.5 COROLLARY. *The representation τ is weakly mixing if and only if for all $\varepsilon > 0$, $F_1, F_2 \in H$, the set $L = \{ g : |\langle \tau(g)F_1, F_2 \rangle| < \varepsilon \}$ is permanently syndetic.*

Proof. If $S \subset G$, S finite, then for $s \in S$,

$$sL = \{ sg : |\langle \tau(g)F_1, F_2 \rangle| < \varepsilon \}$$

$$= \{ x : |\langle \tau(s^{-1}x)F_1, F_2 \rangle| < \varepsilon \}$$

$$= \{ x : |\langle \tau(x^{-1}s)F_2, F_1 \rangle| < \varepsilon \}.$$

Hence, if L is permanently syndetic, then $\bigcap_{s \in S} sL \ne \varnothing$ and there exists $x \in G$ such that $|\langle \tau(x^{-1}s)F_2, F_1 \rangle| < \varepsilon$ for all $s \in S$. By Theorem 1.3, τ is weakly mixing. Conversely, assume τ is weakly mixing. Fix $\varepsilon > 0$, $F_1, F_2 \in H$, and L as above. Let $\gamma(g) = |\langle \tau(g)F_1, F_1 \rangle|$. Since $M(\gamma) = 0$, for all $g_1, \ldots, g_m \in G$, $M(\sum_{i=1}^{m} {}_{g_i}\gamma) = 0$. Hence, for all $\varepsilon > 0$, there exist $h_1, \ldots, h_n \in G$, $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$, such that

$$\sum_{j=1}^{n} \lambda_j \sum_{i=1}^{m} {}_{g_i}\gamma(h_j^{-1}g) < \varepsilon \quad \text{for all } g \in G.$$

But $\varepsilon 1_{G \setminus L} \le \gamma$, so

$$\varepsilon \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{m} 1_{G \setminus g_i L} \left(h_j^{-1} g \right) < \varepsilon.$$

Hence, for all $g \in G$, there exists $j = 1, \ldots, n$ such that $\sum_{i=1}^{n} 1_{G \setminus g_i L}(h_j^{-1} g) < 1$, that is, $h_j^{-1} g \notin G \setminus g_i L$ for all $i = 1, \ldots, n$. But then for all $g \in G$, there exists $j = 1, \ldots, n$ such that $h_j^{-1} g \in g_i L$ for all $i = 1, \ldots, n$. That is,

$$G = \bigcup_{j=1}^{n} h_j \bigcap_{i=1}^{m} g_i L.$$

Hence, L is permanently syndetic. $\square$

In the same spirit as Theorem 1.3, one has:

1.6 COROLLARY. *The representation τ is weakly mixing if and only if for all $\varepsilon > 0$ and $F_1, \ldots, F_n \in H$, there exists $g \in G$ such that $|\langle \tau(g) F_i, F_i \rangle| < \varepsilon$ for $i = 1, \ldots, n$.*

Remark. It is easy to show that in Proposition 1.2, Theorem 1.3, and Corollary 1.5 there is no loss in assuming that $F_1 = F_2$. Also, if G is amenable, the arguments above show that for τ is weakly mixing if and only if the sets $L = \{ g : |\langle \tau(g) F_1, F_2 \rangle| < \varepsilon \}$ are of density one.
      We now consider a better known type of characterization of weak mixing.

1.7 DEFINITION. A vector $F \in H$ is said to be *compact* if $\{ \tau(g) F : g \in G \}$ *is totally bounded in H.*

It is clear that if τ contains a non-trivial finite-dimensional subrepresentation, then there is a non-zero compact vector. The converse is also true. This is observed and proved in Dye [5], but it is an even more classical fact than he indicated. First, consider this lemma. We omit the straightforward proof.

1.8 LEMMA. *The vector $F \in H$ is compact if and only if $\gamma(g) = \langle \tau(g) F, F \rangle$ is in $AP(G)$.*

Moreover, define $H_c = \{ F \in H : F \text{ is compact} \}$. Then H_c is a closed G-invariant subspace of H and the positive definite functions associated with τ restricted to H_c are in $AP(G)$ by Lemma 1.8. In von Neumann [13], it is implicitly shown, see Godement [9], Theorem 16, that on H_c, τ decomposes as a direct sum of finite-dimensional subrepresentations as part of the corresponding theorem about the spectral form of $AP(G)$. Consequently, if $H_c \neq$

$\{0\}$, then τ contains a non-trivial finite-dimensional subrepresentation. See also Moore [12] and Dye [5]. Putting these facts together with our definition of weakly mixing, we give this new proof of the following theorem.

1.9 THEOREM. *The representation τ is weakly mixing if and only if one of these two equivalent conditions holds:*
  (1) *τ admits no non-zero compact vectors.*
  (2) *τ contains no non-trivial finite-dimensional representations.*

Proof. The remarks above show that the conditions (1) and (2) are equivalent. Suppose τ is weakly mixing and suppose $F \in H$ is compact. Let $\varepsilon > 0$ and choose an ε-net $\{\tau(g_1)F, \ldots, \tau(g_n)F\}$ for $\{\tau(g)F : g \in G\}$. By Theorem 1.3, there exists $g_0 \in G$ such that $|\langle \tau(g_0)F, \tau(g_i)F \rangle| < \varepsilon$ for all $i = 1, \ldots, n$. But then for some $i_0 = 1, \ldots, n$,

$$\varepsilon^2 \geq \|\tau(g_0)F - \tau(g_{i_0})F\|_2^2$$
$$= 2\|F\|^2 - 2\,\mathrm{Re}\langle \tau(g_0)F, \tau(g_{i_0})F \rangle$$
$$\geq 2\|F\|^2 - 2\varepsilon.$$

Hence, $\|F\|^2 \leq \varepsilon + (\varepsilon^2/2)$ for all $\varepsilon > 0$, and $F = 0$.

Conversely, assume τ admits no compact functions. Every function of the form $\langle \tau(g)F_1, F_2 \rangle$ is a linear combination of such functions with $F_1 = F_2$. So to show τ is weakly mixing it is enough to show $M(|\gamma|^2) = 0$ if $\gamma(g) = \langle \tau(g)F, F \rangle$ for $F \in H$. By Godement [9, Theorem 16], there exists F_1, $F_2 \in H$ and associated functions $\gamma_i(g) = \langle \tau(g)F_i, F_i \rangle$, $i = 1, 2$, such that $\gamma = \gamma_1 + \gamma_2$, γ_1 is almost periodic, and $M(|\gamma_2|^2) = 0$. By Lemma 1.8, F_1 is compact. By assumption then $F_1 = 0$, $\gamma = \gamma_2$ and $M(|\gamma|^2) = 0$. $\square$

Remarks. It also follows from Godement [9] that $H = H_c \oplus H_w$ where the subspace H_c consists of the compact vectors and for any F_1, $F_2 \in H_w$, $\langle M(g), |\langle \tau(g)F_1, F_2 \rangle| \rangle = 0$. Here τ restricted to H_w gives the largest weakly mixing subrepresentation of τ, and τ is weakly mixing if and only if $H_c = \{0\}$.

There is yet another classical description of weakly mixing transformations that can be obtained here.

1.10 DEFINITION. The representation τ is *ergodic* if the only $F \in H$, such that $\tau(g)F = F$ for all $g \in G$, is $F = 0$.

1.11 PROPOSITION. *If τ is weakly mixing, then τ is ergodic.*

Proof. Given F_1 which is invariant, for all $F_2 \in H$,

$$0 = \langle M(g), |\langle \tau(g)F_1, F_2 \rangle| \rangle = \langle M(g), |\langle F_1, F_2 \rangle| \rangle = |\langle F_1, F_2 \rangle|.$$

So $0 = \langle F_1, F_1 \rangle$ and $F_1 = 0$. $\square$

1.12 *Remark.* The ergodic theorem for τ says that for any $F \in H$, there is a unique τ-invariant function F_I in the norm closure of $\mathrm{co}\{\tau(g)F\colon g \in G\}$. When τ is ergodic, $F_I = 0$ for all $F \in H$. Moreover, it follows easily that if F_1, $F_2 \in H$, then the mean value

$$\left\langle M(g), \left\langle \tau(g)F_1, F_2 \right\rangle \right\rangle = \left\langle (F_1)_I, F_2 \right\rangle = \left\langle F_1, (F_2)_I \right\rangle,$$

in general. See Eberlein [6].

As in Dye [5], one can also characterize weak mixing in terms of the ergodicity of an associated tensor product representation. Let J be a conjugate linear mapping of H such that $J^2 = I$ and $\langle JF_1, JF_2 \rangle = \langle F_2, F_1 \rangle$ for F_1, $F_2 \in H$. Let $\tau^J = J\tau J$, then τ^J is a continuous unitary representation of G on H. Using directly the definition of weakly mixing in terms of the mean M, one can show easily as in Dye [5] that the following Theorems hold.

1.13 THEOREM. *The representation τ is weakly mixing if and only if $\tau \otimes \tau^J$ is ergodic.*

1.14 COROLLARY. *The representation τ is weakly mixing if and only if, for all other representations τ_1, $\tau \otimes \tau_1$ is weakly mixing.*

There is one other general property worth noting. This follows immediately from Theorems 1.9 and 1.13.

1.15 PROPOSITION. *If G_0 is a closed subgroup of G with G/G_0 compact, then any representation of τ is weakly mixing if and only if $\tau|_{G_0}$ is weakly mixing.*

2. Let T denote the set of all continuous unitary representation τ of G on H. The topology used here for T corresponds to the strong operator topology. This topology has as a basis of open neighborhoods of τ_0, the sets

$$O(F_1, \ldots, F_n, K, \varepsilon) = \left\{ \tau \colon \sum_{i=1}^{n} \|\tau(g)F_i - \tau_0(g)F_i\| < \varepsilon \text{ for all } g \in K \right\}$$

where $F_1, \ldots, F_n \in H$ and $K \subset G$, K compact. So a net (τ_α) converges in T to τ_0 if and only if $\tau_\alpha(g)F \to \tau_0(g)F$ uniformly on compact sets in G, for all $F \in H$.

2.1 PROPOSITION. *The space of representations is a complete metric space.*

Proof. Let (F_n) be a dense sequence in $\{F \in H \colon \|F\| \leq 1\}$. Let $K_n \subset G$, K_n compact, $G = \bigcup_{n=1}^{\infty} K_n$. Define a metric d on T by

$$d(\tau_1, \tau_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sup_{g \in K_m} \|\tau_1(g)F_n - \tau_2(g)F_n\|/2^{n+m}.$$

It is easy to show that this metric determines the topology of T and that T is complete in this metric. □

It is convenient to have another characterization of the topology of T. The proof is standard.

2.2 PROPOSITION. *A sequence* $(\tau_n) \subset T$ *converges to* τ *if and only if for* $F \in H$,

$$\lim_{n \to \infty} \langle \tau_n(g)F, F \rangle = \langle \tau(g)F, F \rangle$$

uniformly on compact sets in H.

Let $\mathcal{W} = \{\tau \in T : \tau$ is weakly mixing$\}$ and let $\mathcal{S} = \{\tau \in T : \tau$ is strongly mixing$\}$. Then $\mathcal{S} \subset \mathcal{W}$ if G is non-compact.

2.3 PROPOSITION. *The set* $\mathcal{W}$ *is a* G_δ *subset of* T.

Proof. Let (F_n) be a dense sequence in $\{F \in H : \|F\| \le 1\}$. The proof of Corollary 1.6 shows that $\tau \in \mathcal{W}$ if and only if for all $m, n \ge 1$, there exists $g \in G$ such that $|\langle \tau(g)F_i, F_j \rangle| < 1/n$ for $i, j = 1, \ldots, m$. Let

$$U(i, j, n, g) = \left\{ \tau \in T : \left| \left\langle \tau(g)F_i, F_j \right\rangle \right| < 1/n \right\}.$$

Then $U(i, j, g, n)$ is an open set and $\mathcal{W}$ equals

$$\bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{g \in G} \bigcap_{i=1}^{m} \bigcap_{j=1}^{n} U(i, j, g, n).$$

Since $\bigcap_{i=1}^{m} \bigcap_{j=1}^{n} U(i, j, n, g)$ is open, this shows $\mathcal{W}$ is a countable intersection of the open sets

$$\bigcup_{g \in G} \bigcap_{i=1}^{m} \bigcap_{j=1}^{n} U(i, j, n, g). \qquad \square$$

2.4 DEFINITION. A group G is said to have *property* C_0 if for every continuous positive definite function ϕ on G, $\varepsilon > 0$, and $K \subset G$, K compact, there exists a continuous positive definite function ψ on G which vanishes at ∞ such that $|\phi(g) - \psi(g)| < \varepsilon$ for $g \in K$.

For example, if G is an amenable locally compact group, then this property holds with functions $\psi(g)$ of the form $\sum_{i=1}^{n} \lambda_i f_i * f_i^*$ where $f_i \in L_2(G)$,

$\lambda_i > 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \lambda_i = 1$. See Greenleaf [10]. At another extreme, groups like $SL(2, R)$ also have this property. Indeed, let $P = \{\phi \in CB(G) : \phi$ is positive definite$\}$. For $SL(2, R)$, any $\phi \in P$ can be written as $c + \phi_0$ where c is a constant and $\phi_0 \in C_0(G) \cap P$. But $SL(2, R)$ fails to have property T and so c can be approximated uniformly on compacta by elements of $C_0(G) \cap P$. This shows $SL(2, R)$ has property C_0. Indeed, any group G which fails to have property T, is minimally almost periodic and also minimally weakly almost periodic group (i.e., $B(G) = C \oplus C_0(G)$ and property T fails to hold) will have property C_0. By a similar argument, any simple Lie group failing property T will have property C_0. As observed by R. Zimmer in a personal communication, this shows that the free group on two generators has property C_0. But also, if G has property T, then it does not have property C_0 since $\phi = 1$ cannot be approximated by $C_0(G) \cap P$. Finally, a group may fail to have property T and property C_0. For example, let G have property T and choose H such that $G \oplus H$ does not have property T. This $G \oplus H$ also does not have property C_0.

The main theorem here shows that for many groups, including non-compact amenable groups, the weakly mixing representations are generic. On the unitary level, this generalizes the result of Halmos; see [11].

2.5 THEOREM. *If G is a non-compact group with property C_0, then $\mathscr{W}$ is a dense G_δ subset of T.*

Proof. In view of Proposition 2.3, it suffices to show $\mathscr{W}$ is dense. Since G is non-compact, $\mathscr{S} \subset \mathscr{W}$; so it suffices to show that $\mathscr{S}$ is dense.

Decompose τ on H into cyclic summands $H = \oplus H_i$ with $F_i \in H_i$ cyclic for τ. By Proposition 2.3, it suffices to show that there is a sequence $(\tau_m) \subset \mathscr{S}$ such that for any $n \geq 1$ and any $F \in H_1 \oplus \cdots \oplus H_n$, $\langle \tau_m(g)F, F \rangle \to \langle \tau(g)F, F \rangle$ uniformly on compact sets in G.

Let $\phi_i(g) = \langle \tau(g)F_i, F_i \rangle$. Since G has property C_0, there exists

$$\psi_i^m \in C_0(G) \cap P, \quad i \geq 1,$$

such that $\lim_{m \to \infty} \psi_i^m(g) = \phi_i(g)$ uniformly on compacta in G. It is sufficient to construct a sequence (τ_m) in $\mathscr{S}$ such for all $\varepsilon > 0$, and

$$V_{it} \in \text{span}\{\tau(g)F_i : g \in G\}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, l,$$

and $K \subset G$, K compact, there is some $M \geq 1$ such that if $m \geq M$, then

$$|\langle \tau_m(g)V_{it}, V_{it} \rangle - \langle \tau(g)V_{it}, V_{it} \rangle| < \varepsilon$$

for $g \in K$, $i = 1, \ldots, n$, $t = 1, \ldots, l$. Choose vectors

$$W_{ir} \in \text{span}\{V_{it} : t = 1, \ldots, l\}, \quad r = 1, \ldots, k_i,$$

which form an orthonormal basis of this span. Each $W_{ir} = \sum_{u=1}^{N_r} \alpha_{uir} \tau(g_{uir}) F_i$ for some scalars α_{uir} and $g_{uir} \in G$, $u = 1, \dots, N_r$.

The inner products

$$\langle W_{ir}, W_{is} \rangle = \sum_{u=1}^{N_r} \sum_{v=1}^{N_s} \alpha_{uir} \overline{\alpha_{vis}} \langle \tau(g_{vir}^{-1} g_{uis}) F_i, F_i \rangle$$

$$= \sum_{u=1}^{N_r} \sum_{v=1}^{N_s} \alpha_{uir} \overline{\alpha_{vis}} \phi_i(g_{vir}^{-1} g_{uis}).$$

Thus, there is then an index $M \geq 1$ such that for all $m \geq M$, $i = 1, \dots, n$,

$$\left| \langle W_{ir}, W_{is} \rangle - \sum_{u=1}^{N_r} \sum_{v=1}^{N_s} \alpha_{uir} \overline{\alpha_{vis}} \psi_i^m(g_{vir}^{-1} g_{uis}) \right| < \varepsilon.$$

Choose Hilbert spaces $\mathscr{H}_i^m$, $K_i^m \in \mathscr{H}_i^m$, and continuous unitary representations τ_i^m such that

$$\langle \tau_i^m(g) K_i^m, K_i^m \rangle = \psi_i^m(g) \quad \text{for all } g \in G, i \geq 1.$$

Let $\mathscr{H}^m = \bigoplus_{i=1}^{\infty} \mathscr{H}_i^m$ and $\tilde{K}_i^m = (0, \dots, K_i^m, \dots)$ with K_i^m in the i-th place. The vectors

$$\tilde{W}_{ir}^m = \sum_{u=1}^{N_r} \alpha_{uir} \tau_i^m(g_{uir}) \tilde{K}_i^m$$

then have $|\langle \tilde{W}_{ir}^m, \tilde{W}_{is}^m \rangle - \langle W_{ir}, W_{is} \rangle| < \varepsilon$ for all $m \geq M$, $r, s = 1, \dots, k_i$, $i = 1, \dots, n$. Since $\langle W_{ir}, W_{is} \rangle = \delta_{rs}$ with δ_{rs} the usual delta function, this shows

$$\left| \langle \tilde{W}_{ir}^m, \tilde{W}_{is}^m \rangle - \delta_{rs} \right| < \varepsilon$$

for all $i = 1, \dots, n$ and any r, s.

Now a standard argument in C^{k_i} shows that there exists $\delta = \delta(\varepsilon)$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for some orthonormal basis $\mathscr{W}_{ir}^m$ of $\text{span}\{\tilde{W}_{ir}^m : r\}$,

$$\left\| \mathscr{W}_{ir}^m - \tilde{W}_{ir}^m \right\|_{\mathscr{H}^m} < \delta(\varepsilon)$$

for all r, i. Notice that $\langle W_{ir}, W_{js} \rangle = 0$ for $i \neq j$ and any r, s, and $\langle \mathscr{W}_{ir}^m, \mathscr{W}_{js}^m \rangle = 0$ for $i \neq j$ and any r, s. Therefore, $U_m(W_{ir}) = \mathscr{W}_{ir}^m$, $r = 1, \dots, k_i$, $i = 1, \dots, n$, $m \geq 1$ defines a unitary mapping from

$$\bigoplus_{i=1}^{n} \text{span}\{V_{it} : t = 1, \dots, l\}$$

to

$$\bigoplus_{i=1}^{n} \operatorname{span}\{\mathscr{W}_{ir}^{m} : r\}.$$

Also, $\| U_m(W_{ir}) - \tilde{W}_{ir}^{m}\|_{\mathscr{H}^m} < \delta(\varepsilon)$ for all r, i. Extend U_m to a unitary mapping of H onto $\mathscr{H}^m$.

Consider the unitary representation $\sigma_m(g) = \sum_{i=1}^{\infty} \tau_i^m(g)$ on $\mathscr{H}^m$. The construction has shown that for $i = 1, \ldots, n$,

$$\left| \langle U_m^{-1}\sigma_m(g)U_m(W_{ir}), W_{is}\rangle_H - \langle \tau(g)W_{ir}, W_{is}\rangle_H \right|$$

$$= \left| \langle \sigma_m(g)U_m(W_{ir}), U_m(W_{is})\rangle_{\mathscr{H}^m} - \langle \tau(g)W_{ir}, W_{is}\rangle_H \right|$$

$$\leq \left| \langle \sigma_m(g)\tilde{W}_{ir}^{m}, \tilde{W}_{is}^{m}\rangle_{\mathscr{H}^m} - \langle \tau(g)W_{ir}, W_{is}\rangle_H \right|$$

$$\quad + \left| \langle \sigma_m(g)U_m(W_{ir}) - \sigma_m(g)\tilde{W}_{ir}^{m}, U_m(W_{is})\rangle_{\mathscr{H}^m} \right|$$

$$\quad + \left| \langle \sigma_m(g)\tilde{W}_{ir}^{m}, U_m(W_{is}) - \tilde{W}_{is}^{m}\rangle_{\mathscr{H}^m} \right|$$

$$= \left| \sum_{u,v} \alpha_{uir}\overline{\alpha_{vis}} \langle \sigma^{m}(g_{vis}^{-1}gg_{uir})\tilde{K}_i^m, \tilde{K}_i^m\rangle_{\mathscr{H}^m} \right.$$

$$\quad \left. - \sum_{u,v} \alpha_{uir}\overline{\alpha_{vis}} \langle \tau(g_{vis}^{-1}gg_{uir})F_i, F_i\rangle_H \right| + \delta(\varepsilon) + \delta(\varepsilon)(1 + \varepsilon)$$

$$\leq \sum_{u,v} |\alpha_{uir}\overline{\alpha_{vis}}| |\psi_i^m(g_{vis}^{-1}gg_{uir}) - \phi_i(g_{uis}^{-1}gg_{uir})| + 2\delta(\varepsilon) + \varepsilon\delta(\varepsilon).$$

By increasing M if necessary, this shows that for all $g \in K$ and any r, s, i,

$$\left| \langle U_m^{-1}\sigma_m(g)U_m(W_{ir}), W_{is}\rangle - \langle \tau(g)W_{ir}, W_{is}\rangle \right| < \varepsilon.$$

But then, by decreasing ε if necessary, for any V_{it},

$$\left| \langle U_m^{-1}\sigma_m(g)U_m(V_{it}), V_{it}\rangle - \langle \tau(g)V_{it}, V_{it}\rangle \right| < \varepsilon.$$

Thus, the sequence of unitary representations $\tau_m = U_m^{-1}\sigma_m U_m$ of G on H converges to τ in T.

Finally, each $U_m^{-1}\tau_m U_m$ is strongly mixing. Indeed,

$$\langle U_m^{-1}\tau_m(g)U_m F, F\rangle_H = \langle \tau_m(g)F_0, F_0\rangle_{\mathscr{H}^m}$$

for some $F_0 \in \mathscr{H}^m$, and so it suffices to treat only τ_m on $\mathscr{H}^m$. Without loss of generality consider some fixed $\mathscr{H}_i^m$ and $\tau_m|_{\mathscr{H}_i^m} = \tau_i^m$. By approximation, it is enough to consider only $F \in \mathscr{H}_i^m$ of the form

$$F = \sum_{j=1}^{L} \alpha_j \tau_i^m(g_j)K_i^m$$

for some scalars α_i and some $g_j \in G$, and show that $\psi(g) = \langle \tau_i^m(g)F, F \rangle$ is in $C_0(G)$. But

$$\psi(g) = \sum_{j,\,k=1}^{L} \alpha_j \bar{\alpha}_k \left\langle \tau_i^m\left(g_k^{-1}gg_j\right)K_i^m, K_i^m \right\rangle$$

$$= \sum_{j,\,k=1}^{L} \alpha_j \bar{\alpha}_k \psi_i^m\left(g_k^{-1}gg_j\right).$$

Since $\psi_i^m \in C_0(G)$, this shows that $\psi \in C_0(G)$. $\square$

The question of which groups have property C_0 seems to be an interesting one. A modification of this question may also be interesting. Let P_0 consist of all $\phi \in P$ such that the associated representation τ_ϕ does not contain the trivial subrepresentation, equivalently, τ_ϕ is ergodic. Then $P \cap C_0(G) \subset P_0$. When is $P \cap C_0(G)$ dense in P_0 in the topology of uniform convergence on compacta?

3. Certain representation properties of groups are equivalent to the existence of weakly mixing or strongly mixing actions. See Schmidt [16] and Dye [5] for some examples.

3.1 PROPOSITION. *Any non-compact group admits a strongly mixing representation.*

Proof. The regular representation of G is strongly mixing if G is non-compact. $\square$

Remark. Since G is non-compact, there exists $0 \neq \phi \in P \cap C_0(G)$. For any such ϕ, $M(|\phi|) = 0$. Let τ_ϕ be the associated unitary representation. Then τ_ϕ is strongly mixing. This can give rise to strongly mixing representations that are not weakly contained in the regular representation.

A group G is *minimally almost periodic* if $AP(G) = C$, the constants. By von Neumann [13], G is minimally almost periodic if and only if G admits no non-trivial finite-dimensional representations. Dye [5] constructed a locally finite (and hence amenable) minimally almost periodic group. See von Neumann and Wigner [14] for other examples. The example in Ol'shanskii [15] is a non-amenable periodic group which is also minimally almost periodic. By Theorem 1.9, the following holds.

3.2 THEOREM. *If G is minimally almost periodic, then any ergodic representation τ is weakly mixing.*

Depending on the nature of $\overline{B(G)}$, there may or may not be weakly mixing representations which are not strongly mixing.

3.3 DEFINITION. A group G is *minimally weakly mixing* if $\overline{B(G)} = AP(G) \oplus C_0(G)$.

By Chou [3], [4], a group G is minimally weakly mixing if it is minimally weakly almost periodic. It is not clear for which groups $B(G) = AP(G) \oplus C_0(G)$, but $WAP(G) \neq \overline{B(G)}$.

3.4 THEOREM. *If G is minimally weakly mixing, then any weakly mixing representation is strongly mixing.*

Proof. Let τ be a weakly mixing representation. Let $\phi(g) = \langle \tau(g)F, F \rangle$ for $F \in H$. It suffices to show $\phi \in C_0(G)$. By Godement [9], $\phi = \phi_c + \phi_w$, ϕ_c, $\phi_w \in P$, such that $\phi_c \in AP(G)$ and $M(|\phi_w|) = 0$. Also, this is the decomposition of ϕ as an element of $AP(G) \oplus C_0(G)$. Hence, $\phi_w \in C_0(G)$. But τ is weakly mixing and so $\phi_c = 0$. Hence $\phi \in C_0(G)$. Since F was arbitrary, τ is strongly mixing.

Remark. Chou [3] has given some interesting examples of solvable locally compact groups which are minimally weakly almost periodic. This shows that some solvable groups have all weakly mixing representations being strongly mixing. Theorem 2.5 shows that for such groups, the strongly mixing representations are generic.

As Schmidt [16] points out, all ergodic representations of G are strongly mixing if a representation τ of G has C_0 coefficient functions as soon as τ does not contain the trivial subrepresentation. These are exactly the groups such that $\overline{B(G)} = C \oplus C_0(G)$, i.e., G is minimally almost periodic and minimally weakly mixing at the same time.

An interesting question arises for specific groups when considering the next corollary.

1.6 COROLLARY. *If any finite-dimensional representation of G admits a common eigenvector, then τ is weakly mixing if and only if for all $F \in H$, $\varepsilon > 0$, there exists $g \in G$ for which $|\langle \tau(g)F, F \rangle| < \varepsilon$.*

This characterizes weakly mixing for abelian groups or connected solvable groups. Does it characterize weakly mixing always?

4. All of the theorems in Section 1 carry over with small changes of notation to a representation τ of G as a group of measure-preserving transformations of the probabililty space (X, β, m). Let τ_X be a homomorphism of G

into the group $M(X)$ of invertible measure-preserving transformations of X. It is assumed that for all $A, B \in \beta$, $g \to m(\tau_X(g)A \cap B)$ is continuous on G. To apply Section 1, let

$$\tau(g)(F) = \tau_X(g)F = F \circ \tau_X(g^{-1})$$

where

$$F \in L_2^0(X) = \left\{ F \in L_2(X) : \int F \, dm = 0 \right\}.$$

The following are the results of Section 1 in this context.

4.1 THEOREM. *The following are equivalent for a representation τ_X:*
(a) *τ_X is weakly mixing.*
(b) *For all $F_1, F_2 \in L_2(X)$,*

$$\left\langle M(g), \left| \int (\tau_X(g)F_1)F_2 \, dm - \int F_1 \, dm \int F_2 \, dm \right| \right\rangle = 0.$$

(c) *For all $F_1, \ldots, F_n \in L_2^0(X)$, $\varepsilon > 0$, there exists $g \in G$ with*

$$\left| \int (\tau_X(g)F_1)F_i \, dm \right| < \varepsilon \quad for \ i = 1, \ldots, n.$$

(d) *For all $g_1, \ldots, g_n \in G$ and $F \in L_2^0(X)$, $\varepsilon > 0$, there exists $g \in G$ with*

$$\left| \int \tau_X(g)F\tau_X(g_i)F \, dm \right| < \varepsilon \quad for \ i = 1, \ldots, n.$$

(e) *For all $F \in L_2(X)$, F not constant, $\{\tau_X(g)F : g \in G\}$ is not relatively compact in $L_2(X)$.*
(f) *τ_X contains no finite-dimensional subrepresentations other than $\tau_X | C$.*
(g) *$\tau_X \times \tau_X$ is ergodic.*
(h) *$\tau_X \times \tau_X$ is weakly mixing.*

Remark. By the mean ergodic theorem as in Remark 1.12, if $F_1 \geq 0$, then

$$\left\langle M(g), \int (\tau_x(g)F_1)F_1 \right\rangle$$

$$= \langle (F_1)_I, (F_1) \rangle = \int (F_1)_I^2 \, dm \geq \left(\int (F_1)_I \, dm \right)^2 = \left(\int F_1 \, dm \right)^2.$$

Hence (b) shows that for weakly mixing representations, if $F_1 \geq 0$, $F_1 \neq 0$, $\varepsilon > 0$, then

$$\int (\tau_X(g) F_1) F_1 \, dm \geq (1 - \varepsilon) \left(\int F_1 \, dm \right)^2$$

for many values of $g \in G$.

For the general representation τ_X, the space $L_2(X)_c$ contains the constants at least. It is not difficult to show that $L_2(X)_c$ is actually the closed span of all 1_A, $A \in \beta$, such that 1_A is compact with respect to τ_X. Indeed, $\beta_c = \{ A \in \beta : 1_A$ is compact with respect to $\tau_X \}$ is a σ-algebra and $L_2(X)_c = L_2(X, \beta_c)$. However, generally $L_2(X)_w$ is not $L_2(X, \beta_0)$ for some σ-algebra $\beta_0 \subset \beta$. This gives the following corollary.

4.2 COROLLARY.   *The representation τ_X is weakly mixing if and only if for all $g_1, \ldots, g_n \in G$, $\varepsilon > 0$, there exists $g \in G$ such that*

$$\left| m(\tau_F X(g) A \cap \tau_X(g_i) A) - m(A)^2 \right| < \varepsilon$$

for $i = 1, \ldots, n$.

The results of Section 3 also carry over to this context. For example, if G is countably infinite, let $X = \Pi_g \{ \{0, 1\} : g \in G \}$ and define

$$\tau_X(g_0)(x_g) = x_{g_0^{-1} g}$$

for all $g \in G$ and functions $g \to x_g \in \{0, 1\}$. It is easy to see that τ_X is strongly mixing in $L_2(X)$ if and only if G is infinite. An appropriately modified version of this for non-discrete groups, or the Gaussian measure space construction, see Schmidt [16], show that:

4.3. PROPOSITION.   *Any non-compact group admits a representation as a strongly mixing group of measure-preserving transformations.*

Similar use of the Gaussian measure space construction applied to Section 3 gives this and parallel results.

4.4 THEOREM.   *The group G is minimally weakly mixing if and only if every representation of G as a weakly mixing group of measure-preserving transformations is strongly mixing.*

Finally, the category theorem of Section 2 does not carry over to T_X, the representations of G as groups of measure-preserving transformations. The argument was definitely on the unitary level. However, the parallel result would be worthwhile here.

Question. For (amenable) groups G, are the weakly mixing representations of G as measure-preserving transformations on (X, β, m) of second category in T_X?

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