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HANKEL OPERATORS IN VON-NEUMANN-SCHATTEN CLASSES

BY

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Introduction

In [3], Bonsall reduced the study of a Hankel operator on the Hardy space H^2 of the disc D to the study of its action on a class of simple elements in H^2 which generate the space. To this end he introduced the unit vectors

$$v_z(w) = \frac{\sqrt{1 - |z|^2}}{1 - \bar{z}w} \quad (w \in D)$$

indexed by the points $z \in D$ and proceeded to show that A is a bounded Hankel operator if and only if $\{\|Av_z\|: z \in D\}$ is bounded. His methods also show that A is compact if and only if $\|Av_z\| \to 0$ uniformly as $|z| \to 1$.

The purpose of this paper is to try to find conditions which relate the quantities $\|Av_z\|$ with the property that A belongs to the von Neumann-Schatten class $\mathscr{C}_p (1 \leq p < \infty)$. We get a complete characterization only when $p = 2$. For other values of p we obtain implications in one direction only but are able to show that the converse implications do not hold.

Bonsall also considered unit vectors $u_n(\zeta)$, $n \geq 0$, $\zeta \in \partial D$, the counterparts of the v_z on the unit circle. We obtain completely analogous conditions in terms of $\|Au_n(\zeta)\|$ as stated above for $\|Av_z\|$ including a necessary condition that $A \in \mathscr{C}_1$. Again the condition is shown to be not sufficient.

Preliminaries

We record some notation we will use and recall some pertinent results. Let D denote the unit disc, ∂D the unit circle, $L^p = L^p(\partial D)$ the usual Lebesgue space, $0 < p \leq \infty$, and H^p, Hardy space, the subspace of L^p of functions analytic in D.

Let $\hat{f}(n)$ be the nth Fourier coefficient of the function f in L^1. We will follow the usual practice of identifying a function f in H^p with its analytic extension to D, $\sum_{n=0}^{\infty} \hat{f}(n) z^n$.

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Let $(a_n)_0^\infty$ be an l^2 sequence, $a_n \in \mathbf{C}$. The matrix $A = (a_{i+j})$ is called a Hankel matrix and the sequence (a_n) the coefficient sequence of A. This matrix defines a bounded operator on l^2 or equivalently on H^2, if and only if

$$f = \sum_0^\infty a_n \chi_{-n-1} \quad \left(\chi_n(e^{it}) = e^{int}, \quad n \in \mathbf{Z}\right),$$

is in BMO and is compact if and only if f is in VMO [8]. With the Hankel operator A we associate the function f above. Defining P to be the orthogonal projection of L^2 onto H^2 and J the unitary, $J\chi_n = \chi_{-n}$ we may represent the action of A as follows, when A is bounded:

$$Ag = PJ\chi_1 fg \quad \left(g \in H^2\right).$$

Let T be a compact operator on H^2 and let $(s_k)_0^\infty$ be the sequence of eigenvalues of $(T^*T)^{1/2}$ arranged in decreasing order of magnitude. The number $s_k = s_k(T)$ is called the k-th s-number of the operator T. We may represent T in the form of a Schmidt series

$$Tg = \sum_0^\infty s_k \langle g, \phi_k \rangle \psi_k \quad \left(g \in H^2\right)$$

where $(\phi_k)_0^\infty$, $(\psi_k)_0^\infty$ are orthonormal systems in H^2. T is said to belong to the von Neumann-Schatten class $\mathscr{C}_p$ $(0 < p < \infty)$ if the sequence $(s_k)_0^\infty$ belongs to l^p. For $1 \le p < \infty$, $\mathscr{C}_p$ is a Banach space with the norm

$$\|T\|_{\mathscr{C}_p} = \|T\|_p = \left(\sum_0^\infty s_k(T)^p\right)^{1/p},$$

and $\|T\|_{\mathscr{C}_\infty} = s_0(T)$ is the operator norm. For further information on these classes see [5], [9].

Besov spaces. The Besov space $B_p^{1/p}(1 \le p < \infty)$ consists of those L^p functions f for which

$$\int_{-\pi}^{\pi} \frac{1}{t^2} \int_{-\pi}^{\pi} \left|f(e^{i(s+t)}) + f(e^{i(s-t)}) - 2f(e^{is})\right|^p ds\, dt < \infty.$$

For $p > 1$ this is equivalent to

$$\int_{-\pi}^{\pi} \frac{1}{t^2} \int_{-\pi}^{\pi} \left|f(e^{i(s+t)}) - f(e^{is})\right|^p ds\, dt < \infty;$$

se [7] and [11].

Let $A_p^{1/p}$ denote the subclass of those f in $B_p^{1/p}$ that are analytic in D. Again we identify a function f in $A_p^{1/p}$ with its analytic extension to D. We may also characterize $A_p^{1/p}$ as follows: $f \in A_p^{1/p}$ if and only if

$$\int\int_D |f''(z)|^p (1 - |z|)^{2p-2}\, dx\, dy < \infty,$$

or if $p > 1$, $f \in A_p^{1/p}$ if and only if

$$\int\int_D |f'(z)|^p (1 - |z|)^{p-2}\, dx\, dy < \infty.$$

The study of the Besov spaces is made easier by the following result proved in [7].

THEOREM A. $PB_p^{1/p} = A_p^{1/p}$ $(1 \leq p < \infty)$.

1. Hankel operators in $\mathscr{C}_p(1 < p < \infty)$

Suppose $1 < p < \infty$ and f is in L^p. We define a function $F(z, p)$ on D by

$$F(z, p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it}) - f(z)|^p P(ze^{-it})\, dt, \tag{1}$$

where

$$P(w) = \frac{1 - |w|^2}{|1 - w|^2} \quad (w \in D)$$

is the Poisson kernel and $f(z)$ is the harmonic extension of f to D. Taking M to be the family of disc automorphisms $\mu_\zeta (\zeta \in D)$ with

$$\mu_\zeta(z) = \frac{z + \zeta}{1 + \bar{\zeta} z} \quad (z \in D)$$

we note that by a change of variable

$$F(z, p) = \|f_z\|_p^p,$$

where $f_z(w) = f \circ \mu_z(w) - f(z)$ $(w \in D)$.

We recall that the space $BMOA$ of analytic functions of bounded mean oscillation may be described for any value of p, $0 < p < \infty$, as the family of functions f in H^p for which

$$\sup\{\|f_z\|_p : z \in D\} < \infty;$$

see [2]. We may define a family of equivalent norms on *BMOA* by choosing p, $0 < p < \infty$, and setting

$$\|f\|_* = |f(0)| + \sup\{\|f_z\|_p : z \in D\}.$$

In his discussion of the boundedness of A [3], Bonsall introduced two families of unit vectors $\{v_z\}$, $\{u_n(\zeta)\}$.

For each $z \in D$,

$$v_z(e^{it}) = \frac{\sqrt{1-|z|^2}}{1-\bar{z}e^{it}},$$

while for each $n \geq 0$ and $\zeta \in \partial D$,

$$u_n(\zeta) = \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} \chi_k \bar{\zeta}^k.$$

If $A \in \mathscr{C}_p$ what can be said about $\|Av_z\|$ beyond the fact that $\|Av_z\| \to 0$ uniformly as $|z| \to 1$? First we recall the fundamental result of Peller [7].

THEOREM B. *The Hankel operator $A = A(f)$ belongs to $\mathscr{C}_p(1 \leq p < \infty)$ if and only if $\bar{f} \in A_p^{1/p}$.*

Suppose $A = A(f)$ is a Hankel operator with f in L^2. From [3] we then have

$$\begin{aligned}\|Av_z\|^2 &= (1-|z|^2) \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} a_{k+j} \bar{z}^j \right|^2 \\ &= \frac{1}{2\pi} \int |f(e^{it})|^2 P(ze^{-it})\, dt - |f(z)|^2 \qquad (2) \\ &= F(z,2). \qquad (3)\end{aligned}$$

The case $p = 2$ is straightforward.

THEOREM 1. *The following are equivalent.*

(a) $A \in \mathscr{C}_2$;

(b) $\int_0^{2\pi} \|Au_n(\zeta)\|^2\, d\theta = O(1/n)$ $(\zeta = e^{i\theta})$;

(c) $\int_0^{2\pi} \|Av_z\|^2\, d\theta = O(1-r)$ $(z = re^{i\theta})$;

Proof (a) $\Rightarrow$ (b). It is elementary that A belongs to $\mathscr{C}_2$ if and only if

$$\sum_0^{\infty} (k+1)|a_k|^2 < \infty.$$

We note that

$$\|Au_n(\zeta)\|^2 = \frac{1}{n+1}\sum_{k=0}^{\infty}\left|\sum_{j=0}^{n} a_{k+j}\bar{\zeta}^j\right|^2.$$

Taking $\zeta = e^{i\theta}$ and integrating over $(0, 2\pi)$ we get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\|Au_n(\zeta)\|^2\,d\theta &= \frac{1}{n+1}\sum_{k=0}^{\infty}\sum_{j=0}^{n}|a_{k+j}|^2 \\ &= \frac{1}{n+1}\left\{\sum_{m=0}^{n}(m+1)|a_m|^2 + (n+1)\sum_{m=n+1}^{\infty}|a_m|^2\right\} \\ &= o\left(\frac{1}{n}\right).\end{aligned}$$

(b) ⇒ (c). To see this we write v_z as follows, with $z = r\zeta$,

$$\begin{aligned}v_z(w) &= \frac{\sqrt{1-|z|^2}}{1-\bar{z}w} \\ &= \sqrt{1-r^2}\sum_{n=0}^{\infty} r^n\bar{\zeta}^n w^n \\ &= \sqrt{1-r^2}(1-r)\sum_{n=0}^{\infty} r^n\sum_{0}^{n}\bar{\zeta}^k w^k \\ &= \sqrt{1-r^2}(1-r)\sum_{n=0}^{\infty}\sqrt{n+1}\,r^n u_n(\zeta)(w) \qquad (4)\end{aligned}$$

which expresses v_z as a linear sum of the $u_n(\zeta)$. It follows that

$$\begin{aligned}\|Av_z\|^2 &\le (1-r^2)(1-r)^2\left\{\sum_{n=0}^{\infty}\sqrt{n+1}\,r^n\|Au_n(\zeta)\|\right\}^2 \\ &\le (1-r^2)(1-r)^2\sum_{n=0}^{\infty}(n+1)r^n\|Au_n(\zeta)\|^2\sum_{n=0}^{\infty}r^n \\ &= (1-r^2)(1-r)\sum_{0}^{\infty}(n+1)r^n\|Au_n(\zeta)\|^2,\end{aligned}$$

where we have used the Cauchy-Schwarz inequality. Hence

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\|Av_z\|^2\,d\theta &\le (1-r^2)(1-r)\sum_{0}^{\infty}(n+1)r^n\left(\frac{1}{2\pi}\int_0^{2\pi}\|Au_n(\zeta)\|^2\,d\theta\right) \\ &= o(1-r^2),\end{aligned}$$

which is (c).

(c) $\Rightarrow$ (a). From (2) we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\|Av_z\|^2\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}|f(e^{it})|^2\,dt - \frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2\,d\theta\\ &= \sum_0^{\infty}|a_k|^2(1-r^{2k+2})\\ &= (1-r^2)\sum_1^{\infty}|a_k|^2(1+r^2+\cdots+r^{2k}),\end{aligned}$$

assuming as usual that $f \sim \Sigma a_k\chi_{-k-1}$. By assumption there exists $K > 0$ such that for $0 < r < 1$

$$\sum_1^{\infty}|a_k|^2(1+r^2+\cdots+r^{2k}) \le K.$$

Fixing N and letting $r \to 1$ we have $\Sigma_1^N(k+1)|a_k|^2 \le K$ which implies that A is in $\mathscr{C}_2$. This completes the proof of Theorem 1.

Let

$$K_n(x) = \frac{\sin^2\left(\frac{n+1}{2}\right)x}{(n+1)\sin^2\frac{x}{2}}$$

be the n-th Fejer kernel and let $\sigma_n(g, x) = g * K_n(x)$, the convolution of g with K_n, be the n-th Cesaro mean of g. We define functions $F_n(\phi, p)$ analogous to the $F(z, p)$ as follows: For $1 < p < \infty$ and $f \in L^p$ let

$$F_n(\phi, p) = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{it}) - \sigma_n(f, e^{i\phi})|^p K_n(\phi - t)\,dt.$$

The next result gives a characterisation of $B_p^{1/p}(1 < p < \infty)$ in terms of both $F(z, p)$ and of $F_n(\phi, p)$. For simplicity we shall write $f(t)$ instead of $f(e^{it})$ etc. First we observe by consideration of the change of variables $u = s + t$, $s = s$ and the periodicity of f, that $f \in B_p^{1/p}$ if and only if

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(u) - f(s)|^p}{\sin^2\left(\frac{u-s}{2}\right)}\,du\,ds < \infty.$$

THEOREM 2. *The following are equivalent for* $1 < p < \infty$:

(a) $f \in B_p^{1/p}$;

(b) $\int_{-\pi}^{\pi}F(z, p)\,d\theta = O(1-r)\ (r \to 1)$;

(c) $\int_{-\pi}^{\pi}F_n(\phi, p)\,d\phi = O(1/n)\ (n \to \infty)$.

Proof. We shall show that (a) and (b) are equivalent and then that (a) and (c) are equivalent. Suppose that $f \in B_p^{1/p}$ and $z = re^{i\theta}$ with $r < 1$. Then

$$f(e^{it}) - f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(e^{it}) - f(e^{iu}))P(ze^{-iu})\,du,$$

and

$$|f(t) - f(z)|^p \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t) - f(u)|^p P(ze^{-iu})\,du$$

by Jensen's inequality. Therefore

$$F(z, p) \leq \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(t) - f(u)|^p P(ze^{-iu})P(ze^{-it})\,du\,dt,$$

from which it follows that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}F(z, p)\,d\theta &\leq \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(t) - f(u)|^p \frac{(1 - r^4)}{1 - 2r^2\cos(t - u) + r^4}\,du\,d\\ &\leq \frac{4(1 - r)}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(t) - f(u)|^p}{(1 - r^2)^2 + 4r^2\sin^2\left(\frac{t - u}{2}\right)}\,du\,dt\\ &\leq \frac{1 - r}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(t) - f(u)|^p}{\sin^2\left(\frac{t - u}{2}\right)}\,du\,dt,\end{aligned}$$

provided $r > \frac{1}{2}$. Since $f \in B_p^{1/p}$ it follows from our remark above that (b) holds. If conversely (b) holds, then from (1) we have

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(t) - f(re^{i\theta})|^p}{(1 - r)^2 + 4r\sin^2\left(\frac{t - \theta}{2}\right)}\,dt\,d\theta \leq k$$

for some constant k and all $r < 1$. Now let $r \to 1$; since $f \in L^p$, $f(re^{i\theta}) \to f(e^{i\theta})$ a.e. and Fatou's Lemma gives

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(t) - f(\theta)|^p}{4\sin^2\left(\frac{t - \theta}{2}\right)}\,dt\,d\theta \leq k,$$

from which we conclude that $f \in B_p^{1/p}$. The proof that (a) $\Leftrightarrow$ (c), though similar is more difficult and we therefore give it in full. Let $f \in B_p^{1/p}$; then

$$f(t) - \sigma_n(f, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t) - f(u)) K_n(\phi - u)\, du,$$

and

$$|f(t) - \sigma_n(f, \phi)|^p \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - f(u)|^p K_n(\phi - u)\, du$$

as before. Therefore

$$F_n(\phi, p) \leq \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(t) - f(u)|^p K_n(\phi - u) K_n(\phi - t)\, du\, dt. \quad (5)$$

The convolution $K_n * K_n(x)$ is

$$\begin{aligned}
&\sum_{-n}^{n} \left(1 - \frac{|k|}{n+1}\right)^2 e^{ikx} \\
&= K_n(x) + \frac{\cos^2\left(\frac{n+1}{2}\right)x}{(n+1)\sin^2 \frac{x}{2}} - \frac{\sin(n+1)x \cos \frac{x}{2}}{2(n+1)^2 \sin^3\left(\frac{x}{2}\right)} \\
&= \frac{1}{(n+1)\sin^2 \frac{x}{2}} - \frac{\sin(n+1)x \cos \frac{x}{2}}{2(n+1)^2 \sin^3\left(\frac{x}{2}\right)}
\end{aligned}$$

and the second term is dominated by

$$\frac{(n+1)\left|\sin x \cos \frac{x}{2}\right|}{2(n+1)^2 \left|\sin^3\left(\frac{x}{2}\right)\right|} = \frac{\cos^2 \frac{x}{2}}{(n+1)\sin^2 \frac{x}{2}}$$

whence

$$|K_n * K_n(x)| \leq \frac{2}{(n+1)\sin^2 \frac{x}{2}}.$$

Integrating (5) through with respect to ϕ and using the above estimate we obtain

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(\phi, p)\, d\phi &\leq \frac{1}{(2\pi)^2} \cdot \frac{2}{n+1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(t) - f(u)|^p}{\sin^2\left(\frac{t-u}{2}\right)}\, du\, dt \\
&= o\left(\frac{1}{n}\right)
\end{aligned}$$

since $f \in B_p^{1/p}$. Next assume that (c) holds and let

$$T_n(t) = \int_{-\pi}^{\pi} |f(t+\phi) - \sigma_n(f,\phi)|^p \, d\phi.$$

By hypothesis we have

$$I_n = \int_0^{2\pi} \frac{\sin^2\left(\frac{n+1}{2}\right)t}{\sin^2 \frac{t}{2}} T_n(t)\, dt = O(1) \quad (n \to \infty).$$

Consequently

$$\frac{1}{N+1} \sum_{n=0}^{N} I_n = O(1) \quad (N \to \infty).$$

Now

$$2 \sum_{n=0}^{N} \sin^2\left(\frac{n+1}{2} t\right) = \sum_0^N (1 - \cos(n+1)t)$$

$$= N + 1 - \frac{\sin\left(\frac{N+1}{2}\right)t \cos\left(\frac{N+2}{2}\right)t}{\sin \frac{t}{2}}$$

so that

$$\frac{1}{N} \sum_0^N \sin^2 \frac{n+1}{2} t \to \frac{1}{2} \quad (N \to \infty) \tag{6}$$

for each t, $0 < |t| < \pi$. Also

$$\lim_{n \to \infty} T_n(t) = T(t) = \int_{-\pi}^{\pi} |f(t+\phi) - f(\phi)|^p \, d\phi \tag{7}$$

uniformly in t, $|t| < \pi$, because by the triangle inequality for norms

$$\left|T_n(t)^{1/p} - T(t)^{1/p}\right| \le \|\sigma_n(f) - f\|_p \text{ for all } t,$$

whence

$$\lim_{n \to \infty} \left|T_n(t)^{1/p} - T(t)^{1/p}\right| = 0$$

uniformly in t. Putting these observations together we see that by (6) and (7),

$$\begin{aligned}&\frac{1}{N+1}\sum_{n=0}^{N} T_n(t)\sin^2\left(\frac{n+1}{2}\right)t\\&= \frac{1}{N+1}\sum_0^N \sin^2\left(\frac{n+1}{2}\right)t(T_n(t)-T(t)) + \frac{T(t)}{N+1}\sum_0^N \sin^2\left(\frac{n+1}{2}\right)t\\&\to \frac{T(t)}{2} \text{ as } N\to\infty \text{ for all } t.\end{aligned}$$

By Fatou's Lemma

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{T(t)}{\sin^2\frac{t}{2}}\,dt &\le \liminf_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N} I_n\\&<\infty,\end{aligned}$$

which says that $f\in B_p^{1/p}$ and finishes the proof of Theorem 2.

We remark that just as $|v_z(e^{it})|^2 = P(ze^{-it})$ we also have $|u_n(\zeta)|^2 = K_n(\bar{\zeta})$ (see [3]). Taking $A = A(f)$ we can now draw the following conclusions.

THEOREM 3. *If* $A\in\mathscr{C}_p(2<p<\infty)$ *then*
(i) $\int_{-\pi}^{\pi}\|Au_n(\zeta)\|^p\,d\phi = O(1/n)$ $(\zeta = e^{i\phi})$,
(ii) $\int_{-\pi}^{\pi}\|Av_z\|^p\,d\theta = O(1-r)$ $(r = e^{i\theta})$.

Proof. Although (ii) can easily be derived independently we remark that (i) $\Rightarrow$ (ii) because of the following useful inequality which we obtain from (4):

$$\|Av_z\| \le \sqrt{1-r^2}(1-r)\sum_0^\infty \sqrt{n+1}\,r^n\|Au_n(\zeta)\|. \tag{8}$$

Thus, assuming (i) we have for any $p\ge 1$

$$\|Av_z\|^p \le (1-r^2)^{p/2}(1-r)\sum_0^\infty (n+1)^{p/2}r^n\|Au_n(\zeta)\|^p,$$

from which we have for some constant C,

$$\begin{aligned}\int_{\pi}^{\pi}\|Av_z\|^p\,d\theta &\le C(1-r^2)^{p/2}(1-r)\sum_0^\infty (n+1)^{p/2}\frac{r^n}{n+1}\\&= C(1-r^2)^{p/2}(1-r)\sum_0^\infty (n+1)^{p/2-1}r^n\\&\sim C(1-r^2)^{p/2}(1-r)\frac{\Gamma(p/2)}{(1-r)^{p/2}}\end{aligned}$$

by a result in [12, p. 225]. This gives (ii). Now we prove (i).

Although it is not true that $\|Au_n(\zeta)\|^2 = F_n(\phi, 2)$, we note that from [6],

$$\begin{aligned}\|Au_n(\zeta)\|^2 &= \|fu_n(\zeta)\|^2 - \|Pfu_n(\zeta)\|^2\\ &= \sigma_n(|f|^2, \zeta) - \|Pfu_n(\zeta)\|^2\\ &\le \sigma_n(|f|^2, \zeta) - |\sigma_n(f, \zeta)|^2\\ &= F_n(\phi, 2),\end{aligned} \tag{9}$$

since $\|Pfu_n(\zeta)\|^2 - |\sigma_n(f, \zeta)|^2 \ge 0$ by the Cauchy-Schwarz inequality. Now by Jensen's inequality

$$F_n(\phi, 2)^{p/2} \le F_n(\phi, p),$$

and so

$$\int_{-\pi}^{\pi} F_n(\phi, 2)^{p/2}\, d\phi \le \int_{-\pi}^{\pi} F_n(\phi, p)\, d\phi = O(1/n)$$

by Theorem 2. The result follows from (9).

Remark. We have shown that the conclusion for $\|Au_n(\zeta)\|$ implies that for $\|Av_z\|$ above by virtue of inequality (8). We do not know if the converse implication holds.

THEOREM 4. *If $A \in \mathscr{C}_p (1 < p < 2)$ then*
(i) $\int_{-\pi}^{\pi} \|Au_n(\zeta)\|^p\, d\phi = O((1/n)^{p/2})$
(ii) $\int_{-\pi}^{\pi} \|Av_z\|^p\, d\theta = O((1 - r)^{p/2})$

Proof. (i) Since $\mathscr{C}_p \subset \mathscr{C}_2$, $A \in \mathscr{C}_2$ and since L^p norms increase with p we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} \|Au_n(\zeta)\|^p\, d\phi &\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|Au_n(\zeta)\|^2\, d\phi\right)^{p/2}\\ &= O\left(\left(\frac{1}{n}\right)^{p/2}\right)\end{aligned}$$

by Theorem 1. The proof of (ii) is similar.

It is an open question whether (i) is stronger than (ii).

The converses of Theorems 3 and 4 are false and counter examples will be provided later. In relation to Theorem 4 we have in the opposite direction the following result.

THEOREM 5. *Suppose $2 < p < \infty$. If*
(i) $\int_{-\pi}^{\pi} \|Au_n(\zeta)\|^p\, d\phi = O((1/n)^{p/2})$
or
(ii) $\int_{-\pi}^{\pi} \|Av_z\|^p\, d\theta = O((1 - r)^{p/2})$
then $A \in \mathscr{C}_p$.

Proof. Suppose (ii) holds. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\|Av_z\|^2\,d\theta \le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\|Av_z\|^p\,d\theta\right)^{2/p} = O(1-r)$$

which implies that $A \in \mathscr{C}_2$ by Theorem 1 and so $A \in \mathscr{C}_p$. Since (i) is similar, the proof is complete.

We do not know if hypothesis (i) is stronger than hypothesis (ii).

On comparing the conclusions of Theorems 3 and 4 one may wonder whether a stronger result holds than, for example, that stated in Theorem 4 (ii). This is shown to be not the case by the argument below.

We set $g(e^{it}) = f(e^{-it})$ so that g is analytic and $g(0) = 0$. As $A = (a_{i+j})$ we may write $A = A(f) = A(g)$. If $1 < p < \infty$, then for any bounded operator $A = A(f)$ we have

$$\int_{\pi}^{\pi}\|Av_z\|^p\,d\theta \ge C_p(1-r)^{p/2}\|f_r\|_p^p \quad (r \ge \tfrac{1}{2}).$$

To see this, write

$$\begin{aligned} F(\bar{z},2) &= \frac{(1-r^2)}{2\pi}\int_{-\pi}^{\pi}\left|\frac{f(e^{it})-f(\bar{z})}{e^{it}-\bar{z}}\right|^2 dt \\ &= \frac{(1-r^2)}{2\pi}\int_{-\pi}^{\pi}\left|\frac{g(e^{it})-g(z)}{e^{it}-z}\right|^2 dt. \end{aligned}$$

Now

$$h(w) = \frac{g(w)-g(z)}{w-z} = \sum_{k=0}^{\infty}\frac{h^{(k)}(0)w^k}{k!},$$

hence

$$\begin{aligned} F(\bar{z},2) &\ge (1-r^2)\left|\frac{g(z)-g(0)}{z}\right|^2 \\ &= (1-r^2)\left|\frac{f(\bar{z})}{z}\right|^2, \end{aligned}$$

and

$$\int_{\pi}^{\pi}\|Av_z\|^p\,d\theta \ge \frac{(1-r^2)^{p/2}}{r^p}\int_{-\pi}^{\pi}|f(z)|^p\,d\theta,$$

which is the desired conclusion.

2. Hankel operators of class $\mathscr{C}_1$

We give here a necessary condition that the Hankel operator A belongs to the trace class $\mathscr{C}_1$ which is of a slightly different kind to that in Theorem 3. For this purpose we develop an integral representation for A which in turn depends on an integral representation for functions in A_1^1.

We shall write $h \in L^1(D)$ if

$$\|h\|_{L^1(D)} = \iint_D |h(z)|\,dx\,dy < \infty.$$

Suppose $f \in A_1^1$ with $f(z) = \sum_0^\infty a_n z^n$ $(z \in D)$. Let

$$F(z) = \sum_0^\infty (n+2)(n+3)a_n z^n.$$

It is not hard to see that $F \in L^1(D)$ also and that there are constants C_1, C_2, C_3 such that

$$\int_0^1 \int_0^{2\pi} |F(re^{i\theta})|r\,d\theta\,dr \leq C_1 \iint_D |f''(re^{i\theta})|r\,d\theta\,dr + C_2|a_0| + C_3|a_1|. \quad (10)$$

For instance, on writing $g(z) = z^3 f(z)$ we have $g''(z) = zF(z)$ and $\|g''\|_{L^1(D)}$ is bounded by an expression similar to that on the right-hand side of (10).

For fixed z in D, with $w = \rho e^{i\phi}$ we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^1\int_0^{2\pi} F(w)\frac{(1-|w|^2)^2}{(1-\bar{w}z)^2}\rho\,d\rho\,d\phi \\
&= \frac{1}{2\pi}\int_0^1 (1-\rho^2)^2\rho\left\{\sum_{n=0}^\infty (n+1)\rho^n z^n \int_0^{2\pi} F(\rho e^{i\phi})e^{-in\phi}\,d\phi\right\}d\rho \\
&= \sum_{n=0}^\infty (n+1)(n+2)(n+3)a_n z^n\left(\int_0^1 (1-\rho^2)^2 \rho^{2n+1}\,d\rho\right) \\
&= \sum_0^\infty a_n z^n \\
&= f(z);
\end{aligned}$$

that is

$$f(z) = \frac{1}{2\pi}\iint_D F(w)t_w(z)\rho\,d\rho\,d\phi, \quad (11)$$

which expresses f as a linear $L^1(D)$ sum of the elements

$$t_w(z) = \frac{(1-|w|^2)^2}{(1-\bar{w}z)^2}.$$

It is easily verified that $\{t_w\colon w \in D\}$ is a bounded subset of A_1^1 and in fact that

$$\frac{1}{\pi}\int\int_D |t_w''(z)|r\,d\theta\,dr = 6|w|^2 \quad (w \in D).$$

As in the argument following Theorem 5 we shall continue to write $A = A(f)$ even when f is analytic, $f = \Sigma_0^\infty a_n z^n$, where as usual $A = (a_{i+j})$. Now $A(t_w)$ is a rank two operator [8] whose action on $g(z) \in H^2$ is given by

$$\begin{aligned} A(t_w)g(z) &= \sum_{n=0}^{\infty} z^n \sum_{m=0}^{\infty} (1-|w|^2)^2(n+m+1)\bar{w}^{n+m}\hat{g}_m \\ &= (1-|w|^2)^2 \sum_{n=0}^{\infty} z^n \frac{d}{ds}\left[s^{n+1}g(s)\right]_{s=\bar{w}} \\ &= (1-|w|^2)^2 \frac{d}{ds}\left[\frac{sg(s)}{1-sz}\right]_{s=\bar{w}} \\ &= (1-|w|^2)^2\left[\frac{g(\bar{w})}{(1-\bar{w}z)^2} + \frac{\bar{w}g'(\bar{w})}{1-\bar{w}z}\right]. \end{aligned}$$

Furthermore

$$\begin{aligned} \|A(t_w)g\| &= (1-|w|^2)^2\left\{\sum_{n=0}^{\infty}|w|^{2n}|(n+1)g(\bar{w}) + \bar{w}g'(\bar{w})|^2\right\}^{1/2} \\ &\le (1-|w|^2)^2\Big\{|g(\bar{w})|\Big(\sum(n+1)^2|w|^{2n}\Big)^{1/2} \\ &\qquad + |w||g'(\bar{w})|\Big(\sum|w|^{2n}\Big)^{1/2}\Big\} \\ &= (1-|w|^2)^2\left\{|g(\bar{w})|\left(\frac{1+|w|^2}{(1-|w|^2)^3}\right)^{1/2} + \frac{|w||g'(\bar{w})|}{(1-|w|^2)^{1/2}}\right\} \\ &\le 2(1-|w|^2)^{1/2}|g(\bar{w})| + (1-|w|^2)^{3/2}|g'(\bar{w})|. \end{aligned} \tag{12}$$

Since the function $w \to A(t_w)$ is continuous in the $\mathscr{C}_1$ norm, from (11) we have

$$A = A(f) = \frac{1}{2\pi}\int\int F(w)A(t_w)\rho\,d\phi\,d\rho,$$

where the integral is understood as a limit of Riemann sums. The action of A on $g \in H^2$ is given by

$$Ag(z) = \frac{1}{2\pi}\int\int F(w)A(t_w)g(z)\rho\, d\phi\, d\rho$$
$$= \frac{1}{2\pi}\int\int F(w)(1-|w|^2)^2\left\{\frac{g(\bar{w})}{(1-\bar{w}z)^2} + \frac{\bar{w}g'(\bar{w})}{1-\bar{w}z}\right\}\rho\, d\phi\, d\rho$$

and

$$\|Ag\| \le \frac{1}{2\pi}\int\int |F(w)|\,\|A(t_w)g\|\rho\, d\phi\, d\rho. \tag{13}$$

We are now in a position to give a necessary condition in terms of $Au_n(\zeta)$ that A belong to $\mathscr{C}_1$. It was shown in [6] that A is bounded if and only if

$$\sup\left\{\|Au_n(\zeta)\|: n \in \mathbf{N}, \zeta \in \partial D\right\} < \infty.$$

THEOREM 6. *Suppose the Hankel operator A is in $\mathscr{C}_1$. There exists a constant C such that*

$$\sum_{n=0}^{\infty} \frac{\|Au_n(\zeta)\|}{n+1} \le C\|A\|_1 \quad (\zeta \in \partial D).$$

Proof. From the definition of $u_n(\zeta)$,

$$u_n(\zeta)(w) = \frac{1}{\sqrt{n+1}}\sum_{k=0}^{n} \bar{\zeta}^k w^k \quad (|w| < 1)$$

so

$$|u_n(\zeta)(w)| \le \frac{1}{\sqrt{n+1}}\sum_{k=0}^{n} |w|^k.$$

Taking $g(w) = u_n(\zeta)(w)$ in (12) we have

$$\|A(t_w)u_n(\zeta)\| \le \frac{2(1-|w|^2)^{1/2}}{\sqrt{n+1}}\sum_{k=0}^{n}|w|^k + \frac{(1-|w|^2)^{3/2}}{\sqrt{n+1}}\sum_{k=1}^{n} k|w|^{k-1}.$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\|A(t_w)u_n(\zeta)\|}{n+1}
&\le 2(1-|w|^2)^{1/2} \sum_{n=0}^{\infty} \frac{1}{(n+1)^{3/2}} \sum_{k=0}^{n} |w|^k \\
&\quad + (1-|w|^2)^{3/2} \sum_{n=0}^{\infty} \frac{1}{(n+1)^{3/2}} \sum_{k=1}^{n} k|w|^{k-1} \\
&= 2(1-|w|^2)^{1/2} \sum_{k=0}^{\infty} |w|^k \sum_{n=k}^{\infty} \frac{1}{(n+1)^{3/2}} \\
&\quad + (1-|w|^2)^{3/2} \sum_{k=1}^{\infty} k|w|^{k-1} \sum_{n=k}^{\infty} \frac{1}{(n+1)^{3/2}} \\
&= 2(1-|w|^2)^{1/2} \sum_{k=0}^{\infty} |w|^k O\left(\frac{1}{(k+1)^{1/2}}\right) \\
&\quad + (1-|w|^2)^{3/2} \sum_{k=0}^{\infty} |w|^k O\left((k+1)^{1/2}\right) \\
&= O(1) \quad \text{as } |w| \to 1.
\end{aligned}$$

From (9) it now follows that there is a constant C_1 such that

$$\sum_{0}^{\infty} \frac{\|Au_n(\zeta)\|}{n+1} \le C_1 \|F\|_{L^1(D)}.$$

Now we invoke a result in [4]: Let $f \in H^2$, $h(z) = z^2 f(z)$; then $A(f)$ is trace class if and only if $h'' \in L^1(D)$. Also

$$\frac{\pi}{8} \|h''\|_{L^1(D)} \le \|A(f)\|_1 \le \|h''\|_{L^1(D)}.$$

Applying this result with $g(z) = z^3 f(z)$, $g''(z) = zF(z)$, (and $f(z)$ replaced by $zf(z)$ and h replaced by g) a little manipulation now reveals that there is a constant C_2 such that $C_2 \|F\|_{L^1(D)} \le \|A(f)\|_1$.

Putting these facts together we deduce that there is a constant C such that

$$\sum_{0}^{\infty} \frac{\|Au_n(\zeta)\|}{n+1} \le C \|A\|_1.$$

The proof is complete.

This result implies a similar result for $\|Av_z\|$ as follows.

THEOREM 7. *Let $A \in \mathscr{C}_1$. There is a constant C such that*

$$\int_0^1 \frac{\|Av_z\|}{1-r}\,dr \le C\|A\|_1 \quad (0 < \theta < 2\pi).$$

Proof. We recall the representation (4) for v_z:

$$v_z(w) = \sqrt{1-r^2}(1-r)\sum_0^\infty \sqrt{n+1}\,r^n u_n(\zeta)(w) \quad (z = r\zeta,\ w \in D).$$

From this it follows that

$$\|Av_z\| \le \sqrt{1-r^2}(1-r)\sum \sqrt{n+1}\,r^n\|Au_n(\zeta)\|,$$

and so

$$\begin{aligned}\int_0^1 \frac{\|Av_z\|}{1-r}\,dr &\le \sum\sqrt{n+1}\,\|Au_n(\zeta)\|\int_0^1 r^n(1-r^2)^{1/2}\,dr\\ &= \frac{1}{2}\sum\sqrt{n+1}\,\|Au_n(\zeta)\|\int_0^1(1-s)^{1/2}s^{(n-1)/2}\,ds\\ &= \frac{1}{2}\sum\|Au_n(\zeta)\|\sqrt{n+1}\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}\\ &= O(1)\sum_0^\infty\frac{\|Au_n(\zeta)\|}{n+1}.\end{aligned}$$

The result now follows from the previous theorem.
The converses of Theorems 6 and 7 are false.

3. Counter-examples

The implications in Theorems 3, 4, 5, 6 and 7 are in one direction only. Here we want to construct counter-examples to show that the converses are false. In order to do this we need the following characterization of the Besov classes $B_p^{1/p}$, [7]. Define functions W_n ($n \in \mathbf{Z}$) as follows:

$$\hat{W}_n(k) = \begin{cases} 0, & k \le 2^{n-1},\\ \dfrac{k-2^{n-1}}{2^{n-1}}, & 2^{n-1} \le k < 2^n,\\ \dfrac{2^{n+1}-k}{2^n}, & 2^n \le k \le 2^{n+1},\\ 0, & k \ge 2^{n+1},\end{cases}$$

if $n \ge 2$, $W_1(z) = z + z^2 + \frac{1}{2}z^3$ $W_0 \equiv 1$, $W_n = \overline{W_{-n}}$ for $n < 0$.

THEOREM C. *For* $1 \le p < \infty$, $f \in B_p^{1/p}$ *if and only if* $\sum_{-\infty}^{\infty} 2^{|n|} \|f * W_n\|_p^p < \infty$.

Now consider the function f given by the lacunary power series

$$f(z) = \sum_1^\infty a_k z^{2^k}.$$

It follows easily from Theorem C that $f \in A_p^{1/p}$ if and only if $\sum_1^\infty 2^k |a_k|^p < \infty$. For each p, $2 < p < \infty$, we will find a function f of this type such that f is not in $A_p^{1/p}$ but for which

$$\int_{-\pi}^{\pi} \|Au_n(\zeta)\|^p \, d\phi = O\left(\frac{1}{n}\right).$$

This will show that the converse of Theorem 3(i) is false. That the converse of Theorem 3(ii) is also false now follows using inequality (8) and an argument similar to that given in Theorem 3 itself. In what follows C or C_p is an absolute constant though not always the same one.

Fix p, $2 < p < \infty$, and let $f(z) = \sum_0^\infty a_k z^k$, $A = A(f)$; then

$$\begin{aligned} \|Au_n(\bar{\zeta})\|^2 &= \frac{1}{n+1} \sum_{k=0}^{\infty} \left| \sum_{j=0}^{n} a_{k+j} \zeta^j \right|^2 \\ &= \frac{1}{n+1} \sum_0^\infty |f_k(n, \zeta)|^2 \text{ say.} \end{aligned}$$

By the triangle inequality for norms, on putting $G_n(\bar{\zeta}) = \|Au_n(\bar{\zeta})\|^2$ we get

$$\begin{aligned} \|G_n(\bar{\zeta})\|_{p/2} &\le \frac{1}{n+1} \sum_0^\infty \|f_k^2(n, \zeta)\|_{p/2} \\ &= \frac{1}{n+1} \sum_0^\infty \|f_k(n, \zeta)\|_p^2. \end{aligned}$$

Suppose now that

$$a_k = \begin{cases} 0, & k \ne 2^m, \\ 2^{-m/p}, & k = 2^m, \end{cases} \quad (m \ge 1)$$

so that $f \notin A_p^{1/p}$ and f has a lacunary series. By a result on such series [1, p. 243], there is a constant $C = C_p$ independent of f such that $\|f\|_p \le C\|f\|_2$.

Applying this result to each $f_k(n, \zeta)$ it follows that

$$\begin{aligned}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\|Au_n(\bar{\zeta})\|^p\,d\phi\right)^{2/p} &= \|G_n(\bar{\zeta})\|_{p/2}\\ &\le \frac{C_p}{n+1}\sum_{k=0}^{\infty}\|f_k(n,\zeta)\|_2^2\\ &= \frac{C_p}{n+1}\sum_{k=0}^{\infty}\sum_{j=0}^{n}|a_{k+j}|^2\\ &= \frac{C_p}{n+1}\left\{\sum_{j=2}^{n}(j+1)|a_j|^2+(n+1)\sum_{j=n+1}^{\infty}|a_j|^2\right\}.\end{aligned}$$

If $2^m \le n < 2^{m+1}$, then

$$\begin{aligned}\sum_{2}^{n+1}(k+1)a_k^2 \le \sum_{j=1}^{m}\sum_{k=2^j}^{2^{j+1}-1}(k+1)a_k^2 &= \sum_{j=1}^{m}(2^j+1)\frac{1}{2^{2j/p}}\\ &< 2\sum_{j=1}^{m}2^{j(1-2/p)}\\ &= O(n^{1-2/p}),\end{aligned}$$

while

$$(n+1)\sum_{k=n+1}^{\infty}a_k^2 = O(n^{1-2/p})$$

also. Putting these together we get

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\|Au_n(\zeta)\|^p\,d\phi\right)^{2/p} = O(n^{-2/p}),$$

or

$$\int_{-\pi}^{\pi}\|Au_n(\zeta)\|^p\,d\phi = O\left(\frac{1}{n}\right),$$

which is the required result.

Moving on to Theorem 4, fix p, $1 < p < 2$, and choose $A \in \mathscr{C}_2 \setminus \mathscr{C}_p$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \|Au_n(\zeta)\|^p \, d\phi \le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|Au_n(\zeta)\|^2 \, d\phi\right)^{p/2}$$
$$= o\left(\left(\frac{1}{n}\right)^{p/2}\right)$$

by Theorem 1. Conclusion (ii) of Theorem 4 follows from this, completing the demonstration.

Next consider the converse of Theorem 5. Choose p, $2 < p < \infty$; it is enough to show that $\exists A = A(f) \in \mathscr{C}_p$ such that

$$\int_{\pi}^{\pi} \|Av_z\|^p \, d\theta \ne o\left((1-r)^{p/2}\right).$$

As before we take a lacunary series for f,

$$f(z) = \sum_{k=1}^{\infty} a_k z^{2^k} \quad \text{where } a_k = \left(k/2^k\right)^{1/2}.$$

Since $\sum_1^\infty 2^k a_k^p < \infty$ it is clear that $f \in A_p^{1/p}$. Choose $z \in D$ and let

$$f_z(\zeta) = f\left(\frac{\zeta + z}{1 + \bar{z}\zeta}\right) - f(\zeta) \quad \text{for all } \zeta \in D.$$

By a simple change of variable,

$$F(z,2) = \|f_z\|_2^2 \ge |f_z'(0)|^2 = (1 - |z|^2)^2 |f'(z)|^2,$$

and so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \|Av_z\|^p \, d\theta \ge \frac{(1-r^2)^p}{2\pi}\int_{-\pi}^{\pi} |f'(z)|^p \, d\theta$$
$$\ge C_p (1-r)^p \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(z)|^2 \, d\theta\right)^{p/2}$$
$$\ge C_p (1-r)^p \left(\sum_{k=1}^{\infty} 2^{2k} |a_k|^2 r^{2^{k+1}}\right)^{p/2},$$

once more using the equivalence of the L_p and L_2 norms. Assuming $r \ge \frac{1}{2}$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\|Av_z\|^p\,d\theta \geq C_p(1-r)^p\left(\sum_1^\infty 2^{2k}\frac{k}{2^k}r^{2^k}\right)^{p/2}$$
$$= C_p(1-r)^p\left(\sum_1^\infty 2^k k r^{2^k}\right)^{p/2}$$
$$\geq C_p(1-r)^p\left(\sum_2^\infty (\log n) r^n\right)^{p/2}$$
$$\geq C_p(1-r)^p\left(\frac{\log\frac{1}{1-r}}{1-r}\right)^{p/2}$$
$$\geq C_p(1-r)^{p/2}\left(\log\frac{1}{1-r}\right)^{p/2}.$$

This shows that $\int_{-\pi}^{\pi}\|Av_z\|^p\,d\theta \neq O((1-r)^{p/2})$ and completes the proof.

Our last counter-example serves for both Theorems 6 and 7. Once more we consider a lacunary series, this time we take

$$f(z) = \sum_0^\infty a_k z^k = \sum_{m=1}^\infty 2^{-m} z^{2^m}.$$

Since $\sum_1^\infty 2^m a_{2^m} = \infty$, it is plain that f is not in A_1^1 and so $A = (a_{i+j})$ is not in $\mathscr{C}_1$ by Theorem B. Let $u_n = u_n(1)$. We show that nevertheless

$$\sum_0^\infty \frac{\|Au_n\|}{n+1} < \infty,$$

which is clearly sufficient. We have

$$Au_n = \frac{1}{\sqrt{n+1}}\sum_{i=0}^\infty\left\{\sum_{j=0}^n a_{i+j}\right\}\chi_i,$$

$$\|Au_n\| = \frac{1}{\sqrt{n+1}}\left\{\sum_{k=0}^\infty\sum_{i=2^k}^{2^{k+1}-1}\left(\sum_{j=0}^n a_{i+j}\right)^2\right\}^{1/2}$$

since $a_0 = 0$. Fix i, n and choose integers k and m such that $2^k \leq i < 2^{k+1}$, $2^m \leq n < 2^{m+1}$. If $k \leq m$ the sum $\sum_{j=0}^n a_{i+j}$ has at most $m - k + 2$ non-zero terms and

$$\sum_{j=0}^n a_{i+j} \leq \sum_{j=k}^{m+1} 2^{-j} < \frac{1}{2^{k-1}},$$

while if $k > m$ it has at most one non-zero term and is bounded by $1/2^k$.

Consequently, for all i, $2^k \le i < 2^{k+1}$ and all $n \ge 1$ we have

$$\left(\sum_{j=0}^{n-1} a_{i+j}\right)^2 \le 2^{-2(k-1)},$$

so that

$$\|Au_n\| \le \frac{1}{\sqrt{n+1}}\left\{\sum_{k=0}^{\infty} 2^k 2^{-2(k-1)}\right\}^{1/2} = \frac{2\sqrt{2}}{\sqrt{n+1}}.$$

Therefore

$$\sum_0^{\infty} \frac{\|Au_n\|}{n+1} \le 2\sqrt{2}\sum_0^{\infty}(n+1)^{-3/2} < \infty,$$

as required.

Since Theorem 7 follows from Theorem 6 it is clear that this example also serves to show that the converse of Theorem 7 is false.

References

1. N.K. Bary, *A treatise on trigonometric series*, vol. II, Moscow 1961, Pergamon Press, New York, 1964.
2. A. Baernstein, "Analytic functions of bounded mean oscillation" in *Aspects of contemporary complex analysis*, Academic Press, New York 1980.
3. F.F. Bonsall, *Boundedness of Hankel matrices*, J. London Math Soc. (2), vol 29 (1984), pp. 289–300.
4. F.F. Bonsall and D. Walsh, *Symbols for trace-class Hankel operators with good estimates for norms*, Glasgow Math. J., vol 28 (1986), 47–54.
5. I.C. Gohberg and M.G. Krein, *Introduction to the theory of linear non-self-adjoint operators*, Amer. Math. Soc., Providence, R.I., 1969.
6. Finbarr Holland and David Walsh, *Boundedness criteria for Hankel operators*, Proc. Roy. Irish. Acad., vol. 84A, no. 2 (1984), pp. 141–154.
7. V.V. Peller, *Hankel operators of class $\mathscr{C}_p$ and their applications* (Rational approximation, Gaussian processes, the problem of majorizing operators), English translation, Math. USSR Sbornik, vol. 41 (1982), pp. 443–479.
8. S.C. Power, *Hankel operators on Hilbert space*, Research Notes in Mathematics 64, Pitman, London, 1982.
9. J.R. Ringrose, *Compact non-self-adjoint operators*, Van Nostrand Reinhold Mathematical Studies 35, London, 1971.
10. D. Sarason, *Function theory on the unit circle*, Lecture Notes, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, 1978.
11. E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.
12. E.C. Titchmarsh, *Theory of functions*, Oxford University Press, Oxford, England, 1932.

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