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# CONNECTED SUBGROUPS OF LIE GROUPS

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## 1. Introduction

A subgroup $H$ of a Lie group is said to be an *analytic subgroup* if $H$ is a connected Lie group and the inclusion map is an immersion. Various authors have realized that $H$ is an analytic subgroup if and only if it is pathwise connected. This topological characterization of analyticity appears with varying degrees of rigor in [1], [3] and [5]. A complete proof, whose details also provide interesting corollaries relating to iterates of a curve in a Lie group and to étalée measures in a Lie group, has recently been given by Jenkins [2].

In this note it is shown that path-connectedness cannot be replaced by mere connectedness. Although we present our example in the context of Lie groups, generalizations to other topological groups are readily apparent. The construction assumes the continuum hypothesis and uses old-fashioned transfinite induction.

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## 2. Connectedness does not imply analyticity

THEOREM. *Let $G$ be a connected Lie group of dimension greater than one. Then $G$ has a connected subgroup which is dense in $G$ and which is not an analytic subgroup.*

*Proof.* Let $\mathscr{C}$ be the collection of all uncountable closed subsets of $G$. We shall construct a subgroup $H$ with the properties that $H \cap C \neq \varnothing$ for all $C$ in $\mathscr{C}$ and also $H \cap (G - C) \neq \phi$ for all $C$ in $\mathscr{C}$. The first condition implies that $H$ is dense in $G$. Using this, plus the fact that since $\dim G > 1$ any set which separates $G$ must be uncountable, a simple point set topology argument then shows that $H$ must be connected.

The second condition implies that $H$ contains no uncountable closed subset of $G$. Thus $H$ is far from being an analytic subgroup of $G$.

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We assume the continuum hypothesis which implies that the collection of all closed subsets of any separable metric space has the cardinality of the continuum, the same cardinality as $\Omega$, the set of ordinals strictly less than the first uncountable ordinal [5, page 9]. The collection $\mathscr{C}$ also has this cardinality and therefore we may use the elements of $\Omega$, i.e., the countable ordinals, to index $\mathscr{C}$; thus we write $\mathscr{C} = \{C_\alpha | \alpha \in \Omega\}$.

The subgroup $H$ will be the union of a chain of subgroups $\{H_\alpha | \alpha \in \Omega\}$ which will be constructed by transfinite induction so that the following conditions are satisfied:

(a) Each $H_\alpha$ is countable.

(b) $H_\alpha \subset H_\beta$ if $\alpha < \beta$.

(c) $H_\alpha \cap C_\alpha \neq \varnothing$ for each $\alpha$.

(d) There is a set $E$ containing one point, $y_\alpha$, from each $C_\alpha \in \mathscr{C}$ such that $H_\beta \cap E \neq \varnothing$ for all $\beta \in \Omega$.

To get started, let us denote the first element of $\Omega$ by the symbol 1 and let $x_1$ be any point of $C_1$. The subgroup, $H_1$, generated by $x_1$ is countable and, hence, misses most of the points in $C_1$. Pick one such point and denote it by $y_1$.

For the induction step, suppose $\beta_0$ is any countable ordinal and that, for all $\alpha < \beta_0$, we have constructed subgroups $H_\alpha$ so that (a), (b) and (c) hold. At the same time, suppose that for all $\alpha < \beta_0$ we have selected a point $y_\alpha$ from $C_\alpha$ so that any $H_\beta$ with $\beta < \beta_0$ misses the set $\{y_\alpha | \alpha < \beta_0\}$. We need to construct $H_{\beta_0}$ and choose $y_{\beta_0}$ so that these properties still hold.

There are two cases to consider. First suppose $\beta_0$ is not a limit ordinal and thus has an immediate predecessor $\beta_0 - 1$. Since $H_{\beta_0 - 1}$ is countable and $C_{\beta_0}$ is uncountable, a simple counting argument shows that there is a point $x_{\beta_0}$ in $C_{\beta_0}$ such that the group generated by $H_{\beta_0 - 1}$ and $x_{\beta_0}$ misses the countable set $\{y_\alpha | \alpha \leq \beta_0 - 1\}$. Denote this group by $H_{\beta_0}$; it is countable and hence misses most points of $C_{\beta_0}$. Pick one such point and call it $y_{\beta_0}$.

If $\beta_0$ is a limit ordinal, we simply let $H_{\beta_0} = \bigcup\{H_\alpha | \alpha < \beta_0\}$. Since $\beta_0$ is a countable ordinal, $H_{\beta_0}$ is countable and misses most points of $C_{\beta_0}$. Choose one such point and call it $y_{\beta_0}$.

This finishes the inductive step and completes the construction of our chain of subgroups. If we let $H$ be the union of the $H_\alpha$, $\alpha \in \Omega$, and if we let $E$ be the set of points $y_\alpha$, $\alpha \in \Omega$, which have been inductively chosen, the properties (a), (b), (c) and (d) hold and the proof is complete.

We remark that the proof of the theorem makes no explicit use of the Lie group structure and, thus, the construction can be carried out for more general topological groups.

Finally, we note that it would be of some interest to know whether a continuum-wise connected subgroup of a Lie group must be an analytic subgroup. (A *continuum* is a compact, connected set; the subgroup $H$ constructed in the theorem is not continuum-wise connected because it contains no closed uncountable set.)

## References

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