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CONJUGATE FUNCTIONS AND MODULI OF CONTINUITY

BY

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1.1 *Introduction.* We denote by Ω a domain properly contained in n-dimensional Euclidean space R^n. A continuous nondecreasing function

$$\lambda(t)\colon [0,\infty) \to [0,\infty) \quad \text{with } \lambda(0) = 0$$

is called a majorant if $\lambda(t_1 + t_2) \leq \lambda(t_1) + \lambda(t_2)$, $0 < t_1, t_2 < \infty$. When $f\colon \Omega \to R^m$ and $\lambda(t)$ is a majorant we write $f \in \operatorname{Lip}_\lambda(\Omega)$ if there exists a constant $M < \infty$ such that

$$|f(x_1) - f(x_2)| \leq M\lambda(|x_1 - x_2|) \tag{1.1}$$

for all $x_1, x_2 \in \Omega$. We denote the smallest such M by $\|f\|_\lambda$. When $\lambda(t) = t^\alpha$, $0 < \alpha \leq 1$, the classes $\operatorname{Lip}_\lambda(\Omega)$ are the usual Lipschitz classes. We write D for the open unit disk of the complex plane, $\bar{D}$ for its closure and ∂D for its boundary. Theorem 1.2 follows from theorems of Hardy and Littlewood (see [9] and [2]). When $0 < \alpha < 1$, it can be derived from a similar theorem of Privalov concerning conjugate functions on ∂D [15].

1.2 THEOREM. *If $f = u + iv$ is analytic in D, if $\lambda(t) = t^\alpha$, $0 < \alpha \leq 1$, and if $u \in \operatorname{Lip}_\lambda(D)$, then $f \in \operatorname{Lip}_\lambda(D)$. Moreover $\|f\|_\lambda \leq C\|u\|_\lambda$ where C is a constant which depends only on α.*

The papers [6] and [7] by Gehring and Martio show that Theorem 1.2 holds for a wide class of planar domains, so-called $\operatorname{Lip}_\alpha$-extension domains (see Section 2). On the other hand, simple examples show that Theorem 1.2 fails in arbitrary domains.

The main result of this paper, Theorem 3.8, generalizes Theorem 1.2 to quasiregular mappings in certain domains in R^n and to somewhat more general majorants than t^α. Theorem 3.8 reduces to Theorem 1.2 in the case that $\Omega = D$, $K = 1$ and $\lambda(t) = t^\alpha$.

We first show that Theorem 3.8 holds locally in balls. The geometry of $\operatorname{Lip}_\lambda$-extension domains (Section 2) then guarantees that the result holds globally.

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We define the modulus of continuity of a continuous function $f\colon E \to R^m$ over a set $E \subset R^n$ as

$$\omega(f, \delta; E) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in E \text{ and } |x_1 - x_2| \le \delta\}.$$

Moduli of continuity are nondecreasing with $\omega(f, 0; E) = 0$. Moreover, $\omega(f, \delta\colon \overline{D})$ and $\omega(f, \delta; D)$ are majorants. Corollary 3.7 states that the moduli of continuity of the components of a quasiregular mapping are locally equivalent. In Section 4 we give examples of bounded analytic functions $f = u + iv$ in D with

$$\lim_{\delta \to 0} \frac{\omega(v, \delta; D)}{\omega(u, \delta; D)} = \infty.$$

Hence in general the moduli of continuity of the components of a bounded quasiregular mapping are not globally equivalent even in a ball.

2.1 *Lip_λ-extension domains.* When $f\colon \Omega \to R^m$ we write $f \in \mathrm{locLip}_\lambda(\Omega)$ if there exists a constant M such that (1.1) holds whenever $x_1, x_2 \in \Omega$ with $|x_1 - x_2| \le \frac{1}{2} d(x_1, \partial\Omega)$. Here $d(x_1, \partial\Omega)$ denotes the Euclidean distance from x_1 to the boundary of Ω, $\partial\Omega$. We denote the smallest such M by $\|f\|_\lambda^{\mathrm{loc}}$.

2.2 *Remark.* In [11], Lappalainen shows that if (1.1) holds whenever

$$|x_1 - x_2| \le a d(x_1, \partial\Omega)$$

for some $a \le 1$, then $f \in \mathrm{locLip}_\lambda(\Omega)$.

We call Ω a Lip_λ-extension domain if there exists a constant b such that $\|f\|_\lambda \le b\|f\|_\lambda^{\mathrm{loc}}$ for all $f\colon \Omega \to R^m$. That is, $f \in \mathrm{Lip}_\lambda(\Omega)$ whenever $f \in \mathrm{locLip}_\lambda(\Omega)$. Ω is a Lip_λ-extension domain if and only if there exists a constant M such that each pair x_1, $x_2 \in \Omega$ can be joined by a continuous curve $\gamma \subset \Omega$ satisfying

$$\int_\gamma \frac{\lambda(d(\gamma(s), \partial\Omega))}{d(\gamma(s), \partial\Omega)}\, ds \le M\lambda(|x_1 - x_2|) \tag{2.1}$$

(see [11]). These domains were first identified by Gehring and Martio for $\lambda(t) = t^\alpha$ and called Lip_α-extension domains [7].

For certain $\lambda(t)$, the class of Lip_λ-extension domains is wide. All uniform domains are Lip_λ-extension domains if and only if there is a constant A such that

$$\int_0^t \frac{\lambda(s)}{s}\, ds \le A\lambda(t) \tag{2.2}$$

for all $0 \le t < \infty$ (see [11] and [14]). In particular, if $\lambda(t) = t^\alpha$, all balls, half-spaces, wedge-domains and quasiballs are Lip_λ-extension domains for all $0 < \alpha \le 1$.

3.1 *Quasiregular mappings.* We refer to the following spaces of functions:
$W_n^1(\Omega)$, Sobolev space of L^n-integrable functions with L^n-integrable distributional first derivatives over Ω.
$W_{n,\mathrm{loc}}^1(\Omega) = \cap W_n^1(\Omega')$, where the intersection is taken over all Ω' compactly contained in Ω.
We write Df for the Jacobi matrix of f and $|Df|$ for its norm as a linear transformation. J_f is the Jacobian determinant of f.

3.2 DEFINITION. A function f: $\Omega \to R^n$ is K-quasiregular in $\Omega \subset R^n$, $1 \le K < \infty$, if
(a) $f \in W_{n,\mathrm{loc}}^1(\Omega)$
(b) $|Df|^n \le KJ_f$ a.e. in Ω.

When $n = 2$, f is 1-*qr* if and only if it is an analytic function. A function f is a K-quasiregular homeomorphism if and only if it is K-quasiconformal in the usual sense (see [17], [13] and [12]).
We first prove a local version of Theorem 1.2 for quasiconformal mappings. Although this is a special case of Theorem 3.6, its proof is more geometric in character. We assume from here on that $\lambda(t)$ is a majorant.

3.3 THEOREM. *If $f = (f_1, f_2, \dots, f_n)$ is K-quasiconformal in Ω, and if*

$$f_j \in \mathrm{locLip}_\lambda(\Omega)$$

for some $j = 1, 2, \dots, n$, then $f \in \mathrm{locLip}_\lambda(\Omega)$ with $\|f\|_\lambda^{\mathrm{loc}} \le C\|f_j\|_\lambda^{\mathrm{loc}}$. Here C is a constant which depends only on n, λ and K.

Proof. Since f is K-quasiconformal there exists a continuous, strictly increasing function $\theta_K(t)$: $(0,1) \to (0,\infty)$ such that $\lim_{t\to 1}\theta_K(t) = \infty$, $\lim_{t\to 0}\theta_K(t) = 0$ and

$$\frac{|f(x_1) - f(x_2)|}{d(f(x_1), \partial f(\Omega))} \le \theta_K\left(\frac{|x_1 - x_2|}{d(x_1, \partial\Omega)}\right) \tag{3.1}$$

for all x_1, $x_2 \in \Omega$ satisfying $|x_1 - x_2| < d(x_1, \partial\Omega)$ (see [4]). We use the notation

$$l_r(f, x) = \min_{|x-y|=r} |f(x) - f(y)|,$$
$$L_r(f, x) = \max_{|x-y|=r} |f(x) - f(y)|.$$

Choose C_0 so that $\theta_K(C_0) = 1$. Fix $x_1 \in \Omega$. For each $x_2 \in \Omega$ with

$$|x_1 - x_2| \le C_0 d(x_1, \partial\Omega),$$

(3.1) gives

$$|f(x_1) - f(x_2)| \le d(f(x_1), \partial f(\Omega)).$$

In other words, if $B(y, r) = \{x \in R^n \mid |x - y| < r\}$ then

$$B(f(x_1), L_r(f, x_1)) \subset f(\Omega) \tag{3.2}$$

where $r = |x_1 - x_2|$. By a standard distortion theorem (see [17] and for $n = 3$, [4]), we conclude from (3.2) that

$$\frac{L_r(f, x_1)}{l_r(f, x_1)} \le e^{\chi(n)K} = C(n, K), \tag{3.3}$$

where $\chi(n)$ is a constant which depends only on n. Next choose x_3 so that

$$|x_1 - x_3| = |x_1 - x_2| = r$$

and $f(x_1) - f(x_3)$ is a vector in the f_j-direction. Since $f_j \in \operatorname{locLip}_\lambda(\Omega)$, (3.3) gives

$$\begin{aligned} |f(x_1) - f(x_2)| &\le L_r(f, x_1) \\ &\le C(n, K) l_r(f, x_1) \\ &\le C(n, K)|f(x_1) - f(x_3)| \\ &= C(n, K)|f_j(x_1) - f_j(x_3)| \\ &\le C(n, K)\|f_j\|_\lambda^{\mathrm{loc}}\lambda(|x_1 - x_3|) \\ &= C(n, K)\|f_j\|_\lambda^{\mathrm{loc}}\lambda(|x_1 - x_2|). \end{aligned}$$

Hence $f \in \operatorname{locLip}_\lambda(\Omega)$.

We immediately obtain the following theorem.

3.4 THEOREM. *If $f = (f_1, f_2, \ldots, f_n)$ is K-quasiconformal in a* Lip_λ*-extension domain Ω and if $f_j \in \mathrm{Lip}_\lambda(\Omega)$ for some $j = 1, 2, \ldots, n$, then*

$$f \in \mathrm{Lip}_\lambda(\Omega) \quad \textit{with } \|f\|_\lambda \le C\|f_j\|_\lambda,$$

where C is a constant which depends only on n, λ, Ω and K.

We now prove Theorem 3.4 for quasiregular mappings. We use certain integrability results which require the following definitions.

3.5 DEFINITIONS. If $E \subset R^n$ is a measurable set we write $|E|$ for the n-dimensional Lebesgue measure of E. We write $dx = dx_1 dx_2 \dots dx_n$ for Lebesgue measure and assume this measure when it is omitted from an integral. If $B \subset R^n$ is a ball, then σB, $\sigma \geq 1$, denotes the ball with the same center as B and with a radius of σ times that of B.

3.6 THEOREM. *If $f = (f_1, f_2, \dots, f_n)$ is K-quasiregular in Ω and if*

$$f_j \in \operatorname{locLip}_\lambda(\Omega) \quad \textit{for some } j = 1, 2, \dots, n,$$

then

$$f \in \operatorname{locLip}_\lambda(\Omega) \quad \textit{with } \|f\|_\lambda^{\mathrm{loc}} \leq C \|f_j\|_\lambda^{\mathrm{loc}},$$

where C is a constant which depends only on n, λ and K.

Proof. By Remark 2.2, it is sufficient to show that

$$|f(x_1) - f(x_2)| \leq C\lambda(|x_1 - x_2|) \quad \text{whenever } |x_1 - x_2| < \tfrac{1}{4} d(x_1, \partial\Omega).$$

Fix such an x_1 and x_2 and set $B = B(x_1, |x_1 - x_2|)$. Then $4B \subset \Omega$. Next if f is K-quasiregular in Ω_1, then there exists an $s > n$, depending only on n and K, such that $f \in W^1_{s,\mathrm{loc}}(\Omega_1)$. Moreover for every open set A compactly contained in Ω_1 we have

$$\left(\int_A |Df|^s\right)^{1/s} \leq C d(A, \partial\Omega_1)^{(n-s)/s} \left(\int_{\Omega_1} |Df|^n\right)^{1/n} \tag{3.4}$$

where C is a constant which depends only on n and K. Here $d(A, \partial\Omega_1)$ is the Euclidean distance between A and $\partial\Omega_1$. See [1]. When f is quasiconformal, this result is due to Gehring [5]. Since $f \in W^1_s(B)$, the following estimate holds for each x, $y \in B$:

$$|f(x) - f(y)| \leq \frac{sC(n)}{s - n} \operatorname{diam} B \left(\frac{1}{|B|} \int_B |Df|^s\right)^{1/s} \tag{3.5}$$

where $C(n)$ is a constant which depends only on n. Here diam B denotes the Euclidean diameter of B (see [1]).

Using (3.4), with $\Omega_1 = 3/2B$ and $A = B$, and (3.5), with $x = x_1$ and $y = x_2$, we obtain

$$(3.6)\qquad \begin{aligned} |f(x_1) - f(x_2)| &\le \frac{sC(n)}{s-n} \operatorname{diam} B \left(\frac{1}{|B|} \int_B |Df|^s \right)^{1/s} \\ &\le C_1 \operatorname{diam} B \left(\frac{1}{|B|} \int_{3B/2} |Df|^n \right)^{1/n}. \end{aligned}$$

Next if $f = (f_1, f_2, \dots, f_n)$ is K-quasiregular in Ω_1 and if D is a ball with $\sigma D \subset \Omega_1$, $\sigma > 1$, then for every $a \in R$ and $j = 1, 2, \dots, n$ we have

$$(3.7)\qquad \left(\int_D |Df|^n \right)^{1/n} \le CK \frac{\sigma}{\sigma - 1} \left(\frac{1}{|\sigma D|} \int_{\sigma D} |f_j - a|^n \right)^{1/n},$$

where C is a constant which depends only on n (see [10]). Using (3.7) with $D = 3/2B$ and $\Omega_1 = 2B$, (3.6) becomes

$$|f(x_1) - f(x_2)| \le C_2 \left(\frac{1}{|2B|} \int_{2B} |f_j - a|^n \right)^{1/n}.$$

Choose $a = f_j(x_1)$. When $x \in 2B$, $|x - x_1| \le \frac{1}{2} d(x_1, \partial\Omega)$ and since $f_j \in \operatorname{locLip}_\lambda(\Omega)$ we obtain

$$(3.8)\qquad \begin{aligned} |f(x_1) - f(x_2)| &\le C_2 \|f_j\|_\lambda^{\mathrm{loc}} \left(\frac{1}{|2B|} \int_{2B} \lambda(|x - x_1|)^n \, dx \right)^{1/n} \\ &\le C_3 \lambda(2|x_1 - x_2|) \\ &\le 2C_3 \lambda(|x_1 - x_2|). \end{aligned}$$

Hence $f \in \operatorname{locLip}_\lambda(\Omega)$.

3.7 COROLLARY. *If $f = (f_1, f_2, \dots, f_n)$ is K-quasiregular in Ω, then there exists a constant C, depending only on n and K, such that*

$$\omega(f, \delta; B) \le C\omega(f_j, \delta; B)$$

for all $j = 1, 2, \dots, n$ and all balls B with $2B \subset \Omega$.

Proof. Since $2B \subset \Omega$, f_j is bounded in B and $\omega(f_j, \delta; B)$ is a majorant. If $x, y \in \Omega$ with $|x - y| \le \frac{1}{4} d(x, \partial\Omega)$, then (3.8) gives

$$(3.9)\qquad |f(x) - f(y)| \le 2C_3 \omega(f_j, |x - y|; B_0)$$

where $B_0 = B(x, \frac{1}{4} d(x, \partial\Omega))$.

Now fix x_1, $x_2 \in B$. If $x_2 \in B(x_1, \frac{1}{4}d(x_1, \partial\Omega))$, then (3.9) holds with $x = x_1$ and $y = x_2$. Otherwise let $L \subset B$ be the line segment joining x_1 to x_2 and define points y_j and balls B_j as follows:

$$\begin{aligned} y_1 &= x_1 \\ B_j &= B\big(y_j, \tfrac{1}{4}d(y_j, \partial\Omega)\big) \\ y_{j+1} &\in L \cap \partial B_j \quad \text{where } |y_j - y_1| < |y_{j+1} - y_1|, \end{aligned}$$

for $j = 1, 2, \ldots, N-1$ where N is the integer such that $x_2 \in B(y_N, \frac{1}{4}d(y_N, \partial\Omega))$. If $r(B)$ is the radius of B and $y_{N+1} = x_2$, then $d(y_j, \partial\Omega) \geq r(B)$ for $j = 1, 2, \ldots, N+1$ and we have

$$\begin{aligned} \tfrac{1}{4}(N-1)r(B) &\leq \sum_{k=1}^{N-1} \tfrac{1}{4}d(y_k, \partial\Omega) \\ &= \sum_{k=1}^{N-1} |y_{k+1} - y_k| \\ &\leq |x_1 - x_2| \\ &\leq 2r(B) \end{aligned}$$

and $N \leq 9$. Hence

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq \sum_{k=1}^{N} |f(y_{k+1}) - f(y_k)| \\ &\leq 2C_3 \sum_{k=1}^{N} \omega\big(f_j, |y_{k+1} - y_k|; B\big) \\ &\leq 18C_3\omega\big(f_j, |x_1 - x_2|; B\big). \end{aligned}$$

The result follows.

The next result also follows from Theorem 3.6.

3.8 THEOREM. *If $f = (f_1, f_2, \ldots, f_n)$ is K-quasiregular in a* Lip_λ*-extension domain Ω and if $f_j \in \mathrm{Lip}_\lambda(\Omega)$ for some $j = 1, 2, \ldots, n$, then $f \in \mathrm{Lip}_\lambda(\Omega)$ with $\|f\|_\lambda \leq C\|f_j\|_\lambda$, where C is a constant which depends only on n, Ω, λ and K.*

We next present an example which shows that Theorem 3.8 fails in arbitrary domains.

3.9 *Example.* The function $f(re^{i\theta}) = \log r + i\theta$ is analytic in

$$\Omega = \{re^{i\theta} | 1 < r < \infty \quad \text{and } 0 < \theta < 2\pi\}.$$

When $0 < \alpha \leq 1$ and $1 < r_1 < r_2 < \infty$ we have

$$\log r_2 - \log r_1 \leq \frac{1}{\alpha}(r_2 - r_1)^{\alpha}.$$

Hence $\operatorname{Re} f \in \mathrm{Lip}_\lambda(\Omega)$, $\lambda(t) = t^{\alpha}$, for all $0 < \alpha \leq 1$. However if $z_1 = re^{i\varepsilon}$ and $z_2 = re^{i(2\pi - \varepsilon)}$, then

$$\lim_{\varepsilon \to 0} \frac{\operatorname{Arg} z_1 - \operatorname{Arg} z_2}{|z_1 - z_2|^{\alpha}} = \infty \quad \text{for all } 0 < \alpha \leq 1.$$

4.1 *Other moduli of continuity.* The unit disk D is a Lip_λ-extension domain if and only if there exist constants C and b such that

$$\int_0^{\delta} \frac{\lambda(t)}{t}\, dt \leq C\lambda(\delta) \tag{4.1}$$

whenever $0 < \delta <$ b. See [11]. The following theorem shows that the moduli of continuity of the components of an analytic function in D need not be equivalent when (4.1) fails. We use the notation $a = e^{-(p+2)}$,

$$\Lambda_p(t) = \begin{cases} 0 & \text{if } t = 0, \\ \dfrac{1}{\log\dfrac{1}{t}\left(\log\log\dfrac{1}{t}\right)^{p}} & \text{if } 0 < t \leq a, \\ \Lambda_p(a) & \text{if } a < t, \end{cases}$$

and

$$\lambda_p(t) = \begin{cases} 0 & \text{if } t = 0, \\ \log\left(\dfrac{1}{t}\right)\Lambda_p(t) & \text{if } 0 < t \leq a, \\ \lambda_p(a) & \text{if } a < t, \end{cases}$$

for $p \geq 1$. $\Lambda_p(t)$ and $\lambda_p(t)$ are concave majorants.

4.2 THEOREM. *For each $p \geq 1$ there exists an analytic function in D, $f = u + iv$, with $u \in \mathrm{Lip}_{\lambda_p}(D)$ and $v \notin \mathrm{Lip}_{\lambda_p}(D)$. Moreover when $p > 1$, $f \in \mathrm{Lip}_{\lambda_{p-1}}(D)$.*

Proof. First we define the boundary values of u as follows with $m = \Lambda_p(a)/a$. For $t \in [-\pi, \pi]$,

$$u(e^{it}) = \begin{cases} \Lambda_p(t) & \text{if } 0 < t \le a, \\ -m(t-2a) & \text{if } a < t < 2a, \\ 0 & \text{otherwise.} \end{cases}$$

If $|t_1 - t_2| < a$, then

$$|u(e^{it_1}) - u(e^{it_2})| \le (1 + \pi/2)\Lambda_p(|e^{it_1} - e^{it_2}|).$$

If $|t_1 - t_2| \ge a$, then

$$|u(e^{it_1}) - u(e^{it_2})| \le 2\Lambda_p(a) \le 2(1 + \pi/2)\Lambda_p(|e^{it_1} - e^{it_2}|).$$

Hence $u \in \mathrm{Lip}_{\Lambda_p}(\partial D)$. Since u is harmonic, when $\delta \le \frac{1}{2}$ Poisson's formula gives

$$\omega(u, \delta; \overline{D}) \le C_1 \log\left(\frac{1}{\delta}\right) \omega(u, \delta; \partial D) \quad \text{where } C_1 = \frac{3\pi \log \pi}{\log 2}.$$

Hence $u \in \mathrm{Lip}_{\lambda_p}(\overline{D})$. The conjugate function on the ∂D is given by

$$v(e^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\left(\frac{t-s}{2}\right) u(e^{is})\, ds.$$

$v(t)$ is bounded when $p > 1$ and assuming $0 < \delta < a$ we have

$$\begin{aligned} v(e^{-i\delta}) - v(1) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\cot\left(\frac{t}{2}\right) - \cot\left(\frac{t+\delta}{2}\right)\right] u(e^{it})\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\left(\frac{t}{2}\right)\left(u(e^{it}) - u(e^{i(t-\delta)})\right) dt. \end{aligned}$$

Hence

$$\begin{aligned} 2\pi|v(e^{-i\delta}) - v(1)| \ge & \int_0^{\delta} \cot\left(\frac{t}{2}\right) u(e^{it})\, dt \\ & - \left|\int_a^{a+\delta} \cot\left(\frac{t}{2}\right)\left(u(e^{it}) - u(e^{i(t-\delta)})\right) dt\right| \\ & - \left|\int_{a+\delta}^{2a+\delta} \cot\left(\frac{t}{2}\right)\left(u(e^{it}) - u(e^{i(t-\delta)})\right) dt\right| \\ \ge & \int_0^{\delta} \cot\left(\frac{t}{2}\right) u(e^{it})\, dt \\ & - 2\Lambda_p(a)\cot\left(\frac{a}{2}\right)\delta - \Lambda_p(a)\cot\left(\frac{a+\delta}{2}\right)\delta. \end{aligned}$$

We write $C_0 = 3\Lambda_p(a)\cot(a/2)$. Next

$$\begin{aligned}\int_0^\delta \cot\left(\frac{t}{2}\right)u(e^{it})\,dt &> \int_0^\delta \frac{u(e^{it})}{t}\,dt\\ &= \frac{1}{p-1}\left(\frac{1}{\log\log\frac{1}{\delta}}\right)^{p-1}\\ &= \frac{\lambda_{p-1}(\delta)}{p-1}.\end{aligned}$$

Since $\lim_{t\to 0^+}\lambda'_{p-1}(t) = \infty$ there exists $t_0 > 0$ such that $2(p-1)C_0 t \le \lambda_{p-1}(t)$ whenever $0 < t \le t_0$. Hence for δ sufficiently small,

$$2\pi\left|v(e^{-i\delta}) - v(1)\right| \ge \frac{1}{2(p-1)}\lambda_{p-1}(\delta).$$

Since

$$\lim_{t\to 0^+}\frac{\lambda_{p-1}(t)}{\lambda_p(t)} = \lim_{t\to 0^+}\log\log\frac{1}{t} = \infty,$$

$v \notin \mathrm{Lip}_{\lambda_p}(D)$.

To show that $f \in \mathrm{Lip}_{\lambda_{p-1}}(\overline{D})$ we apply the following general inequality (see [3, p. 106]). There is a constant C_1, independent of f and $\omega(u, t; \partial D)$, such that

$$\omega(v, \delta; \partial D) \le C_1\left(\int_0^\delta \frac{\omega(u, t; \partial D)}{t}\,dt + \delta\int_\delta^\pi \frac{\omega(u, t; \partial D)}{t^2}\,dt\right).$$

Since $\Lambda_p(t)/t$ is nonincreasing, we obtain

$$\begin{aligned}\omega(v, \delta; \partial D) &\le C_2\left(\int_0^\delta \frac{\Lambda_p(t)}{t}\,dt + \Lambda_p(\delta)\int_\delta^\pi \frac{1}{t}\,dt\right)\\ &= C_2\left(\left(\frac{1}{p-1}\right)\lambda_{p-1}(\delta) + \Lambda_p(\delta)(\log\pi - \log\delta)\right).\end{aligned}$$

Hence $v \in \mathrm{Lip}_{\lambda_{p-1}}(\partial D)$ and so $f \in \mathrm{Lip}_{\lambda_{p-1}}(\partial D)$. Since f is analytic it follows that $f \in \mathrm{Lip}_{\lambda_{p-1}}(\overline{D})$. Proofs of this last assertion can be found in [16] and [8]. This completes the proof of Theorem 4.2.

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