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SPECTRAL PROPERTIES OF TRANSLATION OPERATORS IN CERTAIN FUNCTION SPACES

BY

G.I. GAUDRY AND W. RICKER

0. Introduction

Let G be a locally compact abelian group and $1 \leq p \leq \infty$, $p \neq 2$. Then, except in trivial cases, translation operators in $L^p(G)$ are not spectral operators in the sense of N. Dunford [3]; see [2; Chapter 20] and [6], for example. However, it is natural to expect translations to be spectral in some sense because they are isometries of $L^p(G)$ onto itself and, hence, analogues of unitary operators in Hilbert space. For $1 < p < \infty$, this point was taken up in [1; §4] and [6], where it is shown that translations can indeed be expressed in the form

$$\int_{\mathbf{R}} e^{i\lambda}\, dQ(\lambda), \tag{1}$$

where $\{Q(\lambda); \lambda \in \mathbf{R}\}$ is an associated spectral family of commuting projections satisfying certain properties, [1; §4], and the "integral" (1), which can be interpreted as being over the unit circle $\mathbf{T}$ of the complex plane $\mathbf{C}$, exists in a certain well defined sense. On the other hand it should be stressed that in general the spectral family does not generate a σ-additive, projection-valued spectral measure.

However, as shown in the recent note [5], once it is realized that translation operators fail to be spectral for two very different reasons, then in many cases an alternative interpretation of (1) is possible. It may happen that the operator fails to be spectral simply because its domain space is "too small" to accommodate the projections needed to form its resolution of the identity. Accordingly, if interpreted as acting in a suitable space containing the domain space, it often happens that the operator is spectral in Dunford's sense. This has the advantage that the operator then has associated with it a rich functional calculus. For example, this is always the case if G is any locally compact abelian group and $1 < p \leq 2$, or if G is compact and $1 < p < \infty$ or if $2 < p < \infty$ and the element $g \in G$ defining the translation operator generates a compact metrizable subgroup of G; see [5]. In contrast, it is also the case, for fundamentally different reasons, that there exist non-trivial translations in $L^p(G)$, $2 < p < \infty$, for certain groups G, which are *genuinely*

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non-spectral operators; no change of domain space can remedy the situation, [5; Theorem 4.2].

So, for the case of $1 < p < \infty$, the spectral properties of L^p-translations are well understood [2; Chapter 20], [5], [6] and [7]. However, this is not so for translation operators in some other natural function spaces defined on locally compact abelian groups, such as $L^1(G)$, $L^\infty(G)$ and $C_0(G)$, for example. Indeed, not only are non-trivial translations in $L^1(G)$ and $L^\infty(G)$ non-spectral operators [6; Theorem 2] but, in contrast to the situation when $1 < p < \infty$, an "integral" representation of the form (1) may not even be available. This is already the case for the circle group $\mathbf{T}$ [6; Remark 5] and the additive group of integers $\mathbf{Z}$ [2; p. 394, Note 20.27]. Accordingly, the question of spectrality for translation operators in these spaces, in the wider sense of [5] suggested above (and made precise in §1), is an important one. Its solution is the purpose of this note. Our intention is not to strive for the utmost generality but, to illustrate the phenomena of significance: the distinction between the case $L^1(G)$ and that of $L^\infty(G)$ and $C_0(G)$. Of course, the results include at least the classical groups of analysis such as $\mathbf{R}^n$, $\mathbf{Z}^n$ and $\mathbf{T}^n$. It turns out that whereas translations in $L^1(G)$ are always spectral operators in some suitable space containing it (cf. Theorem 2.1), the situation for the spaces $L^\infty(G)$ and $C_0(G)$ is markedly different and depends on the group G and the subgroup of G generated by the element $g \in G$ determining the translation operator; see §2.

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1. Preliminaries and notation

Let X be a locally convex Hausdorff space, always assumed to be quasicomplete, X' its continuous dual and $L(X)$ the space of all continuous linear operators on X equipped with the topology of pointwise convergence in X. The identity operator is denoted by I. The adjoint of an operator T in X is denoted by T'.

A *spectral measure in* X is an $L(X)$-valued, σ-additive and multiplicative map P: $\mathscr{M} \to L(X)$, whose domain $\mathscr{M}$ is a σ-algebra of subsets of a set Ω, such that $P(\Omega) = I$. Of course, the multiplicativity of P means that $P(E \cap F) = P(E)P(F)$, for every $E \in \mathscr{M}$ and $F \in \mathscr{M}$. It follows from the Orlicz-Pettis lemma that an $L(X)$-valued function P on a σ-algebra $\mathscr{M}$ is σ-additive if and only if the $\mathbf{C}$-valued set function

$$\langle Px, x' \rangle : E \mapsto \langle P(E)x, x' \rangle, \quad E \in \mathscr{M},$$

is σ-additive for each $x \in X$ and $x' \in X'$. It follows that the range $\{P(E); E \in \mathscr{M}\}$, of P, is always a bounded subset of $L(X)$. If the range of P is an equicontinuous subset of $L(X)$, then we say that P is equicontinuous.

Let $P\colon \mathscr{M} \to L(X)$ be a spectral measure. An $\mathscr{M}$-measurable function f on Ω is said to be *P-integrable* if it is $\langle Px, x'\rangle$-integrable for every $x \in X$ and $x' \in X'$, and for each $E \in \mathscr{M}$ there is an operator $\int_E f\,dP$ in $L(X)$ such that

$$\left\langle \left(\int_E f\,dP \right) x, x' \right\rangle = \int_E f\,d\langle Px, x'\rangle,$$

for every $x \in X$ and $x' \in X'$. This definition of integrability agrees with that for more general vector measures [9].

An operator $T \in L(X)$ is called a *scalar-type spectral operator* if there exists a spectral measure P in X and a P-integrable function f such that $T = \int f\,dP$. This is the classical definition due to N. Dunford [3]. Additional hypotheses are needed to ensure that bounded measurable functions are P-integrable. For example, if the locally convex space $L(X)$ is known to be sequentially complete or if P is an equicontinuous measure, then this is the case; for example, see [9; Lemma II 3.1] and [11; Proposition 1.1], respectively. Another result in this direction, needed in the sequel, is the following:

LEMMA 1.1. *Let Y be a barrelled locally convex Hausdorff space and $Q\colon \mathscr{M} \to L(Y)$ be a spectral measure. Let X denote Y' equipped with its weak-star topology and for each $E \in \mathscr{M}$, let $P(E) = Q(E)'$. Then X is quasicomplete and the set function $P\colon \mathscr{M} \to L(X)$ so defined is a spectral measure for which bounded measurable functions are P-integrable.*

Proof. That X is quasicomplete is well known [12; Proposition IV 6.1]. Since $X' = Y$, it follows immediately from the identities

$$\langle Q(E)y, x\rangle = \langle y, P(E)x\rangle, \quad E \in \mathscr{M}, \tag{2}$$

valid for each $y \in X'$ and $x \in X$, and the Orlicz-Pettis lemma that P is σ-additive and hence, is a spectral measure. If f is a bounded $\mathscr{M}$-measurable function, then the equicontinuity of Q [12; Theorem III 4.2] ensures that f is Q-integrable. It follows easily from (2) that f is $\langle Px, x'\rangle$-integrable for each $x \in X$ and $x' \in X' = Y$ and the adjoint operators $(\int_E f\,dQ)' \in L(X)$ satisfy

$$\left\langle \left(\int_E f\,dQ \right) x, x' \right\rangle = \int_E f\,d\langle Px, x'\rangle, \quad E \in \mathscr{M},$$

for each $x \in X$ and $x' \in X'$. Accordingly, f is P-integrable and $\int_E f\,dP = (\int_E f\,dQ)'$ for each $E \in \mathscr{M}$.

Remarks. (1) It is worth noting that the space $L(X)$ in Lemma 1.1 need not be sequentially complete nor need the measure P be equicontinuous. For example, this can be shown to be the case if $Y = c_0(\mathbf{Z})$ and Q is the spectral measure in Y of multiplication (co-ordinatewise) by characteristic functions of subsets of $\mathbf{Z}$.

(2) The proof of Lemma 1.1 actually shows more, namely that any Q-integrable function is P-integrable. For non-normable spaces Y, this may include functions which are unbounded on a set of non-zero measure.

(3) If Y is a Banach space, then operators in $L(X)$ of the form $\int f\,dP$, where f is bounded and measurable, are called prespectral operators of class Y [2; Chapter 5].

LEMMA 1.2. *Let X be a quasicomplete space and P: $\mathscr{M} \to L(X)$ be a spectral measure for which all bounded $\mathscr{M}$-measurable functions are P-integrable. Let Z be a closed subspace of X which is invariant for each operator $P(E)$, $E \in \mathscr{M}$. Then each bounded $\mathscr{M}$-measurable function f is integrable with respect to the measure P_Z: $\mathscr{M} \to L(Z)$, where $P_Z(E)$ is the restriction of $P(E)$ to Z, for each $E \in \mathscr{M}$, and*

$$\int_E f\,dP_Z = \left(\int_E f\,dP\right)\Big|_Z, \quad E \in \mathscr{M}.$$

Proof. Let $E \in \mathscr{M}$. Then there exists a uniformly bounded sequence, $\{s_n\}_{n=1}^\infty$, of $\mathscr{M}$-simple functions converging pointwise to $f\chi_E$ on Ω. If $x \in X$, then it follows from the Dominated Convergence Theorem applied to the X-valued vector measure $P(\cdot)x$ [9; Theorem II 4.2] that

$$\lim_{n\to\infty}\left(\int_\Omega s_n\,dP\right)x = \lim_{n\to\infty}\int_\Omega s_n\,dPx = \int_\Omega f\chi_E\,dPx = \left(\int_E f\,dP\right)x.$$

Accordingly, $\{\int_\Omega s_n\,dP\}_{n=1}^\infty$ converges to $\int_E f\,dP$ in the space $L(X)$. Since Z is invariant for each operator $\int_\Omega s_n\,dP$, $n = 1, 2, \ldots,$ it is also invariant for $\int_E f\,dP$.

Let $z \in Z$ and $z' \in Z'$. Then there exists a functional $x' \in X'$ whose restriction to Z coincides with z'. It follows that the complex measures $\langle P_Z z, z'\rangle$ and $\langle Pz, x'\rangle$ agree on $\mathscr{M}$ and hence, that f is $\langle P_Z z, z'\rangle$-integrable. Furthermore, if $E \in \mathscr{M}$, then it follows (notation as above) that

$$\left\langle\left(\int_E f\,dP\right)\Big|_Z z, z'\right\rangle = \lim_{n\to\infty}\left\langle\left(\int_\Omega s_n\,dP\right)\Big|_Z z, z'\right\rangle = \lim_{n\to\infty}\left\langle\left(\int_\Omega s_n\,dP_Z\right)z, z'\right\rangle$$

$$= \lim_{n\to\infty}\int_\Omega s_n\,d\langle P_Z z, z'\rangle = \int_E f d\,\langle P_Z z, z'\rangle.$$

Remark. Although it will not be needed, a slight modification of the above argument shows that the conclusion of Lemma 1.2 is actually valid for any P-integrable function f.

LEMMA 1.3. *Let X be a quasicomplete space, $\mathscr{B}$ denote the σ-algebra of Borel subsets of the unit circle $\mathbf{T}$ and P: $\mathscr{B} \to L(X)$ be a spectral measure for which bounded $\mathscr{B}$-measurable functions are P-integrable. If $T = \int_{\mathbf{T}} \lambda \, dP(\lambda)$, then*

(i) *the operator T^{-1} exists in $L(X)$ and equals $\int_{\mathbf{T}} \bar{\lambda} \, dP(\lambda)$, and*

(ii) *a closed subspace Z of X is invariant for each operator $P(E)$, $E \in \mathscr{B}$, if and only if, Z is invariant for T and T^{-1}.*

Proof. (i) That the operator $S = \int_{\mathbf{T}} \bar{\lambda} \, dP(\lambda)$ exists is clear since the function $\lambda \to \bar{\lambda}$, $\lambda \in \mathbf{T}$, is bounded and measurable. If f and g are $\mathscr{B}$-simple functions, then it is clear from the multiplicativity of P that $\int_{\mathbf{T}} fg \, dP = (\int_{\mathbf{T}} f \, dP)(\int_{\mathbf{T}} g \, dP)$. Hence, if $\{s_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of $\mathscr{B}$-simple functions converging pointwise to the identity function on $\mathbf{T}$, g is a $\mathscr{B}$-simple function (in which case $gs_n \to \lambda g(\lambda)$ on $\mathbf{T}$) and $x \in X$, then it follows from the Dominated Convergence Theorem applied to the vector measure $P(\cdot)x$ in X, that

$$\begin{aligned}\int_{\mathbf{T}} \lambda g(\lambda) \, dP(\lambda)x &= \lim_{n \to \infty} \int_{\mathbf{T}} s_n(\lambda) g(\lambda) \, dP(\lambda)x \\ &= \lim_{n \to \infty} \left(\int_{\mathbf{T}} g(\lambda) \, dP(\lambda) \right) \int_{\mathbf{T}} s_n(\lambda) \, dP(\lambda)x \\ &= \left(\int_{\mathbf{T}} g(\lambda) \, dP(\lambda) \right) \int_{\mathbf{T}} \lambda \, dP(\lambda)x = \left(\int_{\mathbf{T}} g(\lambda) \, dP(\lambda) \right) Tx.\end{aligned}$$

Since $x \in X$ is arbitrary, it follows that

$$(3) \qquad \int_{\mathbf{T}} \lambda g(\lambda) \, dP(\lambda) = \left(\int_{\mathbf{T}} g(\lambda) \, dP(\lambda) \right) T = T \left(\int_{\mathbf{T}} g(\lambda) \, dP(\lambda) \right)$$

whenever g is a $\mathscr{B}$-simple function. In particular, (3) is true if g is replaced by $\bar{s}_n$, for each $n = 1, 2, \ldots$. Using this fact and applying (3) to an arbitrary element x of X it follows again from the Dominated Convergence Theorem applied to $P(\cdot)x$, observing $\bar{s}_n(\lambda) \to \bar{\lambda}$ pointwise on $\mathbf{T}$, that

$$x = \int_{\mathbf{T}} 1 \, dP(\lambda)x = \int_{\mathbf{T}} \lambda \bar{\lambda} \, dP(\lambda)x = TSx = STx,$$

and hence, $S = T^{-1}$.

(ii) If Z is invariant for each operator $P(E)$, $E \in \mathscr{B}$, then in the notation of (i), Z is invariant for the operator $\int_{\mathbf{T}} s_n \, dP$ and $\int_{\mathbf{T}} \bar{s}_n \, dP$, $n = 1, 2, \ldots$. But, it was shown in (i) that $\{\int_{\mathbf{T}} s_n\, dP\}_{n=1}^{\infty}$ and $\{\int_{\mathbf{T}} \bar{s}_n\, dP\}_{n=1}^{\infty}$ converge, in the topology of $L(X)$, to T and T^{-1}, respectively. Hence, Z is invariant for T and T^{-1}.

Conversely, suppose that Z is invariant for T and T^{-1}. Then Z is also invariant for each operator T^n, $n \in \mathbf{Z}$. If $Z^{\perp}$ denotes the annihilator of Z, then it follows for each $x \in Z$ and $x' \in Z^{\perp}$ that

$$0 = \langle T^n x, x' \rangle = \int_{\mathbf{T}} z^n \langle dP(z)x, x' \rangle, \quad n \in \mathbf{Z}.$$

That is, for each $x \in Z$ and $x' \in Z^{\perp}$ the complex measure $\langle Px, x' \rangle$ has zero Fourier-Stieltjes transform. Hence, if $E \in \mathscr{B}$ and $x \in Z$ are fixed, then $\langle P(E)x, x' \rangle = 0$ for each $x' \in Z^{\perp}$. This shows that $P(E)x \in Z^{\perp\perp} = Z$ whenever $x \in Z$. Accordingly, Z is invariant for each operator $P(E)$, $E \in \mathscr{B}$.

Remark. For equicontinuous measures P the proof of Lemma 1.3 can be substantially simplified using the well known functional calculus for P and continuity of the integration mapping; see [11; Proposition 1.1], for example.

The class of operators relevant to this note are those whose spectrum, in the sense of [12; p. 202], is a part of the unit circle $\mathbf{T}$. Of particular relevance will be the *pseudo-unitary operators*, that is, those scalar-type spectral operators T in $L(X)$ such that there exists a spectral measure $P\colon \mathscr{B} \to L(X)$ for which bounded measurable functions are P-integrable and

$$T = \int_{\mathbf{T}} \lambda \, dP(\lambda).$$

The spectrum of a pseudo-unitary operator is necessarily a subset of $\mathbf{T}$. Such operators are a natural generalization of unitary operators in Hilbert space.

Let $T \in L(X)$. A locally convex Hausdorff space Y is said to be *admissible for T* [11; p. 275] if there exist a continuous linear injection $\iota\colon X \to Y$ such that Y is the completion or quasicompletion of $\iota(X)$, and an operator T_Y in $L(Y)$, necessarily unique, such that

$$T_Y(\iota x) = \iota T x, \quad x \in X.$$

In this case the dual space Y' can be identified with the subspace $\{y' \circ \iota;\ y' \in Y'\}$ of X' which separates points of X. Sets bounded in X remain bounded in Y but, more importantly, sets which are unbounded in X may well be bounded in Y.

An operator T in $L(X)$ is said to be *extended pseudo-unitary* if there exists an admissible space Y for T such that T_Y is pseudo-unitary in Y. We remark that this is slightly more general than the definition of extended pseudo-unitary operator adopted in [5], where it was assumed that the spectral measure

$P\colon \mathscr{B} \to L(Y)$ satisfying $T_Y = \int_{\mathbf{T}} \lambda \, dP(\lambda)$ is equicontinuous. However, an examination of the proofs in [5] shows that the equicontinuity of P was only used to ensure that bounded measurable functions are P-integrable and that

$$\sup\{|\langle P(E)y, y'\rangle|; E \in \mathscr{B}, y \in A\} < \infty$$

whenever $y' \in Y'$ and A is a bounded subset of Y. However, since the range of P is a bounded subset of $L(Y)$ and Y is quasicomplete, the latter requirement is always satisfied; see Proposition III 3.3 and Corollary III 3.4 of [12], for example. Accordingly, if we relax the requirement of equicontinuity of P and substitute for it the hypothesis that bounded measurable functions are P-integrable, then the statements of results in [5] and their proofs are unaffected. This slight extension of the definition in [5] is not an exercise in generality but, as seen in Theorems 2.1 and 2.4, is one that suggests itself quite naturally in practice. It is for this reason that Lemmas 1.1–1.3 above are stated in a form which is perhaps more general than usual.

Let G be a locally compact abelian group. The space of continuous functions on G vanishing at infinity and the space of bounded continuous functions on G, both equipped with the uniform norm, are denoted by $C_0(G)$ and $C_b(G)$, respectively. The dual group of G is denoted by Γ and the value of $\gamma \in \Gamma$ at $g \in G$ is written as (g, γ). The Haar measures on G and Γ are assumed to be normalized so that the Fourier transform $f \mapsto \hat{f}$ is an isometry of $L^2(G)$ onto $L^2(\Gamma)$. If $1 < p < 2$, then the Fourier transform $f \to \hat{f}$ is an injective, norm-decreasing mapping of $L^p(G)$ onto a dense subspace of $L^q(\Gamma)$, where $q > 0$ is the number such that $p^{-1} + q^{-1} = 1$, which agrees with the ordinary Fourier transform

$$\hat{f}(\gamma) = \int_G (-g, \gamma) f(g) \, dg, \quad \gamma \in \Gamma,$$

on $L^1(G) \cap L^p(G)$. For the case of $p = 1$, the Fourier transform is an injective, norm-decreasing mapping of $L^1(G)$ onto a dense subspace of $C_0(\Gamma)$.

Let G be a locally compact abelian group and $g \in G$. Then ${}_gT$ denotes the operator of translation by g in any of the spaces $L^p(G)$, $1 \le p \le \infty$, $C_0(G)$ or $C_b(G)$. That is,

$${}_gT\colon f \mapsto f(\cdot + g)$$

where $f(\cdot + g)$ denotes the element $h \mapsto f(h + g)$ for a.e. $h \in G$ if $f \in L^p(G)$ for $1 \le p < \infty$, locally a.e. $h \in G$ if $f \in L^\infty(G)$, and every $h \in G$ if f is an element of $C_0(G)$ or $C_b(G)$. The notation for the operator ${}_gT$ does not indicate in which of the spaces $L^p(G)$, $1 \le p \le \infty$, $C_0(G)$ or $C_b(G)$ it is to be considered; this will always be clear from the context or will be explicitly specified. In each of these spaces ${}_gT$ is an isometry of the space onto itself and so its spectrum is a part of $\mathbf{T}$.

In the following result U denotes the unit right shift in $l^p(\mathbf{Z})$, $1 \le p \le \infty$.

LEMMA 1.4. *Let G be a locally compact abelian group and $g \in G$ have infinite order. Let $\mathscr{S}(G)$ denote any one of the spaces $C_0(G)$, $C_b(G)$ or $L^\infty(G)$. If there exist $p \in (2, \infty)$ and a continuous injection j: $l^p(\mathbf{Z}) \to \mathscr{S}(G)$ such that*

$$(4) \qquad {}_gT(j\xi) = jU\xi$$

for each $\xi \in l^p(\mathbf{Z})$, and

$$(5) \qquad {}_gT^{-1}(j\xi) = jU^{-1}\xi$$

for each $\xi \in l^p(\mathbf{Z})$, then ${}_gT$ is not an extended pseudo-unitary operator.

Proof. Suppose there is an admissible space Y for ${}_gT$ and ι: $\mathscr{S}(G) \to Y$ is the continuous imbedding of $\mathscr{S}(G)$ onto a dense subspace of Y such that

$$(6) \qquad {}_gT_Y(\iota f) = \iota({}_gTf), \quad f \in \mathscr{S}(G),$$

and ${}_gT_Y$ is pseudo-unitary. Then there is a spectral measure P: $\mathscr{B} \to L(Y)$ for which bounded measurable functions are P-integrable and ${}_gT_Y = \int_{\mathbf{T}} \lambda \, dP(\lambda)$.

Let Z denote the closure of $\iota(j(l^p(\mathbf{Z}))$ in Y. Then Z is quasicomplete or complete, depending on whether Y is quasicomplete or complete, respectively, and the composition $\tau = \iota \circ j$ is a continuous imbedding of $l^p(\mathbf{Z})$ onto a dense subspace of Z. It follows from (4) and (6) that

$$(7) \qquad {}_gT_Y(\tau\xi) = \iota({}_gTj\xi) = \iota(jU\xi) = \tau U\xi, \quad \xi \in l^p(\mathbf{Z}).$$

Since $U\xi \in l^p(\mathbf{Z})$ for each $\xi \in l^p(\mathbf{Z})$, this shows that $\tau(l^p(\mathbf{Z}))$ and hence, also Z, is invariant for ${}_gT_Y$. Lemma 1.3(i) implies that ${}_gT_Y$ is invertible in $L(Y)$ and so it follows that Y is also an admissible space for ${}_gT^{-1}$ with respect to ι and $({}_gT^{-1})_Y = {}_gT_Y^{-1}$, that is,

$$(8) \qquad {}_gT_Y^{-1}\iota f = \iota({}_gT^{-1}f), \quad f \in \mathscr{S}(G);$$

see [5; Lemma 1.1]. Using the identities (5) and (8), similar argument as for ${}_gT_Y$ shows that Z is also invariant for ${}_gT_Y^{-1}$. So, Lemma 1.3 (ii) implies that Z is invariant for each operator $P(E)$, $E \in \mathscr{B}$, and if $P_Z(E)$ denotes the restriction of $P(E)$ to Z, for each $E \in \mathscr{B}$, and ${}_gT_Z$ denotes the restriction of ${}_gT_Y$ to Z, then it follows from Lemma 1.2 that ${}_gT_Z = \int_{\mathbf{T}} \lambda \, dP_Z$. Accordingly, ${}_gT_Z$ is a pseudo-unitary operator. Furthermore, it is clear from (7) and the fact that $\tau\xi \in Z$ whenever $\xi \in l^p(\mathbf{Z})$, that Z is an admissible space for U with respect to τ and

$$ {}_gT_Z(\tau\xi) = \tau(U\xi), \quad \xi \in l^p(\mathbf{Z}). $$

Hence, U is an extended pseudo-unitary operator which is a contradiction; see the remark after Theorem 4.2 in [5]. Accordingly, ${}_gT$ is not an extended pseudo-unitary operator.

Remark. It is *not* assumed in the statement of Lemma 1.4 that $j(l^p(\mathbf{Z}))$ is dense in $\mathscr{S}(G)$.

2. Spectrality of translation operators

Let G be a locally compact abelian group. If $g \in G$ has finite order, then the operator ${}_gT$, in any of the spaces $L^p(G)$, $1 \le p \le \infty$, $C_0(G)$ or $C_b(G)$, is always a pseudo-unitary operator in that given space; see [13; Lemma 1], for example. However, if g has infinite order, which we will assume from now on, then it is known that ${}_gT$ is *not* a pseudo-unitary operator in $L^p(G)$, $1 \le p \le \infty$, $p \ne 2$ [6; Theorem 2]. For G the additive group of integers $\mathbf{Z}$ this result goes back to U. Fixman [4; Theorem 5.7]. As remarked in the introduction, the problem of whether or not the operator ${}_gT$, considered in any of the spaces $L^p(G)$, $1 < p < \infty$, $p \ne 2$, is extended pseudo-unitary was considered in [5]. In this section we consider the same question for ${}_gT$ in the spaces $L^1(G)$, $L^\infty(G)$, $C_0(G)$ and $C_b(G)$.

THEOREM 2.1. *Let G be a locally compact abelian group and $g \in G$ have infinite order. Then the translation operator ${}_gT$: $L^1(G) \to L^1(G)$ is extended pseudo-unitary.*

Proof. Let j: $L^1(G) \to C_0(\Gamma)$ denote the Fourier transform map. Then j is a continuous imbedding of $L^1(G)$ onto a dense subspace of $C_0(\Gamma)$. It follows from properties of the Fourier transform that

$$ j({}_gTf)(\gamma) = (g, \gamma)j(f)(\gamma), \quad \gamma \in \Gamma, \tag{9} $$

for each $f \in L^1(G)$. If Y denotes $L^\infty(\Gamma)$ equipped with its weak-star topology, then Y is a quasicomplete locally convex space [12; Proposition IV 6.1] and the natural inclusion ρ: $C_0(\Gamma) \to Y$ is continuous, injective and its range is dense in Y. Accordingly, the composition

$$ \iota = \rho \circ j \colon L_1(G) \to Y $$

is a continuous imbedding of $L^1(G)$ onto a dense subspace of Y. Furthermore, if ${}_gT_Y \in L(Y)$ denotes the operator of multiplication by $(g, \cdot)$, then it follows from (9) that

$$ {}_gT_Y(\iota f) = \iota({}_gTf), \quad f \in L^1(G). $$

This shows that Y is an admissible space for ${}_gT$.

So, it remains only to show that ${}_gT_Y$ is pseudo-unitary. Let $\mathscr{M}$ denote the σ-algebra of Borel subsets of Γ and define P: $\mathscr{M} \to L(Y)$ by $P(E)h = \chi_E h$, for each $h \in Y$ and $E \in \mathscr{M}$. Since each operator $P(E)$, $E \in \mathscr{M}$, is the adjoint of the operator $Q(E)$ in $L(L^1(\Gamma))$ given by $Q(E)\xi = \chi_E\xi$, for each $\xi \in L^1(\Gamma)$, it follows that Q is a spectral measure, necessarily equicontinuous, and hence Lemma 1.1 implies that P: $\mathscr{M} \to L(Y)$ is a spectral measure for which bounded measurable functions are P-integrable. Furthermore, it is easily established that

$$ {}_gT_Y = \int_\Gamma (g, \gamma)\, dP(\gamma). $$

Hence, if $\tilde{P}$: $\mathscr{B} \to L(Y)$ denotes the spectral measure specified by

$$ \tilde{P}(E) = P(\{\gamma \in \Gamma; (g, \gamma) \in E\}), \quad E \in \mathscr{B}, $$

then it follows that bounded $\mathscr{B}$-measurable functions on $\mathbf{T}$ are $\tilde{P}$-integrable and, in particular, that ${}_gT_Y = \int_{\mathbf{T}} \lambda\, d\tilde{P}(\lambda)$; see [2; Proposition 5.8], for example. Accordingly, ${}_gT$ is pseudo-unitary.

Remark. If Γ is σ-compact, a mild restriction in practice, then it can be shown that any of the Fréchet spaces $Y = L^p_{\mathrm{loc}}(\Gamma)$, $1 \le p < \infty$, of (equivalence classes of) locally p-integrable functions is also an admissible space for ${}_gT$ in which ${}_gT$ is extended pseudo-unitary. This has the advantage that the space of operators $L(Y)$ is then quasicomplete [12; Corollary III 4.4] and the spectral measure P: $\mathscr{B} \to L(Y)$ for which ${}_gT_Y = \int_{\mathbf{T}} \lambda\, dP(\lambda)$ is equicontinuous [12; Theorem III 4.2].

The following result, dealing with the extended pseudo-unitariness of translations in *compact* abelian groups G, should be compared with Theorem 4.3 of [5]. Of course, in this case $C_0(G) = C_b(G)$ is simply the space, $C(G)$, of all continuous functions on G equipped with the uniform norm. Since the proof is obvious it will be omitted.

THEOREM 2.2. *Let G be a compact abelian group and $g \in G$ have infinite order. If $\mathscr{S}(G)$ denotes either of the spaces $L^\infty(G)$ or $C(G)$, then the Hilbert space $L^2(G)$ is an admissible space for the translation operator ${}_gT$: $\mathscr{S}(G) \to \mathscr{S}(G)$ in which ${}_gT$ is extended pseudo-unitary.*

Remark. It is worth noting that under the hypotheses of Theorem 2.2 there are many admissible spaces in which ${}_gT$ is extended pseudo-unitary. For example, let $1 \le p < \infty$. Then the natural inclusion, ι_p, of $\mathscr{S}(G)$ into $L^p(G)$

is a continuous imbedding of $\mathscr{S}(G)$ onto a dense subspace of $L^p(G)$ and the extension, with respect to ι_p, of ${}_gT \in L(\mathscr{S}(G))$ to the admissible space $L^p(G)$ is again the operator of translation by g. Accordingly, if $1 < p < 2$, then it follows from [5; Theorem 2.1] that ${}_gT$ is an extended pseudo-unitary operator in the admissible space $L^q(\Gamma)$. Or, in the case of $p = 1$, it follows (cf. proof of Theorem 2.1 above) that $L^\infty(\Gamma)$ equipped with its weak-star topology is also an admissible space in which ${}_gT$ is extended pseudo-unitary. If, in addition, Γ is σ-compact, then it follows that ${}_gT$ is an extended pseudo-unitary operator in any of the admissible spaces $L^r_{\mathrm{loc}}(\Gamma)$, $1 \leq r < \infty$ (cf. remark after Theorem 2.1).

If the group G is not compact, then Theorems 4.2 and 4.6 in [5] suggest that whether or not ${}_gT$, considered as an operator in one of the spaces $L^\infty(G)$, $C_0(G)$ or $C_b(G)$, is extended pseudo-unitary ought to depend on the nature of the closed subgroup $\overline{\langle g \rangle}$, of G, generated by the element g of G. This turns out indeed to be the case: the operator ${}_gT$ is extended pseudo-unitary whenever $\overline{\langle g \rangle}$ is compact and metrizable (cf. Theorem 2.4). However, if $\overline{\langle g \rangle}$ is isomorphic to $\mathbf{Z}$, then, as the following result shows, exactly the opposite is the case.

THEOREM 2.3. *Let G be a locally compact abelian group and g be an element of G such that $\overline{\langle g \rangle}$ is isomorphic to $\mathbf{Z}$. If $\mathscr{S}(G)$ denotes any one of the spaces $C_0(G)$, $C_b(G)$ or $L^\infty(G)$, then ${}_gT$, considered as an element of $L(\mathscr{S}(G))$, is not an extended pseudo-unitary operator.*

Proof. Since $\overline{\langle g \rangle}$ is isomorphic to $\mathbf{Z}$ there exists a continuous function on G, say Δ, with compact support and values in $[0,1]$ such that $\Delta(e) = 1$, where e is the identity element of G, and the supports of $\{{}_gT^k\Delta;\ k \in \mathbf{Z}\}$ are disjoint. Fix any $p \in (2, \infty)$ and define a linear mapping $j\colon l^p(\mathbf{Z}) \to \mathscr{S}(G)$ by

$$j\colon \xi \mapsto \sum_{n=-\infty}^{\infty} \xi(n)\,{}_gT^n\Delta, \quad \xi \in l^p(\mathbf{Z}).$$

It follows from the disjointness of the supports of $\{{}_gT^k\Delta;\ k \in \mathbf{Z}\}$ that j is injective and continuous. Actually, $\|j\xi\|_{\mathscr{S}(G)} \leq \|\xi\|_p$ for each $\xi \in l^p(\mathbf{Z})$. Furthermore, the identities (4) and (5) follow immediately from the definition of j and hence, the desired conclusion follows from Lemma 1.4.

Remark. It follows from Theorem 2.3, in particular, that the classical bilateral unit shift operators in $c_0(\mathbf{Z})$ and $C_0(\mathbf{R})$ are not scalar-type spectral operators.

THEOREM 2.4. *Let G be an abelian, σ-compact group and $g \in G$ be such that $\overline{\langle g \rangle}$ is compact and metrizable. If $\mathscr{S}(G)$ denotes any one of the spaces $L^\infty(G)$, $C_0(G)$ or $C_b(G)$, then the translation operator ${}_gT\colon \mathscr{S}(G) \to \mathscr{S}(G)$ is extended pseudo-unitary.*

The proof will be via a series of lemmas.

Let G_0 be any closed subgroup of a locally compact abelian group G. A subset B of G is said to be a *Borel section* for the quotient group G/G_0 if B is a Borel measurable subset of G and each coset of G_0 in G contains precisely one point of B. The *associated transversal mapping* τ is the 1-1 mapping of G/G_0 onto B such that

$$\tau(b + G_0) = b, \quad b \in B.$$

It is known that if G_0 is compact and metrizable, then there is a Borel section B for G/G_0 whose associated transversal mapping τ is Borel measurable from G/G_0 onto B [5; Lemma 4.4], and hence τ induces an identification of the Haar measure on G/G_0 with a measure on B denoted by db. It follows from the Borel measurability of τ that the mapping ρ of $B \times G_0$ onto G defined by the formula

$$\rho(b, h) = b + h, \quad (b, h) \in B \times G_0,$$

is a Borel isomorphism.

Let $\mathscr{W}$ denote the Banach space $L^1(db; L^2(G_0))$ of Bochner integrable functions from B into $L^2(G_0)$, realized as (equivalence classes of) Borel measurable functions F on $B \times G_0$ such that

$$|||F|||_1 = \int_B \left(\int_{G_0} |F(b, h)|^2 \, dh \right)^{1/2} db = \int_B \|F(b, \cdot)\|_2 \, db < \infty,$$

where $\| \cdot \|_2$ denotes the norm in $L^2(G_0)$. In the case when G_0 is compact and metrizable, the Hilbert space $L^2(G_0)$ is separable and hence, the weak measurability of an element of $\mathscr{W}$ is equivalent to its strong measurability. If G is σ-compact, then db is a σ-finite measure. Accordingly, since $L^2(G_0)$ has the Radon-Nikodým property, it follows that the dual space $X = \mathscr{W}'$ can be identified with $L^\infty(db; L^2(G_0))$, that is, the space of (equivalence classes of) Borel measurable functions H on $B \times G_0$ such that

$$|||H|||_\infty = \operatorname*{ess\,sup}_{b \in B} \left(\int_{G_0} |H(b, h)|^2 \, dh \right)^{1/2} = \operatorname*{ess\,sup}_{b \in B} \|H(b, \cdot)\|_2 < \infty.$$

The duality of $\mathscr{W}$ and X is given by

$$\langle F, H \rangle = \int_B \int_{G_0} F(b, h) H(b, h) \, dh \, db, \quad F \in \mathscr{W}, \quad H \in X.$$

For the remainder of this section we assume that G is σ-compact, $G_0 = \overline{\langle g \rangle}$ is compact and metrizable, B is a Borel section for G/G_0 and $\mathscr{W}$ and X are the associated Banach spaces as described above.

Let ${}_{-g}\tilde{t}$ be the operator of translation by $-g$ acting in $L^2(G_0)$ and write ${}_{-g}T_{\mathscr{W}}$ for the operator in $\mathscr{W}$ defined by

(10)
$$ {}_{-g}T_{\mathscr{W}}F\colon (b,h) \mapsto F(b,h-g) = \left[{}_{-g}\tilde{t}F(b,\cdot)\right](h), \quad (b,h) \in B \times G_0, $$

for each $F \in \mathscr{W}$. Then ${}_{-g}T_{\mathscr{W}}$ is an isometry of $\mathscr{W}$ onto $\mathscr{W}$.

LEMMA 2.5. *The operator ${}_{-g}T_{\mathscr{W}}$ is pseudo-unitary in $\mathscr{W}$.*

Proof. Since ${}_{-g}\tilde{t}$ is unitary in $L^2(G_0)$ there is a spectral measure

$$ Q\colon \mathscr{B} \to L\left(L^2(G_0)\right) $$

such that ${}_{-g}\tilde{t} = \int_{\mathbf{T}}\lambda\, dQ(\lambda)$. Define a set function $Q_{\mathscr{W}}\colon \mathscr{B} \to L(\mathscr{W})$ by the formula

$$ Q_{\mathscr{W}}(E)F\colon (b,h) \mapsto [Q(E)F(b,\cdot)](h), \quad (b,h) \in B \times G_0,\ F \in \mathscr{W}, $$

for each $E \in \mathscr{B}$. That this component-wise definition of $Q_{\mathscr{W}}(E)$ makes sense can be verified as in the proof of Theorem 4.6 in [5]. Furthermore, since $\|Q(E)\| \le 1$ for each $E \in \mathscr{B}$, it follows that also $\|Q_{\mathscr{W}}(E)\| \le 1$, $E \in \mathscr{B}$. Since $Q_{\mathscr{W}}(\mathbf{T}) = I$ and $Q_{\mathscr{W}}$ is multiplicative, to show that $Q_{\mathscr{W}}$ is a spectral measure it suffices to show that it is σ-additive. This can be established using the uniform boundedness of the family $\{Q_{\mathscr{W}}(E);\, E \in \mathscr{B}\}$, the Orlicz-Pettis lemma and the identities

$$ \text{(11)} \qquad \langle Q_{\mathscr{W}}(E)F, H\rangle = \int_B \langle Q(E)F(b,\cdot), H(b,\cdot)\rangle_2\, db, \quad E \in \mathscr{B}, $$

valid for each $F \in \mathscr{W}$ and $H \in X$, where $\langle\,\cdot\,,\cdot\,\rangle_2$ denotes the duality in $L^2(G_0)$. Finally, that ${}_{-g}T_{\mathscr{W}}$ is the operator $\int_{\mathbf{T}}\lambda\, dQ_{\mathscr{W}}(\lambda)$ can be verified by an argument similar to that in the proof of Theorem 4.6 in [5]. This completes the proof.

Let ${}_{g}t = ({}_{-g}\tilde{t})'$ denote the dual operator to ${}_{-g}\tilde{t}$. Then the dual operator $({}_{-g}T_{\mathscr{W}})' \in L(X)$ is given by

(12)
$$ ({}_{-g}T_{\mathscr{W}})'H\colon (b,h) \mapsto H(b,h+g) = \left[{}_{g}tH(b,\cdot)\right](h), \quad (b,h) \in B \times G_0, $$

for each $H \in X$. If X_* denotes X equipped with its weak-star topology as the dual space to $\mathscr{W}$, then X_* is quasi-complete and $({}_{-g}T_{\mathscr{W}})'$ is an element of $L(X_*)$.

LEMMA 2.6. *Considered as an element of $L(X_*)$ the operator $({}_{-g}T_{\mathscr{W}})'$ is pseudo-unitary.*

Proof. If $P(E) \in L(L^2(G_0))$ denotes the dual operator $Q(E)'$ and $P_*(E) \in L(X_*)$ denotes $Q_{\mathscr{W}}(E)'$, for each $E \in \mathscr{B}$, then it follows that

$$P_*(E)H\colon (b, h) \mapsto [P(E)H(b, \cdot)](h), \quad (b, h) \in B \times G_0, \tag{13}$$

for each $h \in X_*$. Then the identities (10)–(13) and Lemma 1.1 (cf. also its proof) imply that $P_*\colon \mathscr{B} \to L(X_*)$ is a spectral measure for which bounded measurable functions are integrable and $({}_{-g}T_{\mathscr{W}})' = \int_{\mathbf{T}} \lambda \, dP_*(\lambda)$.

Consider now the case of $\mathscr{S}(G) = L^\infty(G)$. Define a linear mapping ι: $L^\infty(G) \to X_*$ by

$$\iota f\colon (b, h) \mapsto f(b + h), \quad (b, h) \in B \times G_0, \tag{14}$$

for each $f \in L^\infty(G)$. It is not immediately obvious that ι is well-defined or assumes its values in X_*. We indicate the details.

First, the σ-compactness of G together with Weil's formula [10; Proposition 3.4.9] imply that if N is a null set in G, then

$$\int_{G_0} \chi_N(t + h) \, dh = 0, \tag{15}$$

for a.e. $\dot{t} \in G/G_0$ (the dot indicates coset). Recalling that db is induced by Haar measure from G/G_0 it follows that if f is an (individual) essentially bounded function on G, then the function $b \mapsto f(b + \cdot)$ makes sense and its equivalence class in $X = L^\infty(db; L^2(G_0))$ remains the same if f is altered on a null set. This shows that (14) is at least well defined. If $\xi \in L^2(G_0)$, then the function

$$(b, h) \mapsto f(b + h)\xi(h),$$

which is measurable on $B \times G_0$, satisfies

$$\int_K \int_{G_0} |f(b + h)\xi(h)| \, dh \, db < \infty$$

whenever $K \subseteq B$ is a set of finite db-measure and so a Fubini Theorem type argument (cf. [8; p. 154], for example) shows that

$$b \to \int_{G_0} f(b + h)\xi(h) \, dh = \langle f(b + \cdot), \xi \rangle_2 = \langle [\iota f](b, \cdot), \xi \rangle, \quad b \in B,$$

is measurable. Since this is so for every $\xi \in L^2(G_0)$ it follows that $b \mapsto$

$[\iota f](b, \cdot)$ is weakly measurable, hence also strongly measurable as $L^2(G_0)$ is separable. Finally, since dh is a probability measure in G_0 it follows that

$$\|[\iota f](b, \cdot)\|_2 = \left(\int_{G_0} |f(b+h)|^2 \, dh \right)^{1/2} \leq \|f(b + \cdot)\|_\infty$$

where $\|\cdot\|_\infty$ is the norm in $L^\infty(G_0)$. This observation, together with Weil's formula and the fact that db is induced by Haar measure on G/G_0 imply that

$$(16) \qquad |||\iota f|||_\infty = \operatorname*{ess\,sup}_{b \in B} \|[\iota f](b, \cdot)\|_2 \leq \operatorname*{ess\,sup}_{t \in G} |f(t)| < \infty.$$

This shows that $\iota f \in X_*$. It then follows from (15) that ι is injective. Furthermore, (16) shows that ι: $L^\infty(G) \to X$ is continuous and hence, so is ι: $L^\infty(G) \to X_*$. Summarizing, we have proved:

LEMMA 2.7. *The linear mapping* ι: $L^\infty(G) \to X_*$ *defined by* (14) *is a continuous injection.*

We can now establish Theorem 2.4 (when $\mathscr{S}(G) = L^\infty(G)$). Indeed, let Y denote the closure of $\iota(L^\infty(G))$ in X_*. Then Y is quasi-complete and $\iota(L^\infty(G))$ is dense in Y. If $f \in L^\infty(G)$, then (12) and (14) imply that

$$(17) \qquad \left({}_{-g}T_{\mathscr{W}}\right)'\iota f = \iota\left({}_{g}Tf\right),$$

from which it is clear that Y is invariant for $({}_{-g}T_{\mathscr{W}})'$. Observing that the inverse operator, in $L(X)$, to $({}_{-g}T_{\mathscr{W}})'$ is given by

$$\left(({}_{-g}T_{\mathscr{W}})'\right)^{-1}H\colon (b, h) \mapsto H(b, h - g), \quad (b, h) \in B \times G_0,$$

for each $H \in X$, it can be similarly argued that Y is also invariant for $(({}_{-g}T_{\mathscr{W}})')^{-1}$. Accordingly, if P_*: $\mathscr{B} \to L(X_*)$ is the measure (13), then it follows from Lemmas 2.6 and 1.3 that Y is invariant for each operator $P_*(E)$, $E \in \mathscr{B}$. Let ${}_gT_Y$ denote the restriction of $({}_{-g}T_{\mathscr{W}})'$ to Y and $P_Y(E)$ denote the restriction of $P_*(E)$ to Y, for each $E \in \mathscr{B}$. Then Lemma 1.2 implies that ${}_gT_Y = \int_{\mathbf{T}} \lambda \, dP_Y(\lambda)$, that is, ${}_gT_Y$ is pseudo-unitary. Since Y is an admissible space for ${}_gT$ with respect to the continuous imbedding ι: $L^\infty(G) \to Y$ (cf. Lemma 2.7 and (17)) it follows that ${}_gT$ is an extended pseudo-unitary operator in Y. This completes the proof of Theorem 2.4 for the case when $\mathscr{S}(G) = L^\infty(G)$.

Suppose now that $\mathscr{S}(G)$ is either $C_0(G)$ or $C_b(G)$. The natural inclusion, κ, of $\mathscr{S}(G)$ into $L^\infty(G)$ is an injective isometry. If $j = \iota \circ \kappa$ where ι is given by (14), and Z denotes the closure of $j(\mathscr{S}(G))$ in X_*, then Z is quasicomplete and j is a continuous injection with range dense in Z. Since $\kappa(\mathscr{S}(G))$ is invariant for translation by g and $-g$ in $L^\infty(G)$ it follows that $j(\mathscr{S}(G))$, hence also Z, is invariant for $({}_{-g}T_{\mathscr{W}})' \in L(X_*)$ and its inverse. A similar argument as in the case of $L^\infty(G)$ shows that Z is an admissible space for ${}_gT \in L(\mathscr{S}(G))$ with respect to j; the operator ${}_gT_Z$ is just the restriction of $({}_{-g}T_{\mathscr{W}})'$ to Z. Since ${}_gT_Z$ is pseudo-unitary in Z (by Lemma 1.2 its resolution of the identity is P_* restricted to Z) it follows that ${}_gT$ is an extended pseudo-unitary operator in Z.

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The Flinders University of South Australia
Bedford Park, Australia,
Macquarie University
North Ryde, Australia