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SOME PROPERTIES OF COMMUTATIVE RING EXTENSIONS

BY

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Introduction

Throughout this paper R will be a commutative ring with one, and all R-modules will be unitary. Our purpose is to study commutative ring extensions S of R. As a key tool in our investigation we introduce a torsion functor t_S on R-modules which is determined by the set of all ideals I of R such that $IS = S$. If R is an integral domain with quotient field Q, then t_Q is the classical torsion functor. We also introduce the category $\mathscr{F}_S$ of all R-modules that are R-submodules of S-modules. Again if R is an integral domain, then $\mathscr{F}_Q$ is the category of classical torsion-free R-modules.

In §1 we show that if an R-module satisfies certain conditions, then it is in fact a commutative ring extension of R. We also derive some of the properties of the category $\mathscr{F}_S$; in particular, that it is closed under direct limits.

In §2 we discuss faithful torsion functors on R-modules in general, and apply these results to the special case of the torsion functor t_S. We let E denote the injective envelope of R, $\mathfrak{A}$ the set of all faithful ideals of R, $t_{\mathfrak{A}}$ the torsion functor determined by $\mathfrak{A}$, and U/R the $t_{\mathfrak{A}}$-torsion submodule of E/R. Using the results of §1 we show that U is a commutative ring extension of R contained in E; and if t is any faithful torsion functor and $t(E/R) = T/R$, then T is a commutative subring of U.

If $\mathscr{J}$ is the set of all ideals of R that contain a faithful, finitely generated ideal of R, then $\mathscr{J}$ is equal to the set of all ideals I of R such that $IE = E$. We denote the associated torsion functor of $\mathscr{J}$ by t_E; and if $t_E(E/R) = V/R$, then V is a commutative subring of U containing R.

Let S be a commutative ring extension of R, and $T/R = t_S(S/R)$; then T is a subring of S and is isomorphic to a unique subring of V that we identify with T. We find that $\mathscr{F}_S$ is equal to the category of all t_S-torsion-free R-modules if and only if $\mathscr{F}_S$ is closed under essential extensions, a property somewhat weaker than S being flat. We prove that S is flat if and only if $\operatorname{Tor}_1^R(B, S/R)$ is t_S-torsion for all R-modules B.

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In §3 we define a commutative ring extension S of R to be a torsion envelope of R if S/R is t_S-torsion. In this case S is isomorphic to a unique subring of V and $t_S(E/R) = S/R$. We prove that S is a torsion envelope of R if and only if S is isomorphic to a flat subring of V. We also prove that S is a torsion envelope of R if and only if S is a flat ring bijection of R, a type of ring that has been studied by Lazard [3]. We establish many of the important properties of torsion envelopes, some of which have already been noted by Lazard [3]. We show that there is a unique maximal torsion envelope of R contained in V that contains a copy of every other torsion envelope of R. We also show that S is a torsion envelope of R if and only if either $PS = S$ or $S_P \cong R_P$ for every $P \in \operatorname{Spec} R$.

In §4 we study integral domains R and their over-rings (i.e., the subrings of the quotient field Q of R that contain R) from the vantage point of the theory we have developed. By the results of §3, S is a torsion envelope of R if and only if S is isomorphic to a flat over-ring of R. We show that S is a flat over-ring of R if and only if $S = \bigcap R_P$ ($P \in \operatorname{Spec} R$ and $PS \neq S$). Part of these two results were shown by Richman [8] using different methods and terminology. More generally we let $\mathscr{C}$ be any set of prime ideals of R and study the properties of $S = \bigcap R_P$ ($P \in \mathscr{C}$).

We define R to be a semi-Krull domain if every non-zero, principal ideal of R is a finite intersection of height one primary ideals of R, and every height one primary ideal of R contains a power of its associated prime ideal. In this case $R = \bigcap R_P$ ($P \in \operatorname{Spec} R$ and height $P = 1$). Semi-Krull domains are simultaneously generalizations of Krull domains and Noetherian, Cohen-Macaulay domains. We characterize the flat over-rings of semi-Krull domains and prove that they are also semi-Krull domains.

If R is a semi-Krull domain, I a non-zero ideal of R, and S the ideal transform of I, then $S = \bigcap R_P$ ($P \in \mathscr{C}_1$) where $\mathscr{C}_1$ is the set of all height one prime ideals of R that do not contain I. A more general result holds for Noetherian domains. If I is a finite intersection of height one primary ideals of R, then S is flat if and only if $S = IS$, a generalization of the case of projective ideals of R.

In §5, R is again an integral domain and we say that an over-ring S of R is a complemented extension of R if S/R is a direct summand of Q/R. Of course S is then a flat over-ring of R. If R is a semi-Krull or Noetherian domain, I a non-zero ideal of R, S the ideal transform of I, and $\mathscr{S} = \{1 - a \mid a \in I\}$, then S and $R_{\mathscr{S}}$ are complementary extensions of R if and only if I is a finite intersection of height one primary ideals of R whose associated prime ideals are maximal ideals of R. This result enables us to show that if R is a semi-Krull or Noetherian domain, $\mathscr{C}'$ a non-empty set of height one maximal ideals of R, and $\mathscr{C} = \operatorname{Spec} R - \mathscr{C}'$, then $S = \bigcap R_P$ ($P \in \mathscr{C}$) is a flat over-ring of R. In this case S is also semi-Krull or Noetherian; and if $P \in \operatorname{Spec} R$, then $PS = S$ if and only if $P \in \mathscr{C}'$. We apply these results to the case where I is a finite intersection of height one primary ideals of R and show that if $E(R/I) \subset Q/I$, then every associated prime ideal of I is a maximal ideal of R. This leads us finally to the corollary that if R is a semi-Krull domain such that $\text{inj.dim}_R R = 1$, then R is a Noetherian, Gorenstein domain of Krull dimension one.

In §6 we turn again to the situation where R is an arbitrary commutative ring. We let $\mathscr{F}$ denote a category of R-modules that is closed under sub-modules. We define a pair (F, ϕ) to be an $\mathscr{F}$-lifting of an R-module B if $F \in \mathscr{F}$, ϕ: $F \to B$, and every other such pair can be factored through (F, ϕ). An $\mathscr{F}$-lifting (F, ϕ) of B is called an $\mathscr{F}$-cover of B if f: $F \to F$ and $\phi f = \phi$ implies that f is an automorphism of F. $\mathscr{F}$-covers, if they exist are unique. Following a proof of Enochs [2] concerning flat covers we show that if $\mathscr{F}$ is closed under direct limits, then every $\mathscr{F}$-lifting contains a direct summand that is an $\mathscr{F}$-cover.

Let S be a commutative ring extension of R, B an R-module, C the injective envelope of B, and ψ: $\text{Hom}_R(S, C) \to C$ the map defined by $\psi(f) = f(1)$. We let $F = \psi^{-1}(B)$ and $\phi = \psi | F$; then (F, ϕ) is an $\mathscr{F}_S$-lifting of B. Thus, since $\mathscr{F}_S$ is closed under direct limits, (F, ϕ) contains a direct summand that is an $\mathscr{F}_S$-cover of B. We prove that S is a torsion envelope of R if and only if (F, ϕ) is the $\mathscr{F}_S$-cover of B for every R-module B. This generalizes a result of Banaschewski for torsion-free covers over integral domains [1]. We also show that $\mathscr{F}_S$ is equal to the category of all t_s-torsion free R-modules if and only if the $\mathscr{F}_S$-cover of an injective R-module is R-injective.

1. Some general properties of commutative ring extensions

DEFINITION. Let A be an R-module such that $R \subset A$, and define the canonical map

$$\psi_A: \text{Hom}_R(A, A) \to A$$

by $\psi_A(f) = f(1)$ for all $f \in \text{Hom}_R(A, A)$.

PROPOSITION 1.1. *ψ_A is an R-isomorphism if and only if A is a commutative ring extension of R such that* $\text{Hom}_R(A/R, A) = 0$. *In this case there is only one ring structure on A that is compatible with its R-module structure.*

Proof. Suppose that ψ_A is an R-isomorphism. Then $\text{Hom}_R(A/R, A) \cong \ker \psi_A = 0$. Let $f \in \text{Hom}_R(A, A)$ and let $x = f(1)$. If $y \in A$, then there exists a unique $g \in \text{Hom}_R(A, A)$ such that $g(1) = y$. We can then define λ_x: $A \to A$ by $\lambda_x(y) = g(x)$. Now λ_x is in the centre of $\text{Hom}_R(A, A)$. For if $h \in \text{Hom}_R(A, A)$, then $(h \circ g)(1) = h(g(1)) = h(y)$; and hence $\lambda_x(h(y)) = (h \circ g)(x) = h(g(x)) = h(\lambda_x(y))$.

Now if e is the identity map on A, then $\psi_A(\lambda_x) = \lambda_x(1) = e(x) = x = f(1) = \psi_A(f)$. Hence $\lambda_x = f$, and so $\mathrm{Hom}_R(A, A)$ is a commutative ring extension of R with composition of functions as multiplication. Thus we can use ψ_A to make A into a commutative ring extension of R.

On the other hand suppose that A is a commutative ring extension of R such that $\mathrm{Hom}_R(A/R, A) = 0$. Then $\ker \psi_A = 0$ and hence ψ_A is one-to-one. If $x \in A$ and f is multiplication by x on A, then $\psi_A(f) = x$, and so ψ_A is onto. Since $\mathrm{Hom}_R(A/R, A) = 0$, it is easy to see that there is only one ring structure on A that is compatible with its R-module structure.

DEFINITION. Let A be an R-module such that $R \subset A$ and define the canonical map

$$\delta_A \colon A \to A \otimes_R A$$

by $\delta_A(x) = x \otimes 1$.

PROPOSITION 1.2. *δ_A is an R-isomorphism if and only if A is a commutative ring extension of R such that $A \otimes_R A/R = 0$. In this case ψ_A is also an R-isomorphism.*

Proof. Suppose that δ_A is an R-isomorphism. Then

$$A \otimes_R A/R \cong \operatorname{Coker} \delta_A = 0.$$

Let $x \in A$ and define $\mathscr{E}_x \colon A \to A$ by $\mathscr{E}_x(y) = \delta_A^{-1}(x \otimes y)$ for all $y \in A$. Then $\mathscr{E}_x$ is an R-homomorphism, and we can define an R-homomorphism

$$\mathscr{E}_A \colon A \to \mathrm{Hom}_R(A, A)$$

by $\mathscr{E}_A(x) = \mathscr{E}_x$ for all $x \in A$. It follows immediately that $\psi_A \circ \mathscr{E}_A$ is the identity map on A, and thus ψ_A is onto and $\mathscr{E}_A$ is one-to-one.

Now $\mathrm{Hom}_R(A/R, \mathrm{Hom}_R(A, A)) \cong \mathrm{Hom}_R(A \otimes_R A/R, A) = 0$; and hence if $g \in \mathrm{Hom}_R(A/R, A)$, then $\mathscr{E}_A \circ g = 0$, and so $g = 0$. Thus $\ker \psi_A \cong \mathrm{Hom}_R(A/R, A) = 0$; and ψ_A is an isomorphism. Therefore, A is a commutative ring extension of R by Proposition 1.1.

On the other hand suppose that A is a commutative ring extension of R such that $A \otimes_R A/R = 0$. Then $\operatorname{coker} \delta_A \cong A \otimes_R A/R = 0$, and so δ_A is onto. Now there exists an R-homomorphism $\eta_A \colon A \otimes_R A \to A$ defined by $\eta_A(x \otimes y) = xy$ for all $x, y \in A$; and it is clear that $\eta_A \circ \delta_A$ is the identity on A. Thus δ_A is one-to-one, and so δ_A is an isomorphism.

Remarks. Propositions 1.1 and 1.2 are not dual to each other for there are many examples of commutative ring extensions S of R such that ψ_S is an isomorphism but δ_S is not. For example let R be an integral domain with quotient field Q, and let S be a subring of Q properly between R and Q. Then ψ_S is an isomorphism by Proposition 1.1. However, if δ_S is an isomorphism, then $S \otimes_R S/R = 0$ by Proposition 1.2; and this is not true if S is a finitely generated R-module.

DEFINITION. Let S be a commutative ring extension of R. Then S is said to be a *ring bijection* of R if the inclusion map: $R \to S$ is an epimorphism in the category of rings. The following Corollary 1.3 is well known and follows immediately from Propositions 1.1 and 1.2.

COROLLARY 1.3. *Let S be a commutative ring extension of R. Then the following statements are equivalent*:

(1) *S is a ring bijection of R.*

(2) *The map δ_S: $S \to S \otimes_R S$ is an R-isomorphism.*

(3) $S \otimes_R S/R = 0$.

(4) $\mathrm{Hom}_R(S/R, B) = 0$ *for every S-module B.*

PROPOSITION 1.4. *Let S be a commutative ring bijection of R. Then the following statements are true*:

(1) *An R-module B has at most one S-module structure compatible with its R-module structure.*

(2) *If A and B are S-modules, then* $\mathrm{Hom}_R(A, B) = \mathrm{Hom}_S(A, B)$ *and* $A \otimes_R B = A \otimes_S B$.

(3) *If an S-module is R-flat (or R-injective), then it is S-flat or S-injective.*

Proof. (1) follows easily from the fact that $\mathrm{Hom}_R(S/R, B) = 0$ for every S-module B; (2) follows from (1); and (3) follows from (2).

DEFINITION. Let S be a commutative ring extension of R. Define $\mathscr{F}_S$ to be the category of all R-modules that are R-submodules of S-modules. For each R-module A define θ_A: $A \to A \otimes_R S$ by $\theta_A(x) = x \otimes 1$ for all $x \in A$.

PROPOSITION 1.5. (1) *Let A be an R-module. Then A is in $\mathscr{F}_S$ if and only if θ_A is one-to-one.*

(2) *$\mathscr{F}_S$ is closed under R-submodules, inverse limits, and direct limits.*

Proof. (1) Of course if θ_A is one-to-one, then A is in $\mathscr{F}_S$. Conversely, suppose that A is an R-submodule of an S-module B. Then we have a commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \Big\downarrow{\scriptstyle \theta_A} & & \Big\downarrow{\scriptstyle \theta_B} \\ A \otimes_R S & \longrightarrow & B \otimes_R S \end{array} .$$

Hence it is sufficient to prove that θ_B is one-to-one. But if we define

$$\lambda_B \colon B \otimes_R S \to B$$

by $\lambda_B(b \otimes_R s) = sb$ for all $s \in S$ and $b \in B$, then $\lambda_B \circ \theta_B$ is the identity map on B, and hence θ_B is one-to-one.

(2) It is obvious that $\mathscr{F}_S$ is closed under R-submodules and inverse limits. Let $\{A_\alpha\}$ be a directed family of R-modules that are in $\mathscr{F}_S$, and let $A = \varinjlim A_\alpha$. For each α we have an exact sequence:

$$0 \to A_\alpha \xrightarrow{\theta_{A_\alpha}} A_\alpha \otimes_R S$$

by part (1). Since $\varinjlim$ is an exact functor, we have the exact sequence

$$0 \to A \to \varinjlim (A_\alpha \otimes_R S).$$

But $\otimes_R$ commutes with direct limits and hence $\varinjlim(A_\alpha \otimes_R S) \cong A \otimes_R S$. Therefore, A is in $\mathscr{F}_S$.

2. Torsion functors

Definition. A subfunctor t of the identity functor on R-modules is said to be a *torsion functor* if it satisfies the following two axioms:

(t1) If B is an R-submodule of the R-module A, then $A = t(A)$ if and only if $B = t(B)$ and $A/B = t(A/B)$.

(t2) $t(t(A)) = t(A)$.

A is said to be *t-torsion* if $A = t(A)$ and *t-torsion-free* if $t(A) = 0$. The following properties of a torsion functor t are easy to verify:

(t3) If $B \subset A$, then $t(A) \cap B = t(B)$.

(t4) $A/t(A)$ is t-torsion-free.

(t5) If A is t-torsion-free, then so is $E(A)$, the injective envelope of A.

(t6) Direct limits of t-torsion R-modules are t-torsion; and inverse limits of t-torsion-free R-modules are t-torsion-free.

(t7) If A is t-torsion, then so is $\operatorname{Tor}_n^R(A, C)$ for all R-modules C and $n \geq 0$.

Let t be a torsion-functor and define $\mathscr{I}_t$ to be the family of all ideals I of R such that R/I is t-torsion. It is clear that if A is an R-module and $x \in A$, then $x \in t(A)$ if and only if $(0\colon x) \in \mathscr{I}_t$. It is easy to check that $\mathscr{I}_t$ satisfies the following two axioms for ideals I and J of R:

$\mathscr{I}(1)$ If $I \in \mathscr{I}_t$ and $I \subset J$, then $J \in \mathscr{I}_t$.

$\mathscr{I}(2)$ If $J \in \mathscr{I}_t$ and $(I\colon r) \in \mathscr{I}_t$ for all $r \in J$, then $I \in \mathscr{I}_t$.

We note that it follows from ($\mathscr{I}1$) and ($\mathscr{I}2$) that if I and J are in $\mathscr{I}_t$, so is IJ.

On the other hand let $\mathscr{I}$ be a family of ideals of R that satisfy Axioms ($\mathscr{I}1$) and ($\mathscr{I}2$). We can then define its associated torsion functor $t_{\mathscr{I}}$ on R-modules by

$$t_{\mathscr{I}}(A) = \{x \in A \mid (0:x) \in \mathscr{I}\}$$

for all R-modules A. It is readily verified that $t_{\mathscr{I}}$ is indeed a torsion functor; and we say that $\mathscr{I}$ is a *torsion family of ideals of* R.

We say that a torsion functor t on R-modules is *faithful* if $t(R) = 0$. Clearly t is faithful if and only if every ideal in $\mathscr{I}_t$ is faithful (i.e., has 0 annihilator). We then say that $\mathscr{I}_t$ is faithful.

Examples. (1) Let $\mathfrak{A}$ be the set of all faithful ideals of R. Then $\mathfrak{A}$ is a torsion family of ideals of R and $t_{\mathfrak{A}}$ is a faithful torsion functor. Furthermore, if t is any faithful torsion functor, then $\mathscr{I}_t \subset \mathfrak{A}$ and hence $t(A) \subset t_{\mathfrak{A}}(A)$ for every R-module A.

(2) Let $\mathscr{S}$ be a multiplicatively closed subset of R and let

$$\mathscr{I}_{\mathscr{S}} = \{I \subset R \mid I \cap \mathscr{S} \neq \varnothing\}.$$

Then $\mathscr{I}_{\mathscr{S}}$ is a torsion family of ideals of R and

$$\mathscr{I}_{\mathscr{S}} = \{I \subset R \mid IR_{\mathscr{S}} = R_{\mathscr{S}}\}.$$

In particular if J is an ideal of R, and $\mathscr{S} = \{1 - a \mid a \in J\}$, then

$$\mathscr{I}_{\mathscr{S}} = \{I \subset R \mid I + J = R\}.$$

Also if P is a prime ideal of R and $\mathscr{S} = R - P$, then $\mathscr{I}_{\mathscr{S}} = \{I \subset R \mid I \not\subset P\}$.

(3) Let $\mathscr{C}$ be a non-empty set of prime ideals of R and let

$$\mathscr{I}_{\mathscr{C}} = \{I \subset R \mid I \not\subset P \text{ for any } P \in \mathscr{C}\}.$$

Then $\mathscr{I}_{\mathscr{C}}$ is a torsion family of ideals of R, and we denote its associated torsion functor by $t_{\mathscr{C}}$.

In the next proposition we generalize these examples and lay the basis for the use of torsion functors to study commutative ring extensions of R.

PROPOSITION 2.1. (1) *Let A be an R-module and $\mathscr{I}_A = \{I \subset R \mid IA = A\}$. Then $\mathscr{I}_A$ is a torsion family of ideals of R and its associated torsion functor will be denoted by t_A. If A is a faithful R-module, then t_A is a faithful torsion functor.*

(2) *Let $\mathscr{J}$ be the set of all ideals of R that contain a faithful finitely generated ideal of R; and let E be the injective envelope of R. Then $\mathscr{J} = \mathscr{I}_E$ (i.e., $I \in \mathscr{J}$ if and only if $IE = E$) and so $\mathscr{J}$ is a faithful torsion family of ideals of R.*

Proof. (1) It is clear that $\mathscr{I}_A$ satisfies Axiom $\mathscr{I}(1)$. Hence suppose that $J \in \mathscr{I}_A$ and that I is an ideal of R such that $(I:r) \in \mathscr{I}_A$ for every $r \in J$. We wish to show that $I \in \mathscr{I}_A$; i.e., that $IA = A$. Let $x \in A$; then $x = \sum_{i=1}^{n} r_i x_i$, where $r_i \in J$ and $x_i \in A$. Let

$$K = (I:r_1) \cap \cdots \cap (Ir_n);$$

then $KA = A$ and so $x_i = \sum_{j=1}^{m} t_{ij} y_j$ where $t_{ij} \in K$ and $y_j \in A$. Hence we have

$$x = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} r_i t_{ij}\right) y_j.$$

But $r_i t_{ij} \in I$ for all i and j, and so $A = IA$. It is clear that if A is a faithful R-module, then $\mathscr{I}_A$ is a faithful family of ideals of R.

(2) Since $1 \in E$, we have $\mathscr{I}_E \subset \mathscr{I}$. On the other hand suppose that

$$I = Ra_1 + \cdots + Ra_n$$

is a faithful, finitely generated ideal of R. Let $x = (a_1, a_2, \ldots, a_n) \in R^n$; and let $e_1, \ldots, e_n$ be the standard free basis of R^n. Let $y \in E$; then since $(0:x) = 0$ and E is injective, there exists an R-homomorphism $f\colon R^n \to E$ such that $f(x) = y$. But then $y = \sum_{i=1}^{n} a_i f(e_i) \in IE$, and so $E = IE$. Hence $\mathscr{I}_E = \mathscr{I}$, and by part (1), $\mathscr{I}$ is a faithful torsion family of ideals of R.

DEFINITION. Let $\mathfrak{A}$ be the set of all faithful ideals of R; let E be the injective envelope of R; and let $U = \{x \in E \mid (R:x) \in \mathfrak{A}\}$. Then $U/R = t_{\mathfrak{A}}(E/R)$. Let t be a faithful torsion functor on R-modules and let

$$T = \{x \in E \mid (R:x) \in \mathscr{I}_t\}.$$

Then $T/R = t(E/R)$, and $T \subset U$. The next proposition demonstrates that U is a commutative ring extension of R and that T is a subring of U.

PROPOSITION 2.2. (1) *The canonical maps*

$$\psi_U\colon \mathrm{Hom}_R(U,U) \to U \quad \text{and} \quad \psi_T\colon \mathrm{Hom}_R(T,T) \to T$$

are R-isomorphisms. Thus U is a commutative ring extension of R and T is a subring of U.

(2) *If A is a t-torsion free R-module, then* $A \in \mathscr{F}_T$.

(3) *If C is a t-torsion-free injective R-module, then the canonical map*

$$\phi_C\colon \mathrm{Hom}_R(T,C) \to C$$

defined by $\phi_C(f) = f(1)$ *for all* $f \in \mathrm{Hom}_R(T,C)$ *is an R-isomorphism. Thus C is an injective T-module.*

Proof. (1) We have an exact sequence:

$$\mathrm{Hom}_R(T/R, T) \to \mathrm{Hom}_R(T, T) \xrightarrow{\psi_T} T \to \mathrm{Ext}^1_R(T/R, T).$$

T is an essential extension of R and thus is t-torsion-free. Hence because T/R is t-torsion we have $\mathrm{Hom}_R(T/R, T) = 0$. We have an exact sequence

$$\mathrm{Hom}_R(T/R, E/T) \to \mathrm{Ext}^1_R(T/R, T) \to 0.$$

Since E/T is t-torsion-free, we have $\mathrm{Hom}_R(T/R, E/T) = 0$, and thus

$$\mathrm{Ext}^1_R(T/R, T) = 0$$

also. Therefore, the first exact sequence shows that ψ_T is an R-isomorphism. Hence T is a commutative ring extension of R by Proposition 1.1. Similarly ψ_U is an R-isomorphism and U is a commutative ring extension of R. It is obvious that T is an R-submodule of U. It remains to show that T is a subring of U.

Let $x, y \in T$ and let $x \cdot y$ denote their product in T and $x \circ y$ their product in U. Define f: $T \to U$ by $f(y) = x \cdot y - x \circ y$ for all y in T. Then f is an R-homomorphism and $f(R) = 0$, and hence f induces an R-homomorphism $\bar{f}$: $T/R \to U$. But T/R is t-torsion and U is t-torsion-free. Hence $\bar{f} = 0$, and so $f = 0$, and thus T is a subring of U.

(2) We have an exact sequence

$$\mathrm{Tor}^R_1(A, T/R) \to A \xrightarrow{\theta_A} A \otimes_R T.$$

Since T/R is t-torsion, so is $\mathrm{Tor}^R_1(A, T/R)$; and thus its image in the t-torsion-free R-module A is 0; and so θ_A is one-to-one.

(3) We have an exact sequence

$$0 \to \mathrm{Hom}_R(T/R, C) \to \mathrm{Hom}_R(T, C) \xrightarrow{\phi_C} C \to 0.$$

But T/R is t-torsion and C is t-torsion-free. Thus $\mathrm{Hom}_R(T/R, C) = 0$ and so ϕ_C is an R-isomorphism.

Remarks. (1) We have $U = E$ if and only if the canonical map ψ_E: $\mathrm{Hom}_R(E, E) \to E$ is an isomorphism. For if $x \in E$, $I = (R : x)$ and $c \in (0 : I)$, then there exists $f \in \mathrm{Hom}_R(E, E)$ such that $f(x) = c$ and $f(1) = 0$.

(2) It follows from Proposition 2.2 that E is both the T-injective envelope of T and the U-injective envelope of U.

(3) Let $x \in E$ and $J = \{u \in U | ux \in U\}$. It can be shown that if J is a faithful ideal of U, then $x \in U$. Thus if σ is a faithful torsion functor on U-modules, then E/U is σ-torsion-free.

PROPOSITION 2.3. *Let S be a commutative ring extension of R.*

(1) *Every R-module in $\mathscr{F}_S$ is t_S-torsion-free.*

(2) *If B is an R-module and θ_B: $B \to B \otimes_R S$ is the canonical map, then*

$$t_S(B) \subset \operatorname{Ker} \theta_B.$$

(3) *Let $T/R = t_S(S/R)$; then T is a subring of S and is both ring and R-isomorphic to a unique subring of U. If we identify T with its image in U, then $T/R = t_S(E/R)$.*

Proof. (1) Let A be an S-module and C an R-submodule of A. Let $x \in t_S(C)$ and $I = (0 : x)$; then $SI = S$ and hence $x \in Sx = S(Ix) = 0$. Therefore, C is t_S-torsion-free.

(2) We have $B/\operatorname{Ker} \theta_B \cong \operatorname{Im} \theta_B \in \mathscr{F}_S$. Thus $B/\operatorname{Ker} \theta_B$ is t_S-torsion-free by part (1), and so $t_S(B) \subset \operatorname{Ker} \theta_B$.

(3) It is obvious that T is a subring of S. Let $0 \neq x \in T$ and $I = (R : x)$; then $SI = S$ and so $Sx = S(Ix)$. Therefore Ix is a non-zero ideal of R and hence T is an essential extension of R. Therefore we can assume that $T \subset E$. To complete the proof of (3) it is sufficient by Proposition 2.2 to prove that $t_S(E/R) = T/R$. Hence it is sufficient to prove that E/T is t_S-torsion-free.

Let I be an ideal of R such that $IS = S$. Then it is sufficient to prove that $\operatorname{Hom}_R(R/I, E/T) = 0$. We have the exact sequence

$$\operatorname{Hom}_R(R/I, E) \to \operatorname{Hom}_R(R/I, E/T) \to \operatorname{Ext}^1_R(R/I, T)$$

Since E is an essential extension of R, it is t_S-torsion-free. But R/I is t_S-torsion and so $\operatorname{Hom}_R(R/I, E) = 0$. Therefore it is sufficient to prove that

$$\operatorname{Ext}^1_R(R/I, T) = 0.$$

We have an exact sequence

$$\operatorname{Hom}_R(R/I), S/T) \to \operatorname{Ext}^1_R(R/I, T) \to \operatorname{Ext}^1_R(R/I, S).$$

S/T is t_S-torsion-free and R/I is t_S-torsion, and thus

$$\operatorname{Hom}_R(R/I, S/T) = 0.$$

Also $\operatorname{Ext}^1_R(R/I, S)$ is t_S-torsion and is an S-module. Hence by part (1),

$$\operatorname{Ext}^1_R(R/I, S) = 0.$$

Thus we have $\operatorname{Ext}^1_R(R/I, T) = 0.$

Remarks. Let $\mathscr{J}$ be the set of all ideals of R that contain a faithful finitely generated ideal of R. By Proposition 2.1, $\mathscr{J} = \mathscr{I}_E$ is a faithful torsion family of ideals of R. Let $V/R = t_E(E/R)$; then by Proposition 2.2, V is a commutative subring of U. Let S be a commutative ring extension of R and $T/R = t_S(S/R)$. By Proposition 2.3, $T/R = t_S(E/R)$. Since $\mathscr{I}_S \subset \mathscr{J} = \mathscr{I}_E$, we have

$$t_S(E/R) \subset t_E(E/R).$$

Thus in fact T is a commutative subring of V.

PROPOSITION 2.4. *Let S be a commutative ring extension of R. Then the following statements are equivalent*:

(1) $\mathscr{F}_S$ *is equal to the cateogry of all t_S-torsion-free R-modules.*

(2) $t_S(B) = \operatorname{Ker} \theta_B$ *for all R-modules B.*

(3) $\mathscr{F}_S$ *is closed under essential extensions.*

In this case if P is a prime ideal of R such that $PS \neq S$ and J is a P-primary ideal of R, then $SJ \cap R = J$.

Proof. (1) $\Rightarrow$ (2) Let B be an R-module, $\bar{B} = B/t_S(B)$, and Π: $B \to \bar{B}$ the canonical map. Then we have a commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{\theta_B} & B \otimes_R S \\ \Big\downarrow{\scriptstyle \Pi} & & \Big\downarrow{\scriptstyle \Pi \otimes 1} \\ \bar{B} & \xrightarrow{\theta_{\bar{B}}} & \bar{B} \otimes_R S \end{array} .$$

Since $\bar{B}$ is t_S-torsion-free, it follows from our assumption that $\bar{B} \in \mathscr{F}_S$. Hence by Proposition 1.5 (1), $\theta_{\bar{B}}$ is one-to-one. It follows from the diagram that if $x \in \operatorname{Ker} \theta_B$, then $x \in \operatorname{Ker} \Pi = t_S(B)$; and so $\operatorname{Ker} \theta_B \subset t_S(B)$. The reverse inclusion is provided by Proposition 2.3 (2).

(2) $\Rightarrow$ (1) If B is a t_S-torsion-free R-module, then $\operatorname{Ker} \theta_B = t_S(B) = 0$, and hence B is isomorphic to an R-submodule of the S-module $B \otimes_R S$, i.e., $B \in \mathscr{F}_S$. On the other hand if $B \in \mathscr{F}_S$, then B is t_S-torsion-free by Proposition 2.3 (1).

(1) $\Rightarrow$ (3) It is obvious that essential extensions of t_S-torsion-free R-modules are again t_S-torsion-free.

(3) $\Rightarrow$ (1) Let A be a t_S-torsion-free R-module. By Proposition 2.3 (1) it is sufficient to prove that A is in $\mathscr{F}_S$. Let $0 \neq x \in A$ and $I = (0:x)$; since $t_S(A) = 0$ we have $S/IS \neq 0$. Let C_x be the R-injective envelope of S/IS; then by assumption $C_x \in \mathscr{F}_S$. We let $C = \Pi C_x$ $(0 \neq x \in A)$; then C is an injective R-module and $C \in \mathscr{F}_S$. We have an R-homomorphism f_x: $A \to C_x$

such that $f_x(x) = 1 + IS \neq 0$; and we define an R-homomorphism $f\colon A \to C$ by $f(y) = \langle f_x(y) \rangle$ for all $y \in A$. Then f is one-to-one, and hence A is in $\mathscr{F}_S$.

Now assume that S satisfies the three equivalent conditions of the proposition; let P be a prime ideal of R such that $PS \neq S$, and J a P-primary ideal. Since R/J is t_S-torsion-free, $\operatorname{Ker}\theta_{R/J} = t_S(R/J) = 0$. But $\operatorname{Ker}\theta_{R/J} = (SJ \cap R)/J$, and so $SJ \cap R = J$.

PROPOSITION 2.5. *Let S be a flat commutative ring extension of R and B an R-module. Then the following statements are true*:

(1) *$\mathscr{F}_S$ is equal to the category of all t_S-torsion-free R-modules.*

(2) $t_S(B) = \operatorname{Ker}\theta_B \cong \operatorname{Tor}_1^R(B, S/R)$.

(3) *B is t_S-torsion-free if and only if* $\operatorname{Tor}_1^R(B, S/R) = 0$; *and B is t_S-torsion if and only if $B \otimes_R S = 0$.*

Proof. (1) Let $A \in \mathscr{F}_S$ and let C be an essential extension of A. By Proposition 2.4 it is sufficient to prove that $C \in \mathscr{F}_S$. Because S is flat we have a commutative diagram with exact rows:

$$\begin{array}{ccc} 0 \to A & \longrightarrow & C \\ {\scriptstyle\theta_A}\downarrow & & \downarrow{\scriptstyle\theta_C} \\ 0 \to A \otimes_R S & \longrightarrow & C \otimes_R S \end{array} .$$

Since θ_A is one-to-one by Proposition 1.5, we have $\operatorname{Ker}\theta_C \cap A = 0$, and hence $\operatorname{Ker}\theta_C = 0$; and thus $C \in \mathscr{F}_S$.

(2) Since S is flat, $\operatorname{Ker}\theta_B \cong \operatorname{Tor}_1^R(B, S/R)$; and we have $t_S(B) = \operatorname{Ker}\theta_B$ by Proposition 2.4.

(3) This follows immediately from part (2).

COROLLARY 2.6. *Let S be a commutative ring extension of R. Then S is R-flat if and only if* $\operatorname{Tor}_1^R(B, S/R)$ *is t_S-torsion for all R-modules B.*

Proof. If S is R-flat, then $\operatorname{Tor}_1^R(B, S/R)$ is t_S-torsion by Proposition 2.5. On the other hand if $\operatorname{Tor}_1^R(B, S/R)$ is t_S-torsion, then the exact sequence

$$0 \to \operatorname{Tor}_1^R(B, S) \to \operatorname{Tor}_1^R(B, S/R)$$

shows that $\operatorname{Tor}_1^R(B, S)$ is t_S-torsion also. But $\operatorname{Tor}_1^R(B, S)$ is an S-module and hence t_S-torsion-free by Proposition 2.3 (1). Thus $\operatorname{Tor}_1^R(B, S) = 0$ for all R-modules B, and so S is R-flat.

Remarks. (1) It is not true in general that if $\mathscr{F}_S$ is equal to the category of t_S-torsion-free R-modules, then S is flat. For let R be a Noetherian local ring with maximal ideal M and let $S = \Pi R/M^n$; then S is a commutative ring, and since $\bigcap M^n = 0$, we have a canonical ring injection of R into S. Now $MS \neq S$, and hence every R-module is t_S-torsion-free. If A is a finitely generated R-module, then $\bigcap M^n A = 0$ and hence $A \subset \Pi A/M^n A$; and $\Pi A/M^n A$ is an S-module. Since $\mathscr{F}_S$ is closed under direct limits by Proposition 1.5, every R-module is in $\mathscr{F}_S$. Moreover, if I is an ideal of R, then $IS \cap R = \bigcap_n (I + M^n) = I$. However, S is not R-flat.

A more general example is to let R be any commutative ring with 1 that has an ideal J such that R/J is not R-flat. We let $S = R \oplus R/J$ and $R \to S$ the canonical diagonal ring homomorphism. Then $S/R \cong R/J$; and if I is any ideal of R we have $IS \cap R = I$. If A is any R-module, we define λ_A: $A \otimes_R S \to A$ by

$$\lambda_A(x \otimes (r, t + J)) = rx$$

where $x \in A$ and $r, t \in R$. Then $\lambda_A \circ \theta_A$ is the identity on A, and so θ_A is 1-1. Thus every R-module is in $\mathscr{F}_S$, but S is not R-flat.

(2) It is not true in general that if S is a commutative ring bijection of R, then S is R-flat; or even that $\mathscr{F}_S$ is equal to the cateogry of all t_S-torsion-free R-modules. For let R be a quasi-local integral domain with quotient field Q, and let J be a non-zero ideal of R. Let $S = Q \oplus R/J$, and $R \to S$ the canonical diagonal ring homomorphism. Then $S/R \cong Q/J$, and so $S \otimes_R S/R = 0$. Thus by Corollary 1.3, S is a ring bijection of R. Clearly S is not R-flat. If I is any ideal of R, then $IS \cap R = I + J \neq R$, and so every R-module is t_S-torsion-free. However, since S is a non-flat ring bijection of R, $\mathscr{F}_S$ is not equal to the category of all t_S-torsion-free R-modules as we shall see in Corollary 3.5.

(3) It is not true in general that if S is a commutative ring extension of R such that $PS \cap R = P$ for every prime ideal of P of R, then $\mathscr{F}_S$ is equal to the category of all t_S-torsion-free R-modules. For let R be a quasi-local ring that has two non-zero principal ideals $J_1 = Rx_1$, and $J_2 = Rx_2$ such that $J_1 \cap J_2 = 0$. Let $S = R/J_1 \oplus R/J_2$, and let $R \to S$ be the canonical diagonal ring homomorphism. If I is an ideal of R, then

$$IS \cap R = (I + J_1) \cap (I + J_2) \neq R.$$

Thus every R-module is t_S-torsion-free. If P is a prime ideal of R, then either $J_1 \subset P$ or $J_2 \subset P$ and so $PS \cap R = P$. However, if $I = R(x_1 + x_2)$, then

$$IS \cap R = Rx_1 + Rx_2 \neq I.$$

Hence if $A = R/I$ and θ_A: $A \to A \otimes_R S$ is the canonical map, then

$$\operatorname{Ker} \theta_A = (IS \cap R)/I \neq 0,$$

and so $A \notin \mathscr{F}_S$ by Proposition 1.5(1).

A similar example is obtained if R is an integral domain that is not integrally closed, and S is its integral closure, then $PS \cap R = P$ for every prime ideal P of R. But if $a/b \in S - R$, where $a, b \in R$, then $a \in bS \cap R$ and $a \notin Rb$.

In general if S is a commutative ring extension of R such that $IS \cap R = I$ for every ideal I of R, then every R-module is t_S-torsion-free; and every cyclic R-module is in $\mathscr{F}_S$. But it is an open question whether this implies that $\mathscr{F}_S$ is equal to the category of all R-modules.

DEFINITION. R is said to be *reduced* if R has no non-zero nilpotent elements. If R is reduced and E is the injective envelope of R, then E is a commutative, von-Neumann regular ring extension of R; and if min R denotes the minimal prime spectrum of R, then min R is compact if and only if E is R-flat (see [6, Propositions 1.12 and 1.16]).

PROPOSITION 2.7. *There exists a commutative ring extension S of R such that $\operatorname{Tor}_n^R(A, B)$ is t_S-torsion for all R-modules A and B and $n > 0$ if and only if R is reduced and* min R *is compact. In this case E is such an extension.*

Proof. Suppose that such a ring S exists. Let A be an S-module and B an R-module. Then $\operatorname{Tor}_1^R(A, B)$ is an S-module and hence t_S-torsion-free by Proposition 2.3 (1); but it is t_S-torsion by assumption and so $\operatorname{Tor}_1^R(A, B) = 0$. Therefore, every S-module is R-flat. Now E is t_S-torsion-free and hence $E \in \mathscr{F}_S$ by Proposition 2.5 (1). Therefore, E is an R-direct summand of an S-module. Thus E is R-flat.

Let I be an ideal of R; then $I/I^2 \cong \operatorname{Tor}_1^R(R/I, R/I)$ is t_S-torsion. Hence, since S is R-flat, $SI/SI^2 \cong S \otimes_R I/I^2 = 0$ by Proposition 2.5 (3). Therefore, $SI = SI^2$, and it follows that R is a reduced ring. Since E is R-flat, min R is compact.

Conversely, suppose that R is reduced and that min R is compact. Then E is an R-flat, commutative, von Neumann regular ring extension of R. Let I and J be ideals of R. Then

$$E \otimes_R \operatorname{Tor}_1^R(R/I, R/J) \cong E \otimes_R (I \cap J)/IJ \cong E(I \cap J)/(EI)(EJ) = 0.$$

Hence $\operatorname{Tor}_1^R(R/I, R/J)$ is t_E-torsion by Proposition 2.5 (3). It follows easily that $\operatorname{Tor}_n^R(A, B)$ is t_E-torsion for all R-modules A and B and $n > 0$.

3. Torsion envelopes

DEFINITION. We shall say that an R-module A is a *torsion envelope* of R if A is a commutative ring extension of R such that A/R is t_A-torsion.

Remarks. (1) Let A be a torsion envelope of R and $\mathscr{S}$ a multiplicatively closed subset of R. It is easy to see that $A_{\mathscr{S}}$ is a torsion envelope of $R_{\mathscr{S}}$. Furthermore, if I is a proper ideal of R such that $IA \cap R = I$, then A/IA is a torsion envelope of R/I.

(2) If $\mathscr{S}$ is a multiplicatively closed set of non-zero-divisors in R, then $R_{\mathscr{S}}$ is a torsion envelope of R. Hence if Q is the total ring of quotients of R, then Q is a torsion envelope of R. It will follow from the next proposition that a subring of Q containing R is a torsion envelope of R if and only if it is R-flat.

(3) Let E be the injective envelope of R and $V/R = t_E(E/R)$. We recall that V is a commutative ring extension of R; and if S is a commutative ring extension of R and $T/R = t_S(S/R)$, then T is a commutative subring of V and $T/R = t_S(E/R)$ (see the remarks following Proposition 2.3).

PROPOSITION 3.1. *Let A be an R-module such that $R \subset A$. Then the following statements are equivalent*:

(1) *A is a torsion-envelope of R.*

(2) *A is an R-flat subring of V.*

(3) *A is an R-flat commutative ring extension of R and there exists an R-module $B \supset A$ such that $B \otimes_R A/R = 0$.*

(4) *A is an R-flat ring bijection of R.*

(5) *A is a flat R-module and $A \otimes_R A/R = 0$.*

(6) *A is an essential extension of R and for all $x \in A$ we have $A = (R : x)A$.*

Proof. (1) ⇒ (2) A is a subring of V by the preceding Remark (3). Since A/R is t_A-torsion, so is $\operatorname{Tor}_1^R(B, A/R)$ for all R-modules B. Hence A is R-flat by Corollary 2.6.

(2) ⇒ (3) Let $x \in E$, $y \in A/R$, and $I = (0 : y)$. Since $y \in V/R = t_E(E/R)$, we have $E = IE$; and hence $x = \sum_{i=1}^{n} a_i x_i$ where $x_i \in E$ and $a_i \in I$. Thus

$$x \otimes y = \sum_{i=1}^{n} a_i x_i \otimes y = \sum_{i=1}^{n} x_i \otimes a_i y_i = 0,$$

and so $E \otimes_R A/R = 0$.

(3) ⇒ (4) Since A is R-flat, $\operatorname{Tor}_1^R(B/A, A/R)$ is t_A-torsion by Corollary 2.6. Hence the exact sequence

$$\operatorname{Tor}_1^R(B/A, A/R) \to A \otimes_R A/R \to B \otimes_R A/R = 0$$

shows that $A \otimes_R A/R$ is t_A-torsion. But $A \otimes_R A/R$ is an A-module and hence is t_A-torsion-free by Proposition 2.3(1). Thus $A \otimes_R A/R = 0$, and so A is a ring bijection of R by Corollary 1.3.

(4) ⇔ (5) This follows from Proposition 1.2 and Corollary 1.3.

(4) ⇒ (1) $A \otimes_R A/R = 0$ by Corollary 1.3 and hence A/R is t_A-torsion by Proposition 2.5(3).

(1) ⇒ (6) Since A is a subring of $V \subset E$ by the preceding Remark (3), A is an essential extension of R. And $A = (R : x)A$ for all $x \in A$ since A/R is t_A-torsion.

(6) ⇒ (1) We can assume that $A \subset E$, and then $A/R \subset t_A(E/R) = T/R$. T is a commutative ring extension of R by Proposition 2.2, and $A \subset T$. Let $y \in T$ and $I = (R : y)$; then $IA = A$ by definition. Hence we have $1 = \sum_{i=1}^{n} a_i x_i$ where $a_i \in I$ and $x_i \in A$. Let $r_i = a_i y$; then $r_i \in R$ and

$$y = \sum_{i=1}^{n} (a_i y) x_i = \sum_{i=1}^{n} r_i x_i \in A.$$

Hence $A = T$, and so A is a commutative ring extension of R and thus a torsion envelope of R.

PROPOSITION 3.2. *Let $R \subset T \subset S$ be commutative ring extensions of R. Then the following statements are true*:

(1) *If S is a torsion envelope of R, then T is a torsion envelope of R if and only if T is R-flat.*

(2) *If S is a torsion envelope of T and T is a torsion envelope of R, then S is also a torsion envelope of R.*

(3) *If S is a torsion envelope of R, then S is also a torsion envelope of T.*

Proof. (1) This follows from Proposition 3.1.

(2) Let $x \in S$ and $J = \{a \in T | ax \in T\}$. Then $JS = S$ and hence

$$1 = \sum_{i=1}^{n} a_i y_i \quad \text{where } a_i \in J \text{ and } y_i \in S.$$

Since T/R is t_T-torsion, there exists an ideal I of R such that $IT = T$ and $Ia_i \subset R$ and $I(a_i x) \subset R$ for all $i = 1, \dots, n$. Let $K = Ia_1 + \cdots + Ia_n$; then K is an ideal of R, $Kx \subset R$ and $KS = S$. Thus S/R is t_S-torsion, and so S is a torsion envelope of R.

(3) Let $x \in S$ and $I = (R : x)$; then $IS = S$. Since $(TI)x \subset T$ and $S(TI) = S$, it follows that S is a torsion envelope of T.

A portion of the next Proposition was also proved by Lazard [3, Proposition IV, 3.3] in a different form.

COROLLARY 3.3. *There exists a unique maximal torsion envelope M of R contained in V. M contains a unique copy of every torsion envelope of R; and M itself has no torsion envelopes.*

Proof. It is obvious that if S and T are torsion envelopes of R in V, then ST is also a torsion envelope of R. The proposition now follows from Zorn's Lemma and Propositions 3.1 and 3.2.

It is interesting to compare the next Proposition with the equivalence of statements (1) and (4) in Corollary 1.3.

PROPOSITION 3.4. *Let S be a commutative ring extension of R. Then S is a torsion envelope of R if and only if* $\operatorname{Hom}_R(S/R, A) = 0$ *for every t_S-torsion-free R-module A.*

Proof. If S is a torsion envelope of R, then S/R is t_S-torsion, and the statement follows immediately. Conversely, suppose that $\operatorname{Hom}_R(S/R, A) = 0$ for every t_S-torsion-free R-module A. Let $T/R = t_S(S/R)$; then S/T is t_S-torsion-free and S/R maps onto S/T. Hence $S/T = 0$, and so S is a torsion envelope of R.

COROLLARY 3.5. *Let S be a commutative ring bijection of R. Then the following statements are equivalent:*

(1) *S is a torsion envelope of R.*

(2) *S is R-flat.*

(3) *$\mathscr{F}_S$ is equal to the category of all t_S-torsion-free R-modules.*

Proof. The corollary follows from Propositions 3.1, 2.5, 3.4 and Corollary 1.3.

In the following proposition we collect some facts that we have developed separately.

PROPOSITION 3.6. *Let S be a torsion envelope of R. Then the following statements are true:*

(1) *If $A \in \mathscr{F}_S$, then the R-injective envelope of A is S-injective.*

(2) *An R-module has at most one S-structure that extends its R-module structure.*

(3) *If A and B are S-modules, then* $\operatorname{Hom}_R(A, B) = \operatorname{Hom}_S(A, B)$ *and* $A \otimes_R B = A \otimes_S B$.

(4) *An S-module is R-flat (or R-injective) if and only if it is S-flat (or S-injective).*

(5) *If A is an S-module and B is a t_S-torsion R-module, then* $\operatorname{Tor}_n^R(B, A) = 0 = \operatorname{Ext}_R^n(B, A)$ *for all $n \geq 0$.*

(6) *If R is Noetherian (or coherent) so is S.*

Proof. (1) This follows from Proposition 2.2.

(2) and (3) follow from Proposition 1.4; and since S is R-flat so does (4).

(5) With the assumptions on A and B, $\operatorname{Tor}_n^R(B, A) = 0$ for all $n \geq 0$ for any commutative ring S. By (4) we can compute $\operatorname{Ext}_R^n(B, A)$ by taking an S-injective resolution X of A. But then $\operatorname{Hom}_R(B, X) = 0$, because B is t_S-torsion and X is t_S-torsion-free. Hence $\operatorname{Ext}_R^n(B, A) = 0$.

(6) This follows from (4) and the fact that S is Noetherian if and only if a direct sum of S-injective modules is S-injective (and S is coherent if and only if a direct produce of S-flat modules is S-flat).

LEMMA 3.7. *Let S be a commutative ring extension of R and let I be an ideal of R such that $IS \neq S$. Then there exists a prime ideal P of R such that $I \subset P$, P is maximal with respect to the property $PS \neq S$, and $PS \cap R = P$.*

Proof. By Zorn's Lemma there exists an ideal P of R such that $I \subset P$ and P is maximal with respect to the property $PS \neq S$. Since $P \subset PS \cap R$ and $PS \cap R$ has the same properties as P, we have $P = PS \cap R$. Now suppose that $a, b \in R$, $a \notin P$ and $ab \in P$. Then $(P + Ra)S = S$, and multiplying by b, we see that $b \in PS \cap R = P$. Hence P is a prime ideal of R.

The following Proposition has also been proved by Lazard [3, Proposition IV, 2.4].

PROPOSITION 3.8. *Let S be a commutative ring extension of R. Then S is a torsion envelope of R if and only if for every prime ideal P of R either $PS = S$ or $S_P = R_P$.*

Proof. Assume that S is a torsion envelope of R and that P is a prime ideal of R such that $PS \neq S$. Because S is R-flat, $PS \cap R = P$. Suppose that $S_P \neq R_P$. Now S_P is a torsion envelope of R_P and hence $PS_P = S_P$. But then $PS \cap R \neq P$; and this contradiction shows that $S_P = R_P$.

Conversely, assume that for every prime ideal P of R either $PS = S$ or $S_P = R_P$. Suppose that S is not a torsion envelope of R. Then there exists $x \in S$ such that if $I = (R : x)$, then $IS \neq S$. By Lemma 3.7 there exists a prime ideal P of R such that $I \subset P$ and $PS \neq S$. Hence by assumption $S_P = R_P$. It follows that there exists $u \in R - P$ such that $ux \in R$. This contradiction shows that S is a torsion envelope of R.

COROLLARY 3.9. *Let S be a torsion envelope of R; let $\operatorname{Spec}_S R$ be the set of all prime ideas P of R such that $PS \neq S$; and let $\operatorname{Spec} S$ be the set of all prime ideals of S. Then:*

(1) *There exists a one to one, order preserving correspondence between $\operatorname{Spec}_S R$ and $\operatorname{Spec} S$ given by $P \to PS$ for $P \in \operatorname{Spec}_S R$ and $\mathcal{P} \to \mathcal{P} \cap R$ for $\mathcal{P} \in \operatorname{Spec} S$.*

(2) *If $P \in \operatorname{Spec}_S R$, then S/PS is a subring of R_P/PR_P that contains R/P.*

(3) *P is maximal in $\operatorname{Spec}_S R$ if and only if $S/PS = R_P/PR_P$.*

(4) *$\operatorname{Spec}_S R$ contains every minimal prime ideal of R.*

(5) *If $P \in \operatorname{Spec} R$, then $P \in \operatorname{Spec}_S R$ if and only if $S_P = R_P$.*

Proof. (1) Let $P \in \operatorname{Spec}_S R$; then $PS \cap R = P$ because S is R-flat. Hence S/PS is a torsion envelope of R/P. Since R_P/PR_P is the injective envelope of R/P, it follows from Proposition 3.1 that S/PS is a subring of R_P/PR_P that contains R/P. Hence S/PS is an integral domain, and thus $PS \in \operatorname{Spec} S$.

Let $\mathscr{P} \in \operatorname{Spec} S$ and $P = \mathscr{P} \cap R$. Since S/PS is contained in the quotient field of R/P by the preceding paragraph and $(\mathscr{P}/PS) \cap R/P = 0$, we have $\mathscr{P} = PS$.

(2) This was proved in part (1).

(3) This follows directly from parts (1) and (2).

(4) Let P be a minimal prime ideal of R. Then every element of PR_P is nilpotent, and hence P does not contain a faithful, finitely generated ideal of R. Therefore $PS \neq S$.

(5) This follows from Proposition 3.8.

We have remarked earlier that if E is the injective envelope of R, then $E = U$ if and only if the canonical map ψ_E: $\operatorname{Hom}_R(E, E) \to E$ is an isomorphism. The next corollary gives necessary and sufficient conditions in the case where R is a reduced ring for $E = V$; i.e., for E to be a torsion envelope of R.

COROLLARY 3.10. *Let R be a reduced ring. Then E is a torsion envelope of R if and only if* min R *is compact and $E_P = R_P$ for every minimal prime ideal P of R.*

Proof. This follows from Proposition 3.8 and the fact that E is R-flat if and only if min R is compact, if and only if the only prime ideals P of R such that $PE \neq E$ are the minimal prime ideals of R (see [6, Proposition 1.6]).

Remarks. (1) Let R be a reduced ring such that the total ring of quotients Q of R is a von Neumann regular ring. Then E is a torsion envelope of R if and only if $E = Q$. For if E is a torsion envelope of R, then E is a torsion envelope of Q by Proposition 3.2(3). In this case, since every prime ideal of Q is minimal, we would have $E_P = Q_P$ for every prime ideal P of Q by Corollary 3.10, and hence $E = Q$.

(2) If R is a reduced ring and P is a non-essential prime ideal of R, then $E_P = R_P$ by [6, Proposition 3.9]. Now there exists a reduced ring R with an infinite number of minimal prime ideals and every one of them non-essential (see [6, Example 3]). Hence in this case $E_P = R_P$ for every minimal prime ideal P of R. But min R is not compact [6, Proposition 3.15] and hence by Corollary 3.10, E is not a torsion envelope of R.

(3) In general if R is any commutative ring, then it follows from Proposition 2.1 that E is a torsion envelope of R if and only if $(R : x)$ contains a faithful finitely generated ideal of R for every $x \in E$.

DEFINITION. R is said to be a *semi-regular ring* if every R-module is a submodule of a flat R-module. If R is a reduced ring, then it is semi-regular if and only if it is von-Neumann regular; and if R is Noetherian then it is semi-regular if and only if it is quasi-Frobenius (see [7, Propositions 2.7 and 3.4]). The next proposition generalizes the preceding Remark (1).

PROPOSITION 3.11. *Assume that the total ring of quotients Q of R is a semi-regular ring. Then Q is the unique maximal torsion envelope of R. Moreover an R-module is t_Q-torsion-free if and only if it is an R-submodule of a flat R-module.*

Proof. Let M be the unique maximal torsion envelope of R. Then $Q \subset M$ and M is the unique maximal torsion envelope of Q. Thus without loss of generality we can assume that $R = Q$. Then M/R is contained in a flat R-module; and every flat R-module is an R-submodule of an M-module. Hence M/R is both t_M-torsion and t_M-torsion-free, and thus $M = R$. The last statement of the Proposition follows from Proposition 2.5(1).

DEFINITION. Let I be a faithful ideal of R and define $I^* = \{x \in E | Ix \subset R\}$. As in the case of an integral domain we have $I^* \cong \mathrm{Hom}_R(I, R)$. We then define $S = \bigcup_n (I^n)^*$. By analogy with the integral domain case we call S the *ideal transform* of I. If I contains a faithful, finitely generated ideal of R, then I is a projective ideal of R if and only if $II^* = R$ in which case I itself is finitely generated. For such an ideal we also have $S = IS$, and S is R-flat.

Akiba [0] has given an example of an integral domain R and a non-projective prime ideal P of R such that if S is the ideal transform of P, then $PS = S$ and S is R-flat.

PROPOSITION 3.12. *Let I be a faithful ideal of R and S the ideal transform of I. Then*:

(1) *S is a commutative subring of U that contains R.*

(2) *If I contains a faithful, finitely generated ideal of R, then S is a torsion envelope of R if and only if S is R-flat.*

(3) *If $S = IS$, then I contains a faithful finitely generated ideal of R and S is a torsion envelope of R. In this case* Spec S *is equal to the set of all ideals PS such that P is a prime ideal of R that does not contain I.*

Proof. (1) It is clear that S is an R-submodule of U. If x and y are in S, then their product in U is again in S, and hence S is a subring of U that contains R.

(2) If I contains a faithful, finitely generated ideal of R, then $S \subset V$; and the proposition now follows from Proposition 3.1.

(3) Assume that $S = IS$: then clearly I contains a faithful, finitely generated ideal of R. Moreover, since $I^nS = S$ for all $n > 0$, S/R is a t_S-torsion R-module; i.e., S is a torsion envelope of R.

By Corollary 3.9 Spec S is equal to the set of all ideals of the form PS, where P is a prime ideal of R such that $PS \neq S$. Now if $PS \neq S$, then $I \not\subset P$ because $IS = S$. On the other hand if P is a prime ideal of R such that $PS = S$, then $1 = \sum_{i=1}^{k} p_i x_i$, where $p_i \in P$ and $x_i \in S$ for $i = 1, \ldots, k$. Now there exists an integer $n > 0$ such that $I^n x_i \subset R$ for all $i = 1, \ldots, k$. Thus $I^n \subset P$ and so $I \subset P$. Thus Spec S is equal to the set of all ideals PS where P is a prime ideal of R such that $I \not\subset P$.

The following example shows that the ideal transform S of an ideal I of R is not necessarily a torsion envelope of R, even when I is a faithful, projective ideal of R and S is R-flat.

Example. Let k be a field, K a countably infinite direct product of copies of k, and let R be the set of sequences in K that are constant except for a finite number of coordinates. Let e_n be the element of R that is 0 everywhere except for the nth-coordinate where it is equal to 1. Let $J = \sum_n + Re_n$; then J is a faithful projective ideal of R. Now K is the injective envelope of R and $J^* = K$. Thus K is the ideal transform of J. Since R is a von Neumann regular ring, it has no proper torsion envelopes by Proposition 3.11. Thus K is R-flat, but is not a torsion envelope of R.

We note that if R is a Noetherian local domain of Krull dimension one with quotient field Q, and if I is any non-zero ideal of R, then the ideal transform of I is Q, Q is flat, and of course $IQ = Q$.

4. Semi-Krull domains

Throughout this section R will be an integral domain with quotient field Q. A subring of Q that contains R is called an *over-ring* of R. It follows from Proposition 3.1 that a commutative ring is a torsion envelope of R if and only if it is a flat over-ring of R. Richman has proved part of this result by a different technique [8, Theorem 1]. He has also proved Proposition 4.1 [8, Corollary to Theorem 2], but we shall append a proof using our results for the sake of completeness.

PROPOSITION 4.1. *Let S be a commutative ring extension of R and let $\mathscr{C}$ be the set of prime ideals P of R such that $PS \neq S$. Then S is a flat over-ring of R if and only if $S = \bigcap R_P (P \in \mathscr{C})$.*

Proof. If $S = \bigcap R_P (P \in \mathscr{C})$, then $S_p = R_p$ for all $P \in \mathscr{C}$. Hence S is a flat over-ring of R by Proposition 3.8. Conversely, assume that S is a flat over-ring of R. If $P \in \mathscr{C}$, then $S_P = R_P$ by Proposition 3.8 and hence

$$S \subset \bigcap R_P (P \in \mathscr{C}).$$

On the other hand let $x \in \bigcap R_P (P \in \mathscr{C})$ and $J = (R : x)$. If $JS \neq S$, then by

Lemma 3.7 there exists $P \in \mathscr{C}$ such that $J \subset P$. But since $x \in R_P$, we have $J \not\subset P$. This contradiction shows that $JS = S$, and so $x + R \in t_S(Q/R)$. But $t_S(Q/R) = S/R$ by Proposition 2.3(3), and hence $x \in S$. Thus

$$S = \bigcap R_P(P \in \mathscr{C}).$$

Remarks. We note that if $P_1, \dots, P_n$ is a finite set of height 1 prime ideals of R and $\mathscr{T} = R - \bigcup_{i=1}^n P_i$, then $R_{\mathscr{T}} = \bigcap_{i=1}^n R_{P_i}$ by Proposition 4.1.

DEFINITION. Let $\mathscr{C}$ be a non-empty set of prime ideals of R and let

$$\mathscr{I}_{\mathscr{C}} = \{ I \subset R \mid I \not\subset P \text{ for any } P \in \mathscr{C} \}.$$

Then $\mathscr{I}_{\mathscr{C}}$ is a torsion family of ideals of R and we denote its associated torsion functor by $t_{\mathscr{C}}$. Let $S = \bigcap R_P(P \in \mathscr{C})$, let $\mathscr{I}_S = \{I \subset R \mid IS = S\}$, and let $\overline{\mathscr{C}}$ be the set of prime ideals P of R such that $S \subset R_P$. We call $\overline{\mathscr{C}}$ the *closure* of $\mathscr{C}$. With this notation we have the following:

PROPOSITION 4.2. (1) $t_{\mathscr{C}}(Q/R) = S/R$, $S = \bigcap R_P(P \in \overline{\mathscr{C}})$; *and* $\mathscr{I}_S \subset \mathscr{I}_{\overline{\mathscr{C}}} \subset \mathscr{I}_{\mathscr{C}}$.

(2) *S is a flat over-ring of R if and only if* $\mathscr{I}_S = \mathscr{I}_{\overline{\mathscr{C}}}$.

Proof. (1) Let $T/R = t_{\mathscr{C}}(Q/R)$. Let $x \in S$ and $P \in \mathscr{C}$; then $(R : x) \not\subset P$ and so $(R : x) \in \mathscr{I}_{\mathscr{C}}$. This $S \subset T$. On the other hand let $x \in T$ and $P \in \mathscr{C}$. Since $(R : x) \in \mathscr{I}_{\mathscr{C}}$, we have $(R : x) \not\subset P$ and so $x \in R_P$. Therefore, $x \in S$ and so $S = T$.

It is obvious that $S = \bigcap R_P(P \in \overline{\mathscr{C}})$ and that $\mathscr{C} \subset \overline{\mathscr{C}}$. Hence $\mathscr{I}_{\overline{\mathscr{C}}} \subset \mathscr{I}_{\mathscr{C}}$. Let $I \in \mathscr{I}_S$ and $P \in \overline{\mathscr{C}}$. Then $S_P = R_P$; and since $S = IS$ we have $IR_P = R_P$. Therefore, $I \not\subset P$ and so $I \in \mathscr{I}_{\overline{\mathscr{C}}}$. Thus $\mathscr{I}_S \subset \mathscr{I}_{\overline{\mathscr{C}}}$.

(2) Assume that S is a flat over-ring of R and let $I \in \mathscr{I}_{\overline{\mathscr{C}}}$. Suppose that $I \notin \mathscr{I}_S$; i.e., $IS \neq S$. Then by Lemma 3.7 there exists a prime ideal P of R such that $I \subset P$ and $PS \neq S$. By Proposition 3.8 $S_P = R_P$ and so $P \in \overline{\mathscr{C}}$. But then $I \not\subset P$ by the definition of $\mathscr{I}_{\overline{\mathscr{C}}}$. This contradiction proves that $I \in \mathscr{I}_S$. Hence $\mathscr{I}_S = \mathscr{I}_{\overline{\mathscr{C}}}$.

Conversely, assume that $\mathscr{I}_S = \mathscr{I}_{\overline{\mathscr{C}}}$; then $t_S = t_{\overline{\mathscr{C}}}$. Since $S = \bigcap R_P(P \in \overline{\mathscr{C}})$ we have by part (1) that $t_{\overline{\mathscr{C}}}(Q/R) = S/R$. Therefore S/R is t_S-torsion; i.e., S is a torsion envelope of R; i.e., S is a flat over-ring of R.

We remark that $\overline{\mathscr{C}}$ not only contains every prime ideal of R that is contained in some element of $\mathscr{C}$, but may contain other prime ideals as well.

DEFINITION. If J is an ideal of R, we let rad J denote the intersection of the prime ideals of R that contain J.

PROPOSITION 4.3. *Let R be a Noetherian domain, I a proper, non-zero ideal of R, S the ideal transform of I, and $\mathscr{C}$ the set of prime ideals of R that do not contain I. Then*:

(1) $S = \bigcap R_P(P \in \mathscr{C})$, *and so* $t_{\mathscr{C}}(Q/R) = S/R$.

(2) *If* rad I *is an intersection of prime ideals of R of height one, then S is a flat over-ring of R if and only if $S = IS$.*

Proof. (1) Let $x \in S$ and $P \in \mathscr{C}$. There exists $t > 0$ such that $I^t x \subset R$. Since $I^t \not\subset P$, we have $x \in R_P$; and so

$$S \subset \bigcap R_P(P \in \mathscr{C}).$$

On the other hand let $x \in \bigcap R_P(P \in \mathscr{C})$, $x \notin R$, and let $J = (R : x)$. Then $J \not\subset P$ for any $P \in \mathscr{C}$, and so every prime ideal of R that contains J also contains I. Thus $I \subset \operatorname{rad} J$. Since R is Noetherian, there exists $t > 0$ such that $I^t \subset (\operatorname{rad} J)^t \subset J$. Hence $I^t x \subset R$, and so $x \in S$. Thus

$$S = \bigcap R_P(P \in \mathscr{C}).$$

By Proposition 4.2(1) we have $t_{\mathscr{C}}(Q/R) = S/R$.

(2) If $S = IS$, then S is a flat over-ring of R by Proposition 3.12(3). Assume that rad I is an intersection of prime ideals of R of height one, and let P be one of these prime ideals. Since R is Noetherian and P has height one, there exist elements $a, b \in R$ such that $(Rb : Ra) = P$. Let $x = a/b$; then $(R : x) = P$, and so $x \in P^{-1} \subset I^{-1} \subset S$. If S is a flat over-ring of R; i.e., a torsion envelope of R, then $PS = S$. Hence $(\operatorname{rad} I)S = S$; and since I contains a power of rad I we have $IS = S$.

DEFINITION. We shall say that R is a *semi-Krull domain* if:

(1) Every non-zero, proper, principal ideal of R is a finite intersection of height one primary ideals of R.

(2) Every height one primary ideal of R contains a power of its associated prime ideal.

LEMMA 4.4. *Let R be a semi-Krull domain and a, b non-zero elements of R such that $a \notin Rb$. Let $x = a/b \in Q$ and $J = (R : x)$. Then J is a finite intersection of height one primary ideals of R; and some power of* rad J *is contained in J.*

Proof. We have $Rb = J_1 \cap \cdots \cap J_n$ where J_i is a P_i-primary ideal of R and P_i is a prime ideal of R of height one. Thus

$$J = (R : x) = (Rb : Ra) = (J_1 : Ra) \cap \cdots \cap (J_n : Ra).$$

Now $(J_i : Ra)$ is either a P_i-primary ideal of R, or is equal to R. Hence some power of rad J is contained in J.

The following proposition will provide many examples of semi-Krull domains.

PROPOSITION 4.5. *R is a semi-Krull domain if and only if R satisfies the following three conditions*:

(1) *If b is a non-zero, non-unit element of R, then the set of height one prime ideals of R that contain b is finite and not empty.*

(2) *If P is a height one prime ideal of R, then every non-zero ideal of R_P contains a power of PR_P.*

(3) *$\bigcap R_P = R$, where P ranges over the height one prime ideals of R.*

Proof. Assume that R is a semi-Krull domain. Let b be a non-zero, non-unit element of R. Then $Rb = J_1 \cap \cdots \cap J_n$, where J_i is P_i-primary and P_i is a prime ideal of R of height one. Then Rb contains a power of $P_1P_2 \cdots P_n$, and thus $P_1, P_2, \ldots, P_n$ are the only height one prime ideals of R that contain b.

Let P be a height one prime ideal of R; then every non-zero, proper, principal ideal of R_P is of the form bR_P, where $b \in P$. With the notation of the preceding paragraph we can assume that $P = P_1$, and hence $bR_P = J_1R_P$ contains a power of PR_P.

Let $x = a/b \in \bigcap R_P$, where P ranges over all height one prime ideals of R, and $a, b \in R$. Then $(R : x) = (Rb : Ra)$ is not contained in any height one prime ideal of R. Thus by Lemma 4.4, $a \in Rb$; i.e., $x \in R$.

Conversely, assume that the three conditions of the proposition are satisfied. Let P be a height one prime ideal of R and J a P-primary ideal of R. Since JR_P contains a power of PR_P, J contains a power of P.

Let b be a non-zero, non-unit element of R, and let $P_1, \ldots, P_n$ be the set of height one prime ideals of R that contain b. Since bR_{P_i} is a $P_iR_{P_i}$-primary ideal of R_{P_i}, $J_i = bR_{P_i} \cap R$ is a P_i-primary ideal of R that contains b. If P is a height one prime ideal of R that is not equal to any of the P_i's, then $bR_P = R_P$. Since $R = \bigcap R_P$, where P ranges over all of the height one prime ideals of R, it follows that $Rb = J_1 \cap \cdots \cap J_n$.

Examples of semi-Krull domains. (1) A Krull domain satisfies the three conditions of Proposition 4.5.

(2) A Noetherian integrally closed domain is a Krull domain.

(3) A Noetherian domain with the property that every non-zero principal ideal is unmixed of height one satisfies the axioms of a semi-Krull domain.

(4) A Cohen-Macaulay Noetherian domain satisfies the property of Example (3).

PROPOSITION 4.6. *Let R be a semi-Krull or a Noetherian domain. Let $\mathscr{C}$ be a non-empty set of height one prime ideals of R and let $S = \bigcap R_P(P \in \mathscr{C})$. Then*

$S_P = R_P$ for all $P \in \mathscr{C}$; and $S_{P'} = Q$ for every prime ideal P' of R that does not contain any $P \in \mathscr{C}$.

Proof. If $P \in \mathscr{C}$, then $S \subset R_P$, and so $S_P = R_P$. Let x be a non-zero element of Q; then if R is Noetherian, or if R is a semi-Krull domain by Lemma 4.4, there are only a finite number of height one prime ideals P of R such that $x \notin R_P$. Thus we have an exact sequence

$$0 \to Q/S \to \sum_P \oplus\, Q/R_P(P \in \mathscr{C}).$$

Let P' be a prime ideal of R that does not contain any $P \in \mathscr{C}$. We have a derived exact sequence

$$0 \to Q/S_{P'} \to \sum_P \oplus\, Q/(R_P)_{P'}(P \in \mathscr{C}).$$

Thus to prove that $S_P' = Q$ it is sufficient to prove that $(R_P)_{P'} = Q$ for every $P \in \mathscr{C}$. But if $P \in \mathscr{C}$, then $R - P'$ contains an element of P; and hence $(R_P)_{P'}$ has no non-zero prime ideals. Thus $(R_P)_{P'} = Q$.

PROPOSITION 4.7. *Let R be a semi-Krull domain, $\mathscr{C}_1$ a non-empty set of height one prime ideals of R, and $S = \bigcap R_P(P \in \mathscr{C}_1)$. Then the following statements are equivalent*:

(1) *S is a flat over-ring of R.*

(2) *$\mathscr{F}_S$ is equal to the category of t_S-torsion-free R-modules.*

(3) *If P is a prime ideal of R such that $PS \neq S$, then $PS \cap R = P$.*

(4) *If P is a height one prime ideal of R such that $PS \neq S$, then $PS \cap R = P$.*

In this case $\mathscr{C}_1$ is the set of height one prime ideals P of R such that $PS \neq S$.

Proof. (1) $\Rightarrow$ (2) follows from Proposition 2.5; (2) $\Rightarrow$ (3) follows from Proposition 2.4; and (3) $\Rightarrow$ (4) is trivial. Hence assume that (4) is satisfied. To prove that S is a flat over-ring of R we shall prove that S/R is t_S-torsion. Let $x \in S$, $J = (R : x)$; and suppose that $JS \neq S$. By Lemma 4.4 rad J is a finite intersection of height one prime ideals of R and some power of rad J is contained in J. Thus there exists a height one prime ideal P of R such that $J \subset P$ and $PS \neq S$. By assumption we have $PS \cap R = P$.

If $P \in \mathscr{C}_1$, then $S \subset R_P$, and so $x \in R_P$. But then $J \not\subset P$, and this contradiction shows that $P \notin \mathscr{C}_1$. Hence by Proposition 4.6 we have $S_P = Q$. But then $PS \cap R \neq P$, and this contradiction shows that $JS = S$. Thus S/R is t_S-torsion; i.e., S is a flat over-ring of R.

Now assume that S is a flat over-ring of R. If $P \in \mathscr{C}_1$, then $S \subset R_P$, and so $PS \neq S$. On the other hand suppose that P is a height one prime ideal of R such that $PS \neq S$. If $P \notin \mathscr{C}_1$, then $S_P = Q$ by Proposition 4.6. But since

$PS \neq S$ we have by Proposition 3.8 that $S_P = R_P$. This contradiction shows that $P \in \mathscr{C}_1$.

PROPOSITION 4.8. *Let R be a semi-Krull domain and S a commutative ring extension of R. Let $\mathscr{C}_1$ be the set of height one prime ideals P of R such that $PS \neq S$. Then S is a flat over-ring of R if and only if $S = \bigcap R_P(P \in \mathscr{C}_1)$. In this case S is also a semi-Krull domain.*

Proof. If $S = \bigcap R_P(P \in \mathscr{C}_1)$, then $PS \cap R = P$ for all $P \in \mathscr{C}_1$. Thus S is a flat over-ring of R by Proposition 4.7. On the other hand assume that S is a flat over-ring of R. By Proposition 4.1 we have $S \subset \bigcap R_P(P \in \mathscr{C}_1)$. On the other hand let $x \in \bigcap R_P(P \in \mathscr{C}_1)$, $x \notin R$, and let $J = (R : x)$. By Lemma 4.4

$$\operatorname{rad} J = P_1 \cap \cdots \cap P_n,$$

where the P_i's are height one prime ideals of R; and some power of rad J is contained in J. Since $J \not\subset P$ for any $P \in \mathscr{C}_1$, none of the P_i's are in $\mathscr{C}_1$. Thus by definition of $\mathscr{C}_1$, $P_iS = S$ for all $i = 1, \dots, n$. Therefore, $JS = S$ and so $x + R$ is an element of $t_S(Q/R)$. But $t_S(Q/R) = S/R$ by Proposition 2.3(3), and so $x \in S$. Thus $S = \bigcap R_P(P \in \mathscr{C}_1)$.

Assume that S is a flat over-ring of R. They by Corollary 3.9 the set

$$\{PS | P \in \mathscr{C}_1\}$$

is the set of height one prime ideals of S. If $P \in \mathscr{C}_1$, then $S_P = R_P$ is a quasi-local domain of Krull dimension one with maximal ideal $PS_P = PR_P$. Thus S_P has no proper localization other than Q and so $S_{PS} = S_P$. Therefore

$$\bigcap S_{PS}(P \in \mathscr{C}_1) = \bigcap R_P(P \in \mathscr{C}_1) = S.$$

Moreover, it follows that every non-zero ideal of $S_{PS} = R_P$ contains a power of $PR_P = (PS)S_{PS}$.

Let $x = a/b$ be a non-zero, non-unit element of S, where $a, b \in R$. Let $P_1, \dots, P_n$ be the set of height one prime ideals of R that contain Ra. Since $Sa \subset Sx$ and Ra contains a power of $P_1P_2 \cdots P_n$, it follows that the set of height one prime ideals of S that contain Sx is a subset of $P_1S, \dots, P_nS$. Thus all three conditions of Proposition 4.5 are satisfied by S, and so S is a semi-Krull domain.

PROPOSITION 4.9. *Let R be a semi-Krull domain, I a non-zero ideal of R, and S the ideal transform of I. Let $\mathscr{C}$ be the set of prime ideals of R that do not contain I, and $\mathscr{C}_1$ the set of prime ideals in $\mathscr{C}$ of height one. Then $S = \bigcap R_P(P \in \mathscr{C})$, and $S = R_P(P \in \mathscr{C}_1)$; and $t_{\mathscr{C}}(Q/R) = S/R = t_{\mathscr{C}_1}(Q/R)$.*

Moreover, if I is a finite intersection of height 1 *primary ideals of R, then S is flat if and only if $S = IS$.*

Proof. Let $x \in S$ and $P \in \mathscr{C}$. There exists $t > 0$ such that $I^t x \subset R$. Since $I^t \not\subset P$, we have $x \in R_P$; and so $S \subset \bigcap R_P(P \in \mathscr{C})$. Since $\mathscr{C}_1 \subset \mathscr{C}$ we have

$$\bigcap R_P(P \in \mathscr{C}) \subset \bigcap R_P(P \in \mathscr{C}_1).$$

On the other hand let $x \in \bigcap R_P(P \in \mathscr{C}_1)$, $x \notin R$, and let $J = (R : x)$. Then $J \not\subset P$ for any $P \in \mathscr{C}_1$, and so every height one prime ideal of R that contains J also contains I. By Lemma 4.4, rad J is a finite intersection of height one prime ideals of R and there exists $t > 0$ such that $(\operatorname{rad} J)^t \subset J$. Since $I \subset \operatorname{rad} J$, we have $I^t \subset J$ also. Thus $I^t x \subset R$, and so $x \in S$. Thus

$$S = \bigcap R_P(P \in \mathscr{C}) = \bigcap R_P(P \in \mathscr{C}_1).$$

By Proposition 4.2(1) we have $t_{\mathscr{C}}(Q/R) = S/R = t_{\mathscr{C}_1}(Q/R)$.

If $S = IS$, then S is a flat over-ring of R by Proposition 3.12(3). Conversely, assume that S is a flat over-ring of R. Let P' be a height one prime ideal of R that contains I. Since $S = \bigcap R_P(P \in \mathscr{C}_1)$ and $P' \notin \mathscr{C}_1$, it follows from Proposition 4.7 that $P'S = S$. If I is a finite intersection of height one primary ideals of R, then I contains a power of rad I, and so we have $S = IS$.

Remarks. (1) With the notation of Proposition 4.9, we see that if I is not contained in any height 1 prime ideal of R, then $S = R$.

(2) Let R be a semi-Krull domain and I a non-zero, proper, projective ideal of R. Then I^{-1} is finitely generated by elements $x_1, \ldots, x_n$ of Q; and hence

$$I = \bigcap_{i=1}^{n} (R : x_i)$$

is a finite intersection of height one primary ideals of R by Lemma 4.4. Let S be the ideal transform of I; then by Proposition 4.9, $S = \bigcap R_P$, where P ranges over the height one prime ideals of R that do not contain I; and S is a flat over-ring of R because $S = IS$.

(3) Let S be a commutative ring extension of a semi-Krull domain R. Then it follows from Proposition 4.9 that S is the ideal transform of an ideal that is a finite intersection of height one primary ideals of R if and only if $S = \bigcap R_P(P \in \mathscr{C}_1)$, where $\mathscr{C}_1$ is a set of height one prime ideals of R whose complement in Spec R contains only a finite number of height one prime ideals.

5. Complemented extensions

Throughout this section R will be an integral domain with quotient field Q.

DEFINITION. Let A and A' be R-submodules of Q. Then A and A' are said to be *complementary extensions* of R if $A \cap A' = R$ and $A + A' = Q$; i.e., if

$$Q/R = A/R \oplus A'/R.$$

An over-ring S of R is said to be a *complemented extension* of R if S/R is a direct summand of Q/R. We recall that $\operatorname{Spec}_S R = \{P \in \operatorname{Spec} R \mid PS \neq S\}$.

If S and T are over-rings of R, we let $ST = \{\sum_{i=1}^n s_i t_i \mid s_i \in S, t_i \in T\}$. Then ST is the over-ring of R generated by S and T.

PROPOSITION 5.1. *Let A and A' be complementary extensions of R. Then*:

(1) *A and A' are flat over-rings of R.*

(2) *Let $\mathscr{C} = \operatorname{Spec}_A R$ and $\mathscr{C}' = \operatorname{Spec}_{A'} R$. Then $\mathscr{C} = \overline{\mathscr{C}}, \mathscr{C}' = \overline{\mathscr{C}'}$; $\mathscr{C} \cup \mathscr{C}' = \operatorname{Spec} R$; and $\mathscr{C} \cap \mathscr{C}' = 0$.*

(3) *$A = \bigcap R_P (P \in \mathscr{C})$ and $A' = \bigcap R_{P'} (P' \in \mathscr{C}')$. Thus A' is the unique complement of A.*

(4) *If $P \in \mathscr{C}$, then $A_P = R_P$ and $A'_P = Q$.*

(5) *If I is a non-zero ideal of R then $IA \cap R = I$ if and only if $IA' = A'$.*

Proof. Let $x \in A$; then $x = a/b$ where $a, b \in R$. Since A/R is a direct summand of Q/R, it is a divisible R-module; and hence $A = bA + R$. Thus

$$xA = aA + xR \subset A,$$

and so A is an over-ring of R. Similarly A' is an over-ring of R. We have the exact sequence

$$(*) \qquad 0 \to R \to A \oplus A' \to Q \to 0.$$

This sequence shows that A and A' are flat over-rings of R.

Let P be a non-zero prime ideal of R. Then from the exact sequence $(*)$ we obtain the exact sequence

$$0 \to R_P \to A_P \oplus A_{P'} \to Q \to 0.$$

But R_P is a quasi-local ring, and hence Q/R_P is an indecomposable R_P-module (see [5, Proposition 1.2]). Thus either $A_P = R_P$ and $A'_P = Q$, or vice versa. Statements (2), (3) and (4) now follow from this and Proposition 4.1.

Let I be a non-zero ideal of R; then since A/R is a divisible R-module, we have $A = IA + R$. Thus $A/IA \cong R/(IA \cap R)$. Hence from the exact sequence ($*$) we have

$$R/I \cong A/IA \oplus A'/IA' \cong R/(IA \cap R) \oplus A'/IA'.$$

Since any homomorphism from R/I onto itself is an isomorphism, it follows from this that $IA \cap R = I$ if and only if $A'/IA' = 0$.

PROPOSITION 5.2. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two sets of prime ideals of R such that $\overline{\mathscr{C}} \cup \overline{\mathscr{C}}' = \operatorname{Spec} R$. Let $S = \bigcap R_P(P \in \mathscr{C})$ and $S' = \bigcap R_{P'}(P' \in \mathscr{C}')$. Then the following statements are equivalent*:

(1) *S and S' are complementary extensions of R.*

(2) *$\overline{\mathscr{C}} \cap \overline{\mathscr{C}'} = 0$ and S and S' are flat over-rings of R.*

(3) *$SS' = Q$.*

Proof. Without loss of generality we can assume that $\mathscr{C} = \overline{\mathscr{C}}$ and $\mathscr{C}' = \overline{\mathscr{C}'}$. Since $\mathscr{C} \cup \mathscr{C}' = \operatorname{Spec} R$, we have $S \cap S' = \bigcap R_P(P \in \operatorname{Spec} R) = R$.

(1) $\Rightarrow$ (2) This follows from Proposition 5.1.

(2) $\Rightarrow$ (3) If $SS' \neq Q$, then SS' has a non-zero prime ideal $\mathscr{P}$ and we let $P = \mathscr{P} \cap R$. We can assume that $P \in \mathscr{C}$: and then since $P \notin \mathscr{C}'$ we have $S'_P \neq R_P$. Because S' is a flat over-ring of R we have $PS' = S'$ by Proposition 3.8. Therefore, $P(SS)' = SS'$. But $P(SS') \subset \mathscr{P}$; and this contradiction shows that $SS' = Q$.

(3) $\Rightarrow$ (1) Let P be a prime ideal of R; then we can assume that $P \in \mathscr{C}$ and hence $S_P = R_P$. But then $Q = Q_P = S_P S'_P = S'_P$, and hence

$$(S + S')_P = S_P + S'_P = Q.$$

Since this is true for every prime ideal P of R, we have $S + S' = Q$; i.e., S and S' are complementary extensions of R.

PROPOSITION 5.3. *Let I be a non-zero ideal of R, $\mathscr{C}$ the set of prime ideals of R that do not contain I, $\mathscr{C}'$ the set of prime ideals of R that contain I,*

$$S = \bigcap R_P(P \in \mathscr{C}) \quad \textit{and} \quad \mathscr{S} = \{1 - a \mid a \in I\}.$$

Then:

(1) *$R_{\mathscr{S}} = \bigcap R_{P'}(P' \in \mathscr{C}')$.*

(2) *If $\operatorname{rad} I = P_1 \cap \cdots \cap P_n$ where every P_i is a prime ideal of R of height one, then $R_{\mathscr{S}} = \bigcap_{i=1}^n R_{P_i}$ if and only if every P_i is a maximal ideal of R.*

(3) *S and $R_{\mathscr{S}}$ are complementary extensions of R if and only if $S_{\mathscr{S}} = Q$.*

Proof. (1) Let $x \in R_{\mathscr{S}}$ and $J = (R : x)$; then there exists $a \in I$ such that $1 - a \in J$. Hence $J \not\subset P'$ for any $P' \in \mathscr{C}'$ and so $x \in \bigcap R_{P'}(P' \in \mathscr{C}')$. On the other hand let $x \in \bigcap R_{P'}(P' \in \mathscr{C}')$ and let $J = (R : x)$. Suppose $J \cap \mathscr{S}$

$= \emptyset$; then $I + J \neq R$. Thus there exists a prime ideal P' of R such that $I + J \subset P'$. Therefore $P' \in \mathscr{C}'$, and so $x \in R_{P'}$. But then $J \not\subset P'$, and this contradiction shows that $J \cap \mathscr{S} \neq \emptyset$, and hence $x \in R_{\mathscr{S}}$.

(We note that in fact, since $R_{\mathscr{S}}$ is a flat over-ring of R, we have $R_{\mathscr{S}} = \bigcap R_P$, where P ranges over the prime ideals P of R such that $PR_{\mathscr{S}} \neq R_{\mathscr{S}}$; i.e., the prime ideals P such that $I + P \neq R$, and hence this is the set $\overline{\mathscr{C}'}$.)

(2) Assume that rad $I = P_1 \cap \cdots \cap P_n$ where every P_i is a prime ideal of R of height one. By part (1) we have $R_{\mathscr{S}} \subset \bigcap_{i=1}^n R_{P_i}$; and by the remark following Proposition 4.1 we have $\bigcap_{i=1}^n R_{P_i} = R_{\mathscr{T}}$, where $\mathscr{T} = R - \bigcup_{i=1}^n P_i$.

Assume that $R_{\mathscr{S}} = R_{\mathscr{T}}$, and that one of the P_i's, say P_1, is not a maximal ideal of R. Let P be a maximal ideal of R containing P_1. Then P is not equal to any of the P_i's, and hence $P \cap \mathscr{T} \neq \emptyset$. Therefore $PR_{\mathscr{T}} = R_{\mathscr{T}}$, and so $PR_{\mathscr{S}} = R_{\mathscr{S}}$. Hence there exists $a \in I$ such that $1 - a \in P$. But $I \subset P_1 \subset P$; and this contradiction shows that every P_i is a maximal ideal of R.

Conversely, assume that every P_i is a maximal ideal of R. Let $u \in \mathscr{T}$ and suppose that $I + Ru \neq R$. Then there exists a prime ideal P such that $I + Ru \subset P$. But since $I \subset P$, P is equal to one of the P_i's; and since $u \in P$, P is not equal to any of the P_i's. This contradiction shows that $I + Ru = R$. Hence there exists $r \in R$ such that $ru \in \mathscr{S}$. Thus $1/u = r/ru \in R_{\mathscr{S}}$. Therefore $R_{\mathscr{T}} \subset R_{\mathscr{S}}$, and so $R_{\mathscr{T}} = R_{\mathscr{S}}$.

(3) If $S + R_{\mathscr{S}} = Q$, then clearly $S_{\mathscr{S}} = Q$. Conversely, assume that $S_{\mathscr{S}} = Q$; i.e. $SR_{\mathscr{S}} = Q$. Since $S = \bigcap R_P(P \in \mathscr{C})$, and $R_{\mathscr{S}} = \bigcap R_{P'}(P' \in \mathscr{C}')$ by part (1), and $\mathscr{C} \cup \mathscr{C}' = \operatorname{Spec} R$, we see that S and $R_{\mathscr{S}}$ are complementary extensions of R by Proposition 5.2.

PROPOSITION 5.4. *Let I be an ideal of R, S the ideal transform of I, $\mathscr{C}$ the set of prime ideals of R that do not contain I; and $\mathscr{S} = \{1 - a \mid a \in I\}$.*

(1) *If $S = IS$, then S is a flat over-ring of R and $S = \bigcap R_P(P \in \mathscr{C})$.*

(2) *If S is a complemented extension of R, then the following statements are equivalent*:

(a) $S = IS$.

(b) $S = \bigcap R_P(P \in \mathscr{C})$ *and* $S \not\subset R_P$ *if* $I \subset P$.

(c) $R_{\mathscr{S}}$ *is the complement of S.*

Proof. (1) Assume that $S = IS$. Then S is a flat over-ring of R by Proposition 3.12(3). Hence by Proposition 4.1, $S = \bigcap R_P(P \in \mathscr{D})$, where $\mathscr{D}$ is the set of prime ideals of R such that $PS \neq S$. Let $P \in \mathscr{D}$; then $PS \neq S$ and $IS = S$ and so $I \not\subset P$; i.e., $P \in \mathscr{C}$. Conversely, let $P \in \mathscr{C}$. Let $x \in S$ and $J = (R : x)$; then there exists $t > 0$ such that $I^t \subset J$. Since $I \not\subset P$, we have $J \not\subset P$; thus $x \in R_P$, and so $S \subset R_P$. Therefore, $PS \neq S$, and so $P \in \mathscr{D}$. Hence $\mathscr{C} = \mathscr{D}$, and so $S = \bigcap R_P(P \in \mathscr{C})$.

(2) (a) $\Rightarrow$ (b) Follows from part (1).

(b) $\Rightarrow$ (c) Let A' be the complement of S. Let $\mathscr{C}'$ be the set of prime ideals of R that contain I; by Proposition 5.3(1) we have $R_{\mathscr{S}} = \bigcap R_{P'}(P' \in$

$\mathscr{C}'$). Since $S \not\subset R_P$ for $P \notin \mathscr{C}$, we have $A' = \bigcap R_{P'}(P' \in \mathscr{C}')$ by Proposition 5.1. Thus $R_{\mathscr{S}} = A'$.

(c) $\Rightarrow$ (a) Since $IR_{\mathscr{S}} \cap R = I$, we have $S = IS$ by Proposition 5.1.

PROPOSITION 5.5. *Let R be a semi-Krull domain and $I = I_1 \cap \cdots \cap I_n$, where I_i is a P_i-primary ideal of R and P_i is a height one prime ideal of R. Let S be the ideal transform of I and $S' = \bigcap_{i=1}^n R_{P_i}$. Then $S \cap S' = R$; and if S is a flat over-ring of R we have $SS' = Q$.*

Proof. Let $\mathscr{C}_1$ be the set of height one prime ideals of R that do not contain I and let $\mathscr{C}_1' = \{P_1, \ldots, P_n\}$. By Proposition 4.9, $S = \bigcap R_P(P \in \mathscr{C}_1)$. Hence by Proposition 4.5(3), we have $S \cap S' = R$. By Proposition 4.6, $S_{P_i} = Q$; and if $P \in \mathscr{C}_1$, then $S_P' = Q$. Thus $(SS')_P = Q$ for every height one prime ideal P of R.

By the remarks following Proposition 4.1, we have $S' = R_{\mathscr{T}}$, wher $\mathscr{T} = R - \bigcup_{i=1}^n P_i$; and so S' is a flat over-ring of R. Thus if S is a flat over-ring of R, then SS' is also a flat over-ring of R. In this case, since $SS' \not\subset R_P$ for any height one prime ideal P of R, we have $SS' = Q$ by Proposition 4.8.

Remarks. With the preceding notation Proposition 5.5 seems to suggest that if S is a flat over-ring of R, then S and S' are complementary extensions of R; but there are many examples to show that this is not true in general. For example let R be a Noetherian regular local ring of Krull dimension > 1. Let $I = Rp$, where p is a non-zero prime element of R, and let $\mathscr{S} = \{p^n | n \geq 0\}$. Then $S = R_{\mathscr{S}}$ is flat and so is $S' = R_p$. We have $SS' = Q$ and $S \cap S' = R$; but if q is a non-unit of $R - R_p$, then $1/qp \notin S + S'$. Thus $S + S' \neq Q$ and so S and S' are not complementary extensions of R. However, if S is a complemented extension of R, then by Propositions 4.9, 5.1, 5.3 and 5.4, $R_{\mathscr{S}}$ is the complement of S, $R_{\mathscr{S}} = S'$, and every P_i is a maximal ideal of R. We shall have an even sharper result in the next Proposition.

PROPOSITION 5.6. *Let R be a semi-Krull or a Noetherian domain. Let I be a non-zero, proper ideal of R, $\mathscr{S} = \{1 - a | a \in I\}$, and S the ideal transform of I. Then S and $R_{\mathscr{S}}$ are complementary extensions of R if and only if I is a finite intersection of primary ideals whose associated prime ideals have height one and are maximal ideals of R. In this case if P is a prime ideal of R, then $PS = S$ if and only if P is one of the associated prime ideals of I.*

Proof. Assume that S and $R_{\mathscr{S}}$ are complementary extensions of R. Let M be a maximal ideal of R that contains I, and let P be a non-zero prime ideal of R contained in M. Since $MR_{\mathscr{S}} \neq R_{\mathscr{S}}$, we have $PR_{\mathscr{S}} \neq R_{\mathscr{S}}$ also. Thus by Proposition 5.1(2) we have $PS = S$. Hence $1 = \sum_{j=1}^m p_j x_j$, where $p_j \in P$ and

$x_j \in S$. Now there exists $t > 0$ such that $I^t x_j \subset R$ for all $j = 1, \dots, m$. Hence

$$I^t \subset \sum_{j=1}^{m} p_j\left(I^t x_j\right) \subset P,$$

and so $I \subset P$.

Now I is contained in at most a finite number of height one prime ideals of R, and thus M contains only a finite number of height one prime ideals of R. In either the Noetherian or semi-Krull case M is the union of the height one prime ideals of R that it contains. Thus M has height one. Therefore, if $P_1, \dots, P_n$ are the height one prime ideals of R that contain I, then every P_i is a maximal ideal of R, and no other prime ideal of R contains I. Thus we have

$$I = \bigcap_{i=1}^{n} \left(IR_{P_i} \cap R\right),$$

and $IR_{P_i} \cap R$ is a P_i-primary ideal of R. Now as we have seen in the preceding paragraph, $P_i S = S$ for all $i = 1, \dots, n$; and if P is a prime ideal of R that is not equal to any of the P_i's, then $S \subset R_P$ by either Proposition 4.3 or 4.9, and so $PS \neq S$.

Conversely, assume that $I = I_1 \cap \dots \cap I_n$, where I_i is a P_i-primary ideal of R and P_i is a maximal ideal of R of height one. By Proposition 4.3 or 4.9 we have $S = \cap R_P(P \in \mathscr{C})$ where $\mathscr{C}$ is the set of prime ideals of R that are not equal to any of the P_i's; and by Proposition 5.3(2), $R_{\mathscr{S}} = \bigcap_{i=1}^{n} R_{P_i}$. Thus $S \cap R_{\mathscr{S}} = R$; and it follows from Proposition 4.6 that $(R_{\mathscr{S}})_P = Q$ for every $P \in \mathscr{C}$. Thus to prove that S and $R_{\mathscr{S}}$ are complementary extensions of R it is sufficient to prove that $S_{P_i} = Q$ for all $i = 1, \dots, n$.

If R is a semi-Krull domain, then $S = \bigcap R_P(P \in \mathscr{C}_1)$, where $\mathscr{C}_1$ is the set of height one prime ideals of R that are not equal to any of the P_i's, by Proposition 4.9. Thus in this case $S_{P_i} = Q$ for all $i = 1, \dots, n$ by Proposition 4.6. Hence we can assume that R is a Noetherian domain.

Let P be one of the P_i's, let $x \in Q$, and let $J = \{r \in R \mid rx \in S\}$. If $J \not\subset P$, then $x \in S_P$, and hence we can assume that $J \subset P$. Since P has height one, P is an associated prime ideal of J, and so $J = J_1 \cap J_2$ where J_1 is P-primary and J_2 is a finite intersection of primary ideals of R whose associated prime ideals of R are all different from P. Since P is a maximal ideal of height one, we have $J_1 + J_2 = R$. Thus $1 = a_1 + a_2$ where $a_1 \in J_1$ and $a_2 \in J_2$; and so $x = a_1 x + a_2 x$. We have $J_2 a_1 x \subset S$ and $J_1 a_2 x \subset S$. Since $J_2 \not\subset P$, we have $a_1 x \in S_P$, and thus it is sufficient to show that $a_2 x \in S$. Now there exists $t > 0$ such that $I^t \subset J_1$, and hence $I^t a_2 x$ is a finitely generated R-submodule of S. Thus there exists $m > 0$ such that $I^m(I^t a_2 x) \subset R$. Therefore, $a_2 x \in S$, and hence $x \in S_P$. Thus $S_{P_i} = Q$ for all $i = 1, \dots, n$.

COROLLARY 5.7. *Let R be a semi-Krull (or a Noetherian domain); let $\mathscr{C}'$ be a non-empty set of height one maximal ideals of R; let $\mathscr{C}$ be the complement of*

$\mathscr{C}'$ in Spec R; *and let* $S = \bigcap R_P(P \in \mathscr{C})$. *Then S is a flat over-ring of R and S is a semi-Krull domain (or a Noetherian domain); and if P is a prime ideal of R, then $PS = S$ if and only if $P \in \mathscr{C}'$.*

Proof. If $P \in \mathscr{C}$, then $S_P = R_P$; on the other hand let $P \in \mathscr{C}'$ and let $S(P) = \bigcap R_{P'}$ where P' ranges over all prime ideals of R different from P. Then by Proposition 5.6, we have $PS(P) = S(P)$. Since $S(P) \subset S$, we have $PS = S$ also. Thus S is a flat over-ring of R by Proposition 3.8. If R is a semi-Krull domain, then S is a semi-Krull domain by Proposition 4.8; and if R is a Noetherian domain, then S is a Noetherian domain by Proposition 3.6.

Remarks. With the notation of Corollary 5.7, we note that by Corollary 3.9 there is a one to one, order preserving correspondence between Spec S and $\mathscr{C}$ given by $P \to PS$ for $P \in \mathscr{C}$. Thus Corollary 5.7 shows that we can remove any or all of the height one maximal ideals of R and no other prime ideals by passing to a suitable flat over-ring of R.

DEFINITION. If A is an R-module, let $E(A)$ denote the injective envelope of A.

LEMMA 5.8. *Let R be an integral domain, $I = I_1 \cap \cdots \cap I_n$, where I_i is a P_i-primary ideal of R and P_i is a non-zero prime ideal of R; and assume that $E(R/I) \subset Q/I$. Then:*

(1) *If P is a prime ideal of R, then $P + I \neq R$ if and only if there exists one of the P_i's such that either $P_i \subset P$ or $P \subset P_i$.*

(2) *If R is a Noetherian domain, then every P_i is a height one maximal ideal of R.*

(3) *If R is a semi-Krull domain and every P_i has height one, then every P_i is a maximal ideal of R.*

Proof. (1) Let $\mathscr{S} = \{1 - a \mid a \in I\}$; then by [5, Proposition 2.3], $\text{inj.dim}_{R_{\mathscr{S}}} I_{\mathscr{S}} = 1$. Since $R_{\mathscr{S}}/I_{\mathscr{S}} \subset Q/I_{\mathscr{S}}$, we can without loss of generality assume that $R = R_{\mathscr{S}}$ and that $\text{inj.dim}_R I = 1$. Thus I is contained in the Jacobson radical of R, and hence by [4, Corollary 2.5] we have $E(R/I) = Q/I$.

Let P be a prime ideal of R and assume that $P \not\subset P_i$ for any $i = 1, \ldots, n$. Then there exists $b \in P$ such that $b \notin \bigcup_{i=1}^n P_i$; and we let $x = 1/b + I \in Q/I$. Now there exists a monomorphism: $R/I \to R/I_1 \oplus \cdots \oplus R/I_n$ such that

$$1 + I \to (1 + I_1, \ldots, 1 + I_n).$$

Since Q/I is an essential extension of R/I, this monomorphism extends to a monomorphism

$$Q/I \to E(R/I_1) \oplus \cdots \oplus E(R/I_n).$$

We identify Q/I with its image and we have $x = (x_1, \ldots, x_n)$ where $x_i \in E(R/I_i)$. Then

$$(bx_1, \ldots, bx_n) = bx = (1 + I_1, \ldots, 1 + I_n)$$

and so $(0 : bx_i) = I_i$ for all $i = 1, \ldots, n$. Since $b \notin P_i$ it follows that $(0 : x_i) = I_i$ also. Thus $bI = (0 : x) = \bigcap_{i=1}^n (0 : x_i) = \bigcap_{i=1}^n I_i = I$. Therefore $I \subset P$; and since rad $I = P_1 \cap \cdots \cap P_n$, it follows that there exists P_i such that $P_i \subset P$.

(2) Assume that R is a Noetherian domain. We follow the argument of part (1), except that we allow P to be any height one prime ideal of R and b to be any non-zero element of P. Then as in (1) we have $bI = \bigcap_{i=1}^n (0 : x_i)$ and x_i is a non-zero element of $E(R/I_i)$. But since R is Noetherian and I_i is a P_i-primary ideal of R, $(0 : x_i)$ is also a P_i-primary ideal of R. Hence there exists an integer $k > 0$ such that $(P_1 \cap \cdots \cap P_n)^k \subset bI \subset P$, and thus there exists a P_i such that $P_i \subset P$. But since P has height one it follows that $P_i = P$. Thus R has only a finite number of height one prime ideals, and so R has Krull dimension one. Thus every P_i is a maximal ideal of R of height one.

(3) Assume that R is a semi-Krull domain and that every P_i has height one. We follow the proof in part (1) except that we assume that P is a height one prime ideal of R different from any of the P_i's and that b is an element of P not in any of the P_i's. Then as in (1) we have $I = bI \subset P$; and since I contains a power of $P_1 \cap \cdots \cap P_n$, there exists a P_i such that $P_i \subset P$. Since P has height one, we have $P = P_i$; and this contradiction shows that the P_i's are the only height one prime ideals of R. It follows that R has Krull dimension one, and so every P_i is a maximal ideal of R.

COROLLARY 5.9. *Let $I = I_1 \cap \cdots \cap I_n$, where I_i is a P_i-primary ideal of R and P_i is a non-zero prime ideal of R. Let S be the ideal transform of I and $\mathcal{S} = \{1 - a \mid a \in I\}$. Assume that either R is a Noetherian domain, or that R is a semi-Krull domain and every P_i has height one. Then $E(R/I) \subset Q/I$ if and only if* $\text{inj.dim}_{R_{\mathcal{S}}} I_{\mathcal{S}} = 1$ *and every P_i is a maximal ideal of R of height one. In this case S and $R_{\mathcal{S}}$ are complementary extensions of R.*

Proof. If R is any integral domain then by [5, Proposition 2.3], $E(R/I) \subset Q/I$ if and only if $\text{inj.dim}_{R_{\mathcal{S}}} I_{\mathcal{S}} = 1$ and $R_{\mathcal{S}}$ is a complemented extension of R. Corollary 5.9 now follows immediately from Proposition 5.6 and Lemma 5.8.

DEFINITION. We recall that a Noetherian domain of Krull dimension one is called a *Gorenstein ring* if $\text{inj.dim}_R R = 1$.

PROPOSITION 5.10. *Let R be a semi-Krull or a Noetherian domain such that* $\text{inj.dim}_R R = 1$. *Then R is a Noetherian Gorenstein ring of Krull dimension one.*

Proof. Let b be a non-zero, non-unit element of R. Then $Rb = I_1 \cap \cdots \cap I_n$ where I_i is a P_i-primary ideal of R and P_i is a prime ideal of R. If R is a semi-Krull domain then we can assume that every P_i has height one. Now $\text{inj.dim}_R Rb = 1$, and so Q/Rb is R-injective. Therefore $E(R/Rb) \subset Q/Rb$, and so by Corollary 5.9 every P_i is a maximal ideal of R of height one. It follows that R has Krull dimension one. Thus for the remainder of the proof we can assume that R is a semi-Krull domain, and we shall prove that R is Noetherian.

Let P be a non-zero prime ideal of R and $b \neq 0 \in P$. Then with the preceding notation P is one of the P_i's belonging to Rb. Since every I_i contains a power of P_i it follows that there exists $a \in R$ such that $(Rb : Ra) = P$. Thus if $x = a/b + R$, then $(0 : x) = P$. Thus Q/R contains a copy of R/P for every non-zero prime ideal P of R. Since Q/R is R-injective, Q/R is what we have called in [4] a universal injective module for R. Hence it follows from [4, Theorem 2.1] that P^k/P^{k+1} is a finite dimensional vector space over R/P for every $k > 0$.

We can assume that the P_i's in the decomposition of Rb are all different, and thus $I_i + I_j = R$ for all $i \neq j$. Thus, if we let $\overline{R}_i = R/I_i$, then by the Chinese Remainder Theorem we have $\overline{R} = R/Rb \cong \overline{R}_1 \oplus \cdots \oplus \overline{R}_n$. Now $\overline{R}_i$ has only a single prime ideal $\overline{P}_i = P_i/I_i$ and $\overline{P}_i^k/\overline{P}_i^{k+1}$ is a finite dimensional vector space over $\overline{R}_i/\overline{P}_i$. Moreover, there exists $t > 0$ such that $\overline{P}_i^t = 0$. It follows that $\overline{R}_i$ is an Artinian, and hence Noetherian, ring. Thus $\overline{R} = R/Rb$ is a Noetherian ring for every nonzero, non-unit element b of R. Therefore, R is a Noetherian domain.

COROLLARY 5.11. *Let $I = I_1 \cap \cdots \cap I_n$ where I_i is a P_i-primary ideal of R. Let S be the ideal transform of I and $\mathcal{S} = \{1 - a \mid a \in I\}$. Assume that either R is a Noetherian domain, or that R is a semi-Krull domain and every P_i has height one. Then* $\text{inj.dim}_R I = 1$ *if and only if the following three conditions are satisfied*:

(1) $\text{Inj.dim}_{R_\mathcal{S}} I_\mathcal{S} = 1$.

(2) *S and $R_\mathcal{S}$ are complementary extensions of R.*

(3) *S is a Noetherian Gorenstein ring of Krull dimension* ≤ 1.

In this case the following three conditions are also satisfied:

(4) *R has Krull dimension one.*

(5) $R_\mathcal{S} = \bigcap_{i=1}^{n} R_{P_i}$.

(6) *If $\mathscr{C}_1$ is any set of prime ideals of R, $\mathscr{C}_2 = \operatorname{Spec} R - \mathscr{C}_1$, $S_1 = \bigcap R_P(P \in \mathscr{C}_1)$, and $S_2 = \bigcap R_P(P \in \mathscr{C}_2)$, then S_1 and S_2 are complementary extensions of R.*

Proof. If (1), (2) and (3) are satisfied, then $\text{inj} \cdot \dim_R I = 1$ by [5, Proposition 2.4]. Conversely, assume that $\text{inj} \cdot \dim_R I = 1$.

By Corollary 5.9, S and $R_{\mathscr{S}}$ are complemented extensions of R; and every P_i is a maximal ideal of R of height one. Thus $R_{\mathscr{S}} = \bigcap_{i=1}^{n} R_{P_i}$ by Proposition 5.3. By [5, Proposition 2.4] we have $\text{inj.dim}_S S \leq 1$. If $\text{inj.dim}_S S = 0$, then $S = Q$ and S is certainly a Noetherian Gorenstein ring of Krull dimension 0. Hence we can assume that $\text{inj.dim}_S S = 1$. Now S is a flat over-ring of R. Hence if R is Noetherian, then so is S by Proposition 3.6; and if R is a semi-Krull domain, so is S by Proposition 4.8. Therefore, by Proposition 5.10, S is a Noetherian Gorenstein ring of Krull dimension one. If P is a prime ideal of R different from any of the P_i's, then height P = height $PS \leq 1$ by Corollary 3.9. Thus R has Krull dimension one.

Let S_1 and S_2 be as described in the statement of this corollary. Then

$$S_1 \cap S_2 = \bigcap R_P \ (P \in \operatorname{Spec} R) = R.$$

By Proposition 4.6, $(S_1)_P = Q$ for all $P \in \mathscr{C}_2$ and $(S_2)_P = Q$ for all $P \in \mathscr{C}_1$. Thus

$$(S_1 + S_2) = \bigcap (S_1 + S_2)_P \ (P \in \operatorname{Spec} R) = Q;$$

and so S_1 and S_2 are complementary extensions of R.

COROLLARY 5.12. *Let R be a Krull domain. Then the following statements are equivalent*:

(1) $\text{Inj.dim}_R R = 1$.

(2) *R is a Dedekind ring.*

(3) *R has Krull dimension one.*

Proof. The equivalence of (1) and (2) is a consequence of Proposition 5.10; and the equivalence of (2) and (3) is a fairly obvious and standard result.

6. $\mathscr{F}_S$-cover of R-modules

Throughout this section R will be an arbitrary commutative ring and $\mathscr{F}$ will be a category of R-modules that is closed under submodules.

DEFINITION. If A and B are R-modules, then the symbol $\lambda: A \to B$ will mean that λ is an R-homomorphism from A to B. A pair (F, ϕ) is said to be an *$\mathscr{F}$-lifting of B* if:

(1) $F \in \mathscr{F}$ and $\phi: F \to B$.

(2) If (A, ψ) is a pair such that $A \in \mathscr{F}$ and $\psi: A \to B$, then there exists $\lambda: A \to F$ such that $\phi\lambda = \psi$.

An $\mathscr{F}$-lifting (F, ϕ) of B is said to be *pure* if the only R-submodule P of F such that $P \subset \ker\phi$ and $F/P \in \mathscr{F}$ is $P = 0$.

Proposition 6.1. *An $\mathscr{F}$-lifting (F, ϕ) of B is pure if and only if $f\colon F \to F$ and $\phi f = \phi$ implies that f is one to one.*

Proof. Assume that (F, ϕ) is a pure $\mathscr{F}$-lifting of B and that we have $f\colon F \to F$ and $\phi f = \phi$. Then $\ker f \subset \ker \phi$ and $F/\ker f \subset F$. Thus $F/\ker f \in \mathscr{F}$ and so $\ker f = 0$. Conversely, assume that (F, ϕ) is an $\mathscr{F}$-lifting of B such that if $f\colon F \to \mathscr{F}$ and $\phi f = \phi$, then f is one to one. Let P be an R-submodule of $\ker \phi$ and suppose that $F/P \in \mathscr{F}$. Then ϕ induces $\bar{\phi}\colon F/P \to B$ and hence there exists $\lambda\colon F/P \to F$ such that $\phi\lambda = \bar{\phi}$. Let $\Pi\colon F \to F/P$ be the canonical map and $f = \lambda\Pi$. Then $f\colon F \to F$ and $\phi f = \phi$, and so $\ker f = 0$. But $P \subset \ker f$, and so $P = 0$. Hence (F, ϕ) is pure.

Definition. An $\mathscr{F}$-lifting (F, ϕ) of B is said to be an *$\mathscr{F}$-cover of B* if $f\colon F \to F$ and $\phi f = \phi$ implies that f is an automorphism of F. It is obvious that if $F \in \mathscr{F}$, then $(F, 1_F)$ is an $\mathscr{F}$-cover of F. Two $\mathscr{F}$-liftings (F_1, ϕ_1) and (F_2, ϕ_2) of B are said to be *isomorphic* if there exists an isomorphism $\lambda\colon F_1 \to F_2$ such that $\phi_2\lambda = \phi_1$. We will say that (F_1, ϕ_1) is a *sublifting* in (F_2, ϕ_2) if $F_1 \subset F_2$ and $\phi_1 = \phi_2|F_1$; and that it is a *direct summand* of (F_2, ϕ_2) if F_1 is a direct summand of F_2 and $\phi_1 = \phi_2|F_1$.

Proposition 6.2. (1) *Let (F_1, ϕ_1) and (F_2, ϕ_2) be two $\mathscr{F}$-liftings of B. Then (F_1, ϕ_1) is a direct summand of (F_2, ϕ_2) if and only if $F_2 = F_1 \oplus C$, where $C \subset \ker \phi_2$ and $\phi_1 = \phi_2|F_1$.*

(2) *Let (F, ϕ) be a pure $\mathscr{F}$-lifting of B. Then (F, ϕ) has no proper direct summands. Thus if $f\colon F \to F$, $\phi f = \phi$, and* $\operatorname{Im} f$ *is a direct summand of F, then f is an automorphism of F.*

(3) *If (F, ϕ) is a pure $\mathscr{F}$-lifting of B and F is R-injective, then (F, ϕ) is an $\mathscr{F}$-cover of B.*

Proof. (1) Suppose that (F_1, ϕ_1) is a direct summand of (F_2, ϕ_2). Then there exists an R-submodule A of F_2 such that $F_2 = F_1 \oplus A$ and $\phi_1 = \phi_2|F_1$. By definition there exists $\lambda\colon F_2 \to F_1$ such that $\phi_1\lambda = \phi_2$. Thus $\phi_2\lambda = \phi_2$. Let

$$C = \{x - \lambda(x) | x \in A\}.$$

Then it is easily verified that $F_2 = F_1 \oplus C$ and $C \subset \ker \phi_2$.

(2) It follows immediately from part (1) that a pure $\mathscr{F}$-lifting has no proper direct summands. Suppose that $f\colon F \to F$, $\phi f = \phi$, and that $F_1 = \operatorname{Im} f$ is a direct summand of F. If we let $\phi_1 = \phi|F_1$, then (F_1, ϕ_1) is an $\mathscr{F}$-lifting of B and a direct summand of (F, ϕ). Hence $\operatorname{Im} f = F$ and so f is onto. Since f is one to one by Proposition 6.1, f is an automorphism of F.

(3) This follows immediately from part (2) and Proposition 6.1.

PROPOSITION 6.3. *Let (F_1, ϕ_1) and (F_2, ϕ_2) be two $\mathcal{F}$-liftings of B. Then*

(1) *If (F_1, ϕ_1) is pure, then it is isomorphic to a sublifting in (F_2, ϕ_2).*

(2) *If (F_1, ϕ_1) is an $\mathcal{F}$-cover of B, then it is isomorphic to a direct summand of (F_2, ϕ_2).*

(3) *If (F_1, ϕ) is an $\mathcal{F}$-cover of B and (F_2, ϕ_2) is pure, then (F_1, ϕ_1) is isomorphic to (F_2, ϕ_2).*

Proof. There exist λ_1: $F_1 \to F_2$ and λ_2: $F_2 \to F_1$ such that $\phi_2\lambda_1 = \phi_1$ and $\phi_1\lambda_2 = \phi_2$. Thus $(\lambda_2\lambda_1)$: $F_1 \to F_1$ and $\phi_1(\lambda_2\lambda_1) = \phi$. Parts (1) and (2) are easily deduced from these equations; whereas (3) follows from (2) and Proposition 6.2(2).

PROPOSITION 6.4. *Assume that $\mathcal{F}$ is closed under essential extensions. Let C be an injective R-module and (F, ϕ) a pure $\mathcal{F}$-lifting of C. Then F is an injective R-module; and (F, ϕ) is an $\mathcal{F}$-cover of C.*

Proof. By Proposition 6.2 it is sufficient to prove that F is an injective R-module. Let G be the injective envelope of F; by assumption $G \in \mathcal{F}$. Since C is R-injective, there exists $\psi : G \to C$ which extends ϕ: $F \to C$. It is obvious that (G, ψ) is an $\mathcal{F}$-lifting of C.

Now suppose that $P \subset \ker \psi$ and $G/P \in \mathcal{F}$. Let $P' = P \cap F$; then $P' \subset \ker \phi$ and $F/P' \subset G/P$. Thus $F/P' \in \mathcal{F}$, and since (F, ϕ) is pure, we have $P' = 0$. But G is an essential extension of F, and so $P = 0$. Thus (G, ψ) is a pure $\mathcal{F}$-lifting of C. Hence by Proposition 6.2(3), (G, ψ) is an $\mathcal{F}$-cover of C. Therefore, by Proposition 6.3(3), (G, ψ) is isomorphic to (F, ϕ), and hence $G \cong F$. Thus F is an injective R-module (i.e., $F = G$).

DEFINITION. Let $B \subset C$ be R-modules, and let (G, ψ) be an $\mathcal{F}$-lifting of C. Let $F = \psi^{-1}(B)$ and $\phi : F \to B$ the restriction of ψ to F. Then clearly (F, ϕ) is an $\mathcal{F}$-lifting of B. (F, ϕ) is called the *restriction of* (G, ψ) *to* B.

Remarks. If C is an essential extension of B, then G is an essential extension of F. For $\ker \psi \subset F$; and if $x \in G - \ker \psi$, then there exists $r \in R$ such that $0 \neq r\psi(x) \in B$. Thus $0 \neq rx \in F$, proving that G is an essential extension of F.

PROPOSITION 6.5. *Let C be the R-injective envelope of B, (G, ψ) an $\mathcal{F}$-lifting of C, and (F, ϕ) the restriction of (G, ψ) to B. Then*:

(1) *If (F, ϕ) is a pure $\mathcal{F}$-liting of B, then (G, ψ) is a pure $\mathcal{F}$-lifting of C.*

(2) *If G is R-injective, then both (F, ϕ) and (G, ψ) are $\mathcal{F}$-covers if and only if either one of them is a pure $\mathcal{F}$-lifting.*

Proof. (1) Assume that (F, ϕ) is a pure $\mathcal{F}$-lifting of B. Let $P \subset \ker \psi = \ker \phi$ and suppose that $G/P \in \mathcal{F}$. Then $F/P \subset G/P$, and so $F/P \in \mathcal{F}$. Since (F, ϕ) is pure, $P = 0$; and thus (G, ψ) is pure.

(2) Assume that G is an injective R-module. If (F, ϕ) is pure, then (G, ψ) is pure by part (1); and hence (G, ψ) is an $\mathscr{F}$-cover of C by Proposition 6.2(3). On the other hand assume that (G, ψ) is a pure $\mathscr{F}$-lifting. Then (G, ψ) is an $\mathscr{F}$-cover of C by Proposition 6.2(3). We must show that (F, ϕ) is an $\mathscr{F}$-cover of B. Let $f\colon F \to F$ be a map such that $\phi f = \phi$. Since G is R-injective we can extend f to $g\colon G \to G$. The problem we have to overcome is that we don't necessarily have $\psi g = \psi$.

We shall prove first that $g^{-1}(F) \subset F$. Let $y \in g^{-1}(F)$. Since $\phi f = \phi$, we have $F = \operatorname{Im} f + \ker \phi$; and so $g(y) = f(x) + z$, where $x \in F$ and $z \in \ker \phi$. Hence without loss of generality we can assume that $g(y) = z$. Suppose $y \notin F$; then $y \notin \ker \psi \subset F$, and hence there exists $r \in R$ such that $0 \neq r\psi(y) \in B$. But then $ry \in \psi^{-1}(B) = F$ and so $r\psi(y) = \phi(ry) = \phi f(ry) = \psi g(ry) = \psi(rz) = 0$. This contradiction proves that $g^{-1}(F) \subset F$.

We next define $\mathscr{E}\colon (F + \operatorname{Im} g) \to C$ by $\mathscr{E}(x + g(y)) = \psi(x + y)$ for all $x \in F$ and $y \in G$. Now $\mathscr{E}$ is well-defined; for if $x + g(y) = 0$, then $y \in g^{-1}(F) \subset F$; and hence $\phi(x + y) = \psi(x + y) = \phi(x) + \phi f(y) = \psi(x + g(y)) = 0$. Since C is R-injective, we can extend $\mathscr{E}$ to all of G; and we call the extension $\mathscr{E}$ as well. We then have that $\mathscr{E} = \phi$ on F and $\mathscr{E} g = \psi$.

Since (G, ψ) is an $\mathscr{F}$-lifting of C, there exists $\lambda\colon G \to G$ such that $\psi\lambda = \mathscr{E}$. But then $\psi(\lambda g) = \mathscr{E} g = \psi$; and since (G, ψ) is an $\mathscr{F}$-cover of C it follows that λg is an automorphism of G. Thus λ is onto and g is one to one. Therefore, f is one to one also, and this proves that (F, ϕ) is a pure $\mathscr{F}$-lifting of B by Proposition 6.1.

Now $\lambda(F) \subset F$; for if $x \in F$, then $\psi\lambda(x) = \mathscr{E}(x) = \phi(x) \in B$, and so $\lambda(x) \in \psi^{-1}(B) = F$. Thus on F we have $\phi\lambda = \phi$; and hence λ is one to one on F because (F, ϕ) is pure. But since G is an essential extension of F by the remarks preceding this proposition, it follows that λ is one to one on G. Thus λ is an automorphism, and hence so is $g = \lambda^{-1}(\lambda g)$. Since $g^{-1}(F) \subset F$ and $g = f$ on F, it follows that f is an automorphism of F. Thus (F, ϕ) is an $\mathscr{F}$-cover of B.

PROPOSITION 6.6. *Assume that $\mathscr{F}$ is closed under direct limits, and let (F, ϕ) be an $\mathscr{F}$-lifting of B. Then (F, ϕ) contains a direct summand that is an $\mathscr{F}$-cover of B.*

Proof. Let $\{P_\alpha\}$ be a linearly ordered family of R-submodules of $\ker \phi$ such that $F/P_\alpha \in \mathscr{F}$ for all α. Let $P = \bigcup_\alpha P_\alpha$; then $P \subset \ker \phi$ and $F/P = \varinjlim F/P_\alpha \in \mathscr{F}$. Thus by Zorn's Lemma we can assume that P is maximal with respect to these properties. We let $\bar{\phi}\colon F/P \to B$ be the map induced by ϕ; then it is fairly obvious that $(F/P, \bar{\phi})$ is a pure $\mathscr{F}$-lifting of B. It now follows from Proposition 6.3(2) that we can assume without loss of generality that (F, ϕ) is a pure $\mathscr{F}$-lifting of B. We shall prove that (F, ϕ) is an $\mathscr{F}$-cover of B.

Let $f: F \to F$ be a map such that $\phi f = \phi$. Then f is one-to-one and we shall prove that it is onto. Let δ be a non-limit ordinal such that card $F <$ card δ. For each $\alpha \le \delta$, let $F_\alpha = F$. We shall define by transfinite induction f_α^β: $F_\alpha \to F_\beta$ for all $\alpha \le \beta \le \delta$ such that the following four properties are satisfied:

(1) If $\alpha \le \gamma \le \beta \le \delta$, then $f_\gamma^\beta f_\alpha^\gamma = f_\alpha^\beta$;
(2) $\phi f_\alpha^\beta = \phi$
(3) $f_\alpha^{\alpha+1} = f$
(4) $f_\alpha^\alpha = 1_{F_\alpha}$.

We assume that f_α^γ has been defined for all $\alpha \le \gamma < \beta$ to satisfy these four properties, and we shall define f_α^β: $F_\alpha \to F_\beta$ to satisfy them. If β is not a limit ordinal then we can define $f_\alpha^\beta = f f_\alpha^{\beta-1}$ for all $\alpha \le \beta - 1$ and $f_\beta^\beta = 1_{F_\beta}$. Then it is easy to verify that the four properties are satisfied. Hence assume that β is a limit ordinal. Since the maps $\{f_\alpha^\gamma\}$ for $\alpha \le \gamma < \beta$ are a directed family we can use them to form $L = \varinjlim F_\alpha$. Then by the properties of direct limits there exists j_α: $F_\alpha \to L$ such that $j_\alpha = j_\gamma f_\alpha^\gamma$ for all $\alpha \le \gamma < \beta$; and since $\phi f_\alpha^\gamma = \phi$, there exists ψ: $L \to B$ such that $\psi j_\alpha = \phi$. Since $L \in \mathscr{F}$ by assumption, there exists λ: $L \to F$ such that $\phi\lambda = \psi$. We now define f_α^β: $F_\alpha \to F_\beta$ by $f_\alpha^\beta = \lambda j_\alpha$ for $\alpha < \beta$ and $f_\beta^\beta = 1_{F_\beta}$. Then it is easily verified that f_α^β-satisfies the four listed properties.

By transfinite induction we have defined f_α^δ: $F_\alpha \to F_\delta$ for all $\alpha \le \delta$ to satisfy the four properties. Since $\phi f_\alpha^\delta = \phi$, and (F, ϕ) is pure, f_α^δ is one to one by Proposition 6.1. If $\alpha < \delta$, we have $f_\alpha^\delta = f_{\alpha+1}^\delta f$; and so $\operatorname{Im} f_\alpha^\delta \subset \operatorname{Im} f_{\alpha+1}^\delta$. Since card $F <$ card δ, there exists $\alpha < \delta$ such that $\operatorname{Im} f_\alpha^\delta = \operatorname{Im} f_{\alpha+1}^\delta$. Thus if $x \in F$, there exists $y \in F$ such that

$$f_{\alpha+1}^\delta(x) = f_\alpha^\delta(y) = f_{\alpha+1}^\delta(f(y)).$$

But $f_{\alpha+1}^\delta$ is one to one, and hence $x = f(y)$. Therefore, f is onto, and so (F, ϕ) is an $\mathscr{F}$-cover of B.

Remarks. The proof of Proposition 6.6 was modeled closely after the proof of Lemma 2.3 in [2].

Definition. Let S be a commutative ring extension of R, B an R-module, and C the R-injective envelope of B. For the remainder of this section we shall let

$$\psi: \operatorname{Hom}_R(S, C) \to C$$

denote the canonical map defined by $\psi(f) = f(1)$ for all $f \in \operatorname{Hom}_R(S, C)$. We shall let (F, ϕ) denote the restriction of $(\operatorname{Hom}_R(S, C), \psi)$ to B. Thus

$$F = \{f \in \operatorname{Hom}_R(S, C) | f(R) \subset B\}.$$

PROPOSITION 6.7. *Let S be a commutative ring extension of R. Then:*

(1) *Every R-module B has an $\mathscr{F}_S$-cover.*

(2) *If C is the R-injective envelope of B, then $(\mathrm{Hom}_R(S, C), \psi)$ is an $\mathscr{F}_S$-lifting of C; and (F, ϕ), the restriction of $(\mathrm{Hom}_R(S, C), \psi)$ to B, is an $\mathscr{F}_S$-lifting of B. Thus the $\mathscr{F}_S$-cover of B is a direct summand of (F, ϕ).*

Proof. We shall first prove that $(\mathrm{Hom}_R(S, C), \psi)$ is an $\mathscr{F}_S$-lifting of C. Hence suppose that $A \in \mathscr{F}_S$ and $f\colon A \to C$. Let $\theta_A\colon A \to A \otimes_R S$ be the canonical map. Then θ_A is one to one by Proposition 1.5; and so, since C is R-injective, there exists $g\colon A \otimes_R S \to C$ such that $g\theta_A = f$. Define $\lambda\colon A \to \mathrm{Hom}_R(S, C)$ by $\lambda(x)(s) = g(x \otimes s)$ for all $x \in A$ and $s \in S$. Then clearly $\psi\lambda = f$, and so $(\mathrm{Hom}_R(S, C), \psi)$ is an $\mathscr{F}_S$-lifting of C. It follows immediately that the restriction (F, ϕ) is an $\mathscr{F}_S$-lifting of B.

Now $\mathscr{F}_S$ is closed under direct limits by Proposition 1.5. Hence by Proposition 6.6, (F, ϕ) contains a direct summand that is an $\mathscr{F}_S$-cover of B.

DEFINITION. Let S be a commutative ring extension of R and B an R-module. The $\mathscr{F}_S$-cover of B that exists by Proposition 6.7 is unique up to isomorphism by Proposition 6.3(3); and we shall denote it by $(\mathscr{F}_S(B), \phi_B^S)$.

PROPOSITION 6.8. *Let S be a commutative ring extension of R. Then the following statements are equivalent:*

(1) *$\mathscr{F}_S$ is equal to the category of all t_S-torsion-free R-modules.*

(2) *$\mathscr{F}_S(C)$ is an injective R-module for every injective R-module C.*

Proof. (1) $\Rightarrow$ (2) $\mathscr{F}_S$ is closed under essential extensions by Proposition 2.4. Hence $\mathscr{F}_S(C)$ is R-injective for every injective R-module C by Proposition 6.4.

(2) $\Rightarrow$ (1) Let $B \in \mathscr{F}_S$ and let C be the R-injective envelope of B. By Proposition 2.4 it is sufficient to prove that $C \in \mathscr{F}_S$. By assumption $\mathscr{F}_S(C)$ is R-injective. Let (F, ϕ) be the restriction of $(\mathscr{F}_S(C), \phi_C^S)$ to B. Then by Proposition 6.5(2), (F, ϕ) is an $\mathscr{F}_S$-cover of B. But $(B, 1_B)$ is an $\mathscr{F}_S$-cover of B, and hence $(B, 1_B)$ is isomorphic to (F, ϕ) by Proposition 6.3(3). Therefore $\ker \phi_C^S = \ker \phi = 0$, and so ϕ_C^S is one to one. Since an $\mathscr{F}_S$-lifting map is always onto, ϕ_C^S is an isomorphism. Therefore, $C \cong \mathscr{F}_S(C) \in \mathscr{F}_S$.

Remarks. The equivalence of Proposition 6.8 is not too surprising in the sense that the condition that $\mathscr{F}_S$ is equal to the cateogry of t_S-torsion-free R-modules is a generalization of the condition that S is flat; and S is flat if and only if $\mathrm{Hom}_R(S, C)$ is R-injective for every injective R-module C. Since $\mathscr{F}_S(C) \subset \mathrm{Hom}_R(S, C)$, this raises the question of when the two are equal. More light will be shed on this in the next proposition and its corollary. The next proposition also generalizes Banaschewski's results for torsion-free covers over an integral domain [1, Proposition 1 and its corollary].

PROPOSITION 6.9. *Let S be a commutatitive ring extension of R, B an R-module, and C the R-injective envelope of B. Let (F, ϕ) be the restriction of $(\mathrm{Hom}_R(S, C), \psi)$ to B. Then the following statements are equivalent*:

(1) *S is a torsion envelope of R.*

(2) *(F, ϕ) is an $\mathscr{F}_S$-cover of B for every R-module B.*

Proof. (1) ⇒ (2) By Proposition 3.1, S is R-flat; and so $\mathrm{Hom}_R(S, C)$ is R-injective. Thus by Propositions 6.5(2) and 6.7 it is sufficient to prove that $(\mathrm{Hom}_R(S, C), \psi)$ is pure, for then (F, ϕ) will be an $\mathscr{F}_S$-cover of B. Hence suppose that $0 \neq P \subset \ker \psi$, and let $0 \neq f \in P$. Then there exists $s \in S$ such that $f(s) \neq 0$. Since $\mathrm{Hom}_R(S, C)$ is an S-module, $g = sf$ is also an element of $\mathrm{Hom}_R(S, C)$. We have $\psi(g) = (sf)(1) = f(s) \neq 0$, and thus $g \notin \ker \psi$. Therefore, $g + P$ is a non-zero element of $\mathrm{Hom}_R(S, C)/P$.

Let $I = (R : s)$; then $I \in \mathscr{I}_S$ because S is a torsion envelope of R. Now

$$Ig = (Is)f \subset Rf \subset P,$$

and so $g + P$ is a t_S-torsion element of $\mathrm{Hom}_R(S, C)/P$. Therefore, since every element of $\mathscr{F}_S$ is t_S-torsion-free by Proposition 2.3(1), we have

$$\mathrm{Hom}_R(S, C)/P \notin \mathscr{F}_S.$$

Thus $(\mathrm{Hom}_R(S, C), \psi)$ is pure.

(2) ⇒ (1) Suppose that S is not a torsion envelope of R. Then there exists $s \in S$ such that if $I = (R : s)$, then $S \neq IS$. Let $B = S/IS$ and let C be the R-injective envelope of B. Let $x = s + R$ in S/R; then $(0 : x) = I$, and so there exists $f\colon S/R \to C$ such that $f(x) = 1 + IS$. Thus $\mathrm{Hom}_R(S/R, C) \neq 0$.

Let (F, ϕ) be the restriction of $(\mathrm{Hom}_R(S, C), \psi)$ to B. By assumption (F, ϕ) is an $\mathscr{F}_S$-cover of B. But B is an S-module and so $(B, 1_B)$ is an $\mathscr{F}_S$-cover of B. Thus (F, ϕ) is isomorphic to $(B, 1_B)$ by Proposition 6.3(3). Hence $\ker \phi = 0$. But $\ker \phi = \ker \psi \cong \mathrm{Hom}_R(S/R, C) \neq 0$. This contradiction proves that S is a torsion envelope of R.

COROLLARY 6.10. *Let S be a commutative ring extension of R and assume that $\mathscr{F}_S$ is equal to the category of all t_S-torsion-free R-modules. Then the following statements are equivalent*:

(1) *S is a torsion envelope of R.*

(2) *$(\mathrm{Hom}_R(S, C), \psi)$ is an $\mathscr{F}_S$-cover of C for every injective R-module C.*

Proof. That (1) ⇒ (2) follows from Proposition 6.9. Hence assume (2). Let B be an R-module. C the R-injective envelope of B, and (F, ϕ) the restriction of $(\mathrm{Hom}_R(S, C), \psi)$ to B. By Proposition 6.8, $\mathrm{Hom}_R(S, C)$ is an injective R-module. Hence (F, ϕ) is an $\mathscr{F}_S$-cover of B by Proposition 6.5(2). Thus S is a torsion envelope of R by Proposition 6.9.

DEFINITION. Let $\mathscr{F}_0$ be the class of all R-modules that are submodules of flat R-modules. Then $\mathscr{F}_0$ is closed under direct limits. Hence by Proposition 6.6 any $\mathscr{F}_0$-lifting of an R-module B contains a direct summand that is an $\mathscr{F}_0$-cover of B. It would be interesting to know which rings have the property that every one of its modules has an $\mathscr{F}_0$-cover. The next Proposition will provide some examples. We note first that if S is any commutative ring extension of R, then $\mathscr{F}_0 \subset \mathscr{F}_S$.

PROPOSITION 6.11. *Let S be a commutative ring extension of R and assume that S is R-flat and is a semi-regular ring. Then $\mathscr{F}_0 = \mathscr{F}_S$, and hence every R-module has an $\mathscr{F}_0$-cover. Moreover, if C is an injective R-module, then the $\mathscr{F}_0$-cover of C is a flat, injective R-module.*

Proof. Since S is semi-regular, every S-module is an S-submodule of a flat S-module. Since S is R-flat, every flat S-module is a flat R-module. Thus $\mathscr{F}_S \subset \mathscr{F}_0$; and $\mathscr{F}_0 \subset \mathscr{F}_S$ for any commutative ring extension S. Let C be an injective R-module. Then $\mathscr{F}_S(C)$ is a direct summand of $\operatorname{Hom}_R(S, C)$ by Proposition 6.7. $\operatorname{Hom}_R(S, C)$ is an injective S-module, and hence is a direct summand of a flat S-module. Thus $\operatorname{Hom}_R(S, C)$ is R-flat; and it is R-injective since S is R-flat.

Examples. Next we exhibit two general types of rings that have the property that every one its modules has an $\mathscr{F}_0$-cover.

(1) Let R be a reduced ring such that $\min R$ is compact. Then E, the injective envelope of R, is a flat, von-Neumann regular, commutative ring extension of R, and hence Proposition 6.11 applies.

(2) Let R be a Noetherian Gorenstein ring. Then Q, the total ring of quotients of R, is a flat, quasi-Frobenius, commutative ring extension of R. Hence Q is semi-regular and Proposition 6.11 applies.

PROPOSITION 6.12. *Let R be a Noetherian ring and S a commutative ring extension of R such that S/R has finite length. Then every finitely generated R-module has a finitely generated $\mathscr{F}_S$-cover.*

Proof. Let B be a finitely generated R-module and let C be the R-injective envelope of B. Now $(\mathscr{F}_S(B), \phi_B^S)$ is contained in $(\operatorname{Hom}_R(S, C), \psi)$ by Proposition 6.7. Thus

$$\operatorname{Ker} \phi_B^S \subset \ker \psi \cong \operatorname{Hom}_R(S/R, C).$$

But $\operatorname{Hom}_R(S/R, C)$ has finite length since both S/R and the socle of C have finite length. Thus $\ker \phi_B^S$ has finite length. There exists a finitely generated R-submodule A of $\mathscr{F}_S(B)$ such that $\mathscr{F}_S(B) = A + \ker \phi_B^S$. Hence $\mathscr{F}_S(B)$ is a finitely generated R-module.

Remarks. With the notation of Proposition 6.12 we observe that if socle $B = 0$, then $B \in \mathscr{F}_S$. For in that case $\mathrm{Hom}_R(S/R, C) = 0$, and hence $\ker \phi_B^S = 0$. Since ϕ_B^S is onto, we have $B \cong \mathscr{F}_S(B) \in \mathscr{F}_S$.

DEFINITION. Let S be a commutative ring extension of R. An R-module B will be said to be *S-cotorsion* if $\mathrm{Hom}_R(S, B) = 0$ and $\mathrm{Ext}_R^1(S, B) = 0$. B will be said to have *S-bounded order* if $(0 : B) \in \mathscr{I}_S$ (i.e., $S = (0 : B)S$). Clearly if B has S-bounded order, then it is S-cotorsion. Thus if $I \in \mathscr{I}_S$ (i.e., $IS = S$), then R/I is S-cotorsion.

PROPOSITION 6.13. *Let S be a torsion envelope of R and B an S-cotorsion R-module. Let C be the injective envelope of B. Then* $\mathscr{F}_S(B) \cong \mathrm{Hom}_R(S/R, C/B)$.

Proof. Let (F, ϕ) be the restriction of $(\mathrm{Hom}_R(S, C), \psi)$ to B. Then by Proposition 6.9, $F = \mathscr{F}_S(B)$. We have a natural injection

$$0 \to \mathrm{Hom}_R(S/R, C/B) \xrightarrow{\varepsilon} \mathrm{Hom}_R(S, C/B),$$

and since B is S-cotorsion, the natural map η: $\mathrm{Hom}_R(S, C) \to \mathrm{Hom}_R(S, C/B)$ is an isomorphism. Hence if we let $\nu = \eta^{-1}\varepsilon$, then ν: $\mathrm{Hom}_R(S/R, C/B) \to \mathrm{Hom}_R(S, C)$ is one to one. It is easy to verify that

$$\mathrm{Im}\,\nu = \{ f \in \mathrm{Hom}_R(S, C) | f(1) \in B \} = F.$$

COROLLARY 6.14. *Let S be a torsion envelope of R; and suppose that I is an ideal of R such that $IS = S$, and E/I is the injective envelope of R/I, where E is the injective envelope of R. Then* $\mathscr{F}_S(R/I) \cong \mathrm{Hom}_R(S/R, S/R)$.

Proof. Let $B = R/I$ and $C = E/I$; then $C/B \cong E/R$. Thus

$$\mathscr{F}_S(R/I) \cong \mathrm{Hom}_R(S/R, E/R)$$

by Proposition 6.13. Since S/R is t_S-torsion by assumption, E/S is t_S-torsion-free by Proposition 2.3. Thus $\mathrm{Hom}_R(S/R, E/S) = 0$, and so

$$\mathrm{Hom}_R(S/R, E/R) \cong \mathrm{Hom}_R(S/R, S/R).$$

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