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GROUPS AND CENTRAL ALGEBRAS

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If K is a field and A is a finite dimensional central simple K-algebra, then the Brauer class of A contains a crossed product (cf. [4, page 379], [10, page 474]). The algebra of real quaternions is a twisted group algebra of the Klein-4-group over the field of real numbers $\mathbf{R}$; it is also the algebra obtained from the group algebra $\mathbf{R}[G]$, where G is the group of quaternions, by identifying the center of G with $\{1, -1\}$ of $\mathbf{R}$. The similar construction for the dihedral group of order 8 gives the algebra of 2×2-matrices over $\mathbf{R}$. It turns out that twisted group algebras are the algebras obtained from group algebras by identifying a central subgroup with a subgroup of the field's multiplicative group. We also determine when the algebras obtained from group algebras by such identifications are central.

A group G is called completely central if for every non-central $g \in G$ with only finitely many conjugates, there is a central $1 \neq n \in G$ such that g is a conjugate of ng. The class of completely central groups contains all free groups, all nilpotent class 2-groups, all torsion free nilpotent groups and all groups of central type. However, there are nilpotent class 3-groups that are not completely central groups (e.g., the dihedral group of order 16) and there are nilpotent class 2-groups that are not of central type (e.g., one of the groups of order 64). We characterize groups of central type in the class of finite completely central groups.

K will always denote a non-trivial commutative ring with 1. Let $K^{\times}$ denote the group of units of K. The center of a group G will be denoted by $\zeta(G)$ and the center of an algebra A will be denoted by $\zeta(A)$. The conjugacy class of $g \in G$ will be denoted by $\mathrm{Cl}(g)$.

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1. Suppose K is a commutative ring with 1, G is a group, N is a central subgroup of G and α is a homomorphism of N into $K^{\times}$. The algebra obtained from the group algebra $K[G]$ by identifying n with $\alpha(n)$ for every $n \in N$ will be denoted by $KG\alpha$. The ideal of $K[G]$ generated by $\{n - \alpha(n)1 \mid n \in N\}$ will

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be denoted by $I(\alpha)$. Thus $KG\alpha = K[G]/I(\alpha)$. Since N is central in $K[G]$,

$$I(\alpha) = \Sigma\{(n - \alpha(n)1)K[G] | n \in N\}.$$

If N is generated by A, then $I(\alpha) = \Sigma\{(1 - \alpha(a^{-1})a)K[G] | a \in A\}$. In particular, if $N = \langle a \rangle$ is cyclic, then $I(\alpha)$ is a principal ideal;

$$I(\alpha) = (1 - \alpha(a^{-1})a)K[G].$$

The following two theorems describe $KG\alpha$ as a twisted group algebra:

THEOREM 1. *Suppose K is a commutative ring with* 1, *G is a group, N is a central subgroup of G and α is a homomorphism of N into $K^{\times}$. Then $KG\alpha$ is a twisted group algebra of G/N over K. Furthermore, if B is a transversal of G modulo N, then every element of $K[G]$ is uniquely writable as a K-linear combination in B plus an element of $I(\alpha)$.*

Proof. Let B be a transversal of G modulo N. Define

$$\gamma\colon G/N \times G/N \to K^{\times}$$

as follows: If $b, b', b'' \in B$, $n \in N$ and $bb' = nb''$, then $\gamma(Nb, Nb') = \alpha(n)$. It is routine to check that

$$\gamma(xy, z)\gamma(x, y) = \gamma(x, yz)\gamma(y, z) \quad \text{for all } x, y, z \in G/N;$$

i.e., γ is a $K^{\times}$-factor set for G/N (cf. [5, page 182], [9, page 174], [11, page 13]). So, we can form the twisted group algebra $K^t[G/N]$: the additive group of $K^t[G/N]$ is a free K-module with basis G/N. Multiplication is defined by $kx = xk$, $x \cdot y = \gamma(x, y)(xy)$ for all $k \in K$, $x, y \in G/N$. If $a \in B \cap N$, then $\alpha(a^{-1})N$ is the identity element of $K^t[G/N]$. Thus if $1 \in B$, N is the identity element of $K^t[G/N]$.

Let μ be the K-module homomorphism determined by $\mu(nb) = \alpha(n)(Nb)$, $n \in N$, $b \in B$. The mapping μ is also an algebra homomorphism. Indeed, let $n, n', n'' \in N$, $b, b', b'' \in B$ and $bb' = n''b''$. Then

$$\begin{aligned}\mu((nb)(n'b')) &= \mu(nn'bb') = \mu(nn'n''b'') = \alpha(nn'n'')Nb'',\\ \mu(nb) \cdot \mu(n'b') &= \alpha(n)(Nb) \cdot \alpha(n')(Nb')\\ &= \alpha(n)\alpha(n')(Nb) \cdot (Nb') = \alpha(n)\alpha(n')\alpha(n'')Nb''.\end{aligned}$$

Thus μ is a homomorphism of $K[G]$ onto $K^t[G/N]$. Let $a \in B \cap N$, $n \in N$. Then

$$\mu(n - \alpha(n)1) = \mu(na^{-1}a - \alpha(n)a^{-1}a) = \alpha(na^{-1})N - \alpha(n)\alpha(a^{-1})N = 0$$

in $K^t[G/N]$. Thus $I(\alpha)$ is contained in the kernel of μ.

Conversely, if $\mu(\Sigma\{t_{n,b}nb|n \in N, b \in B\} = 0$ in $K^t[G/N]$, then

$$\Sigma\{t_{n,b}\alpha(n)(Nb)|n \in N, b \in B\} = 0$$

in $K^t[G/N]$. Hence $\Sigma\{t_{n,b}\alpha(n)|n \in N\} = 0$ in K for every $b \in B$. Hence

$$\Sigma\{t_{n,b}nb|n \in N, b \in B\} = \Sigma\{t_{n,b}(n - \alpha(n)1)b|n \in N, b \in B\} \in I(\alpha).$$

Thus $I(\alpha) = \operatorname{Ker}\mu$ and $K^t[G/N] \cong K[G]/I(\alpha) = KG\alpha$.

Since $nb = \alpha(n)b + (n - \alpha(n)1)b \in \alpha(n)b + I(\alpha)$, the set of all K-linear combinations in B is a transversal of the additive group of $K[G\}$ modulo $I(\alpha)$.

The following theorem is a modification of Lemma 2.3 of [11, page 16] and establishes the converse of Theorem 1:

THEOREM 2. *Suppose K is a commutative ring with* 1, *H is a group and $A = K^t[H]$ is a twisted group algebra. Then there is a group G with a central subgroup N such that $G/N \cong H$ and there is an injective homomorphism α of N into $K^\times$ such that $KG\alpha \cong A$.*

Proof. Let $G = \{ax|a \in K^\times,\ x \in H\}$. Then G is a multiplicative subgroup of the group of units of A. Furthermore, the mapping $ax \to x$ is a homomorphism of G onto H with kernel $N = K^\times 1 = \{a1|a \in K^\times\}$. Obviously, N is central in G and $G/N \cong H$. Furthermore, the embedding of G into A extends to a K-linear mapping β of $K[G]$ onto $A = K^t[H]$ since G contains a basis for A. Let J denote the kernel of β and let α be the homomorphism of N into $K^\times$ defined by $\alpha(n) = k$ if $n = k1$. It is clear that α is an isomorphism.

The set H, as a subset of G, is a transversal of G modulo N. Thus, if $C \in J$, then we can write $C = \Sigma\{t_{n,x}nx|n \in N, x \in H\}$ and then

$$0 = \beta(C) = \Sigma\{t_{n,x}\alpha(n)x|n \in N, x \in H\}.$$

Hence, for every $x \in H$, $\Sigma\{t_{n,x}\alpha(n)|n \in N\} = 0$, and so

$$C = \Sigma\{t_{n,x}(n - \alpha(n)1)x|n \in N, x \in H\};$$

i.e., $J \subseteq I(\alpha)$. Conversely, $\beta(n - \alpha(n)1) = \alpha(n)1 - \alpha(n)1 = 0$. Thus $I(\alpha) = J$ and so $A \cong K[G]/I(\alpha) = KG\alpha$.

PROPOSITION 3. *Suppose K, G, N and α are as in Theorem* 1. *If L is the kernel of α and α' is the injective homomorphism of N/L into $K^\times$ induced by α, then $KG\alpha \cong K(G/L)\alpha'$.*

Proof. Let B be a transversal of G modulo N. Then $B' = \{Lb|b \in B\}$ is a transversal of G/L modulo N/L. Proposition 3 follows from Theorem 1. The mapping $b \to Lb$ induces an isomorphism of $KG\alpha$ onto $K(G/L)\alpha'$.

COROLLARY 4. *If K, G, N and α are as in Theorem 1 and N is a direct factor of G, then $KG\alpha \cong K[G/N]$.*

Proof. Let G be the internal direct product of N and B. Then B is a transversal of G modulo N and $\gamma(x, y) = 1$ for all $x, y \in G/N$ since $bb' \in B$ for all $b, b' \in B$.

The converse of Corollary 4 is not true in general. For the field of complex numbers $\mathbf{C}$, and for any finite abelian group G of order n, $\mathbf{C}[G] \cong \mathbf{C}^n$ If N is a subgroup of G and α is a homomorphism of N into $\mathbf{C}^{\times}$, then $\mathbf{C}[G] \cong \mathbf{C}^n \cong \mathbf{C}[N \times G/N]$. Thus $\mathbf{C}G\alpha \cong \mathbf{C}(G/N \times N)\alpha \cong \mathbf{C}[G/N]$.

COROLLARY 5. *Let K be a commutative ring with 1, in which $2 \neq 0$. Then every negacyclic code of odd length over K is equivalent to a cyclic code. Over the field of complex numbers every negacyclic code is equivalent to a cyclic code.*

Proof. A negacyclic code of length n over K can be defined as a submodule M of the K-module K^n such that

$$\langle c_0, c_1, \dots, c_{n-1}\rangle \in M$$

implies

$$\langle -c_{n-1}, c_0, \dots, c_{n-2}\rangle \in M.$$

Rewriting $\langle c_0, c_1, \dots, c_{n-1}\rangle$ as $c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$, negacyclic codes turn out to be the ideals of $KC\alpha$, where C is the cyclic group

$$\{1, x, \dots, x^{2n-1}\}$$

of order $2n$, $N = \{1, x^n\}$ and $\alpha(1) = 1$, $\alpha(x^n) = -1$. If n is odd, $C = N \times B$, where $B = \{1, x^2, \dots, x^{2(n-1)}\}$ is a cyclic group of order n. Since cyclic codes are identified with the ideals of $K[B]$ (cf. [1, page 129], [2, page 41]) every negacyclic code of odd length is equivalent to a cyclic code (by Corollary 4). The equivalence is achieved by $\Sigma t_i x^i \to \Sigma t_i(-x)^i$.

Over the complex numbers $\mathbf{C}C\alpha \cong \mathbf{C}[C/N]$ and every negacyclic code over $\mathbf{C}$ is equivalent to a cyclic code.

THEOREM 6 (Maschke). *Suppose K is a field, G is a group and N is a central subgroup of G. If $[G : N]$ is finite, $[G : N] \neq 0$ in K and α is a homomorphism of N into $K^{\times}$, then $KG\alpha$ is semisimple.*

Proof. By Theorem 1, $KG\alpha \cong K^t[G/N]$. Thus, Theorem 6 follows from the similar theorem for twisted group algebras (cf. [7], [14]). However, due to Theorems 1 and 2, the proof of Maschke's Theorem for group algebras can easily be adapted to twisted group algebras by averaging over a transversal B of G modulo N (cf., [5, page 43], [9, page 5], [13, page 210]).

2. By Theorem 2, twisted group algebras are $KG\alpha$ for appropriate G and α. This simplifies the description of the center. Before we describe the center we need some preliminaries.

Suppose G is a group, N is a central subgroup of G and $g \in G$. Let $T(g) = [g, G] \cap N$. Then $T(g)$ is the set of all $n \in N$ such that ng is a conjugate of g. Also, $T(g)$ is a subgroup of N. Furthermore, if g is a conjugate of g', then $T(g) = T(g')$.

Indeed, let $m, n \in T(g)$. Then there are $h, k \in G$ such that $h^{-1}gh = mg$, $k^{-1}gk = ng$. Hence $(k^{-1}h)^{-1}g(k^{-1}h) = h^{-1}(kgk^{-1})h = h^{-1}n^{-1}gh = mn^{-1}g$. Thus $T(g)$ is a subgroup of N. Now, if g is a conjugate of g' and $m \in T(g)$, there are $h, k \in G$ such that $h^{-1}gh = mg$ and $k^{-1}gk = g'$. Thus $k^{-1}(mg)k = mg'$. Hence g, g', mg, mg' are all conjugates; i.e., g' is a conjugate of mg'. Thus $T(g) \subset T(g')$. By symmetry, $T(g) = T(g')$.

The following theorem describes the center of $KG\alpha$:

THEOREM 7. *Suppose K is an integral domain, G is a group, N is a central subgroup of G and α is an injective homomorphism of N into $K^{\times}$. Suppose $D \subseteq G$ and $\{Nd \mid d \in D\}$ is a transversal for the conjugacy classes of G/N. Then the set*

$$\{I(\alpha) + \Sigma \mathrm{Cl}(d) \mid d \in D, \mathrm{Cl}(d) \text{ is finite and } [d, G] \cap N = \{1\}\}$$

is a basis for the center of $KG\alpha$ as a K-module.

Proof. First, we construct a transversal B of G modulo N such that

$$D \subseteq B \subseteq \bigcup\{\mathrm{Cl}(d) \mid d \in D\}.$$

For every $d \in D$, let $B(d) \subseteq \mathrm{Cl}(\mathrm{d})$ be a transversal of $N\,\mathrm{Cl}(d)$ modulo N such that $d \in B(d)$. Then $B = \bigcup\{B(d) \mid d \in D\}$ is a transversal of G modulo N. Indeed, let $b \in B(d)$, $b' \in B(d')$ and $Nb = Nb'$. As $B(d) \subseteq \mathrm{Cl}(d)$, $B(d') \subseteq \mathrm{Cl}(d')$, b is a conjugate of d and b' is a conjugate of d'. Thus Nb is a conjugate of Nd and Nb' is a conjugate of Nd' in G/N. Hence Nd is a conjugate of Nd' and so $d = d'$. Thus $b, b' \in B(d)$ and $Nb = Nb'$. By the choice of $B(d)$, $b = b'$. Finally, let $g \in G$. Then Ng is a conjugate of Nd in G/N for some $d \in D$. Hene $ng \in \mathrm{Cl}(d)$ for some $n \in N$. Hence, there is $b \in B(d)$ such that $Nb = Nng = Ng$.

The set B' of all $b \in B$ such that $T(b) = [b, G] \cap N = \{1\}$ is closed under conjugates. Indeed, let $b \in B$, $T(b) = \{1\}$ and $b \in B(d) \subseteq \mathrm{Cl}(d)$. Thus $\mathrm{Cl}(b) = \mathrm{Cl}(d)$. Hence $T(d) = \{1\}$ by the preliminaries before Theorem 7. If $c \in \mathrm{Cl}(b) = \mathrm{Cl}(d)$, there is $b' \in B(d)$ such that $Nc = Nb'$; i.e., $c = nb'$ for some $n \in N$. But b, c, b' are all conjugates since they belong to $\mathrm{Cl}(d)$. Thus $n \in T(b') = T(d) = \{1\}$; i.e., $n = 1$ and $c = b' \in B(d)$. Thus $B(d) = \mathrm{Cl}(d) = \mathrm{Cl}(b)$. Thus the mapping $b \to g^{-1}bg$ is a permutation of B' for every $g \in G$.

Suppose $b, c \in B$, $m, n \in N$, $g \in G$ and $g^{-1}bg = mb$, $g^{-1}cg = nb$. Then $b = c$. Indeed $g^{-1}m^{-1}nbg = m^{-1}ng^{-1}bg = m^{-1}nmb = nb = g^{-1}cg$. Thus $g^{-1}m^{-1}nbg = g^{-1}cg$ and so $m^{-1}nb = c$. Hence $b = c$.

Now, we are ready to show that if $A \in \zeta(KG\alpha)$, then A can be uniquely written as

$$I(\alpha) + \Sigma\{t_b b \mid b \in B'\}$$

with only finitely many $t_b \in K$ different from 0. Indeed, by Theorem 1, A is uniquely writable as $I(\alpha) + \Sigma\{t_b b \mid b \in B\}$ with only finitely many $t_b \neq 0$. If $A \in \zeta(KG\alpha)$, then $A = g^{-1}Ag$ for every $g \in G$. Let $c \in B$ and $n \in T(c)$. Then there is $g \in G$ such that $g^{-1}cg = nc$. By the above observation,

$$A - g^{-1}Ag = I(\alpha) + t_c(1 - \alpha(n))c + \Sigma\{u_b b \mid b \in B, b \neq c\}.$$

Since $A \in \zeta(KG\alpha)$, $A - g^{-1}Ag = I(\alpha)$. By Theorem 1, $t_c(1 - \alpha(n)) = 0$ in K. As K is an integral domain, $t_c = 0$ or $\alpha(n) = 1$. Since α is injective $t_c = 0$ or $n = 1$. Thus every element of $\zeta(KG\alpha)$ can be uniquely written as

$$I(\alpha) + \Sigma\{t_b b \mid b \in B'\} = I(\alpha) + \Sigma\{t_b b \mid b \in B, T(b) = \{1\}\}$$

with only finitely many $t_b \neq 0$.

If $A = I(\alpha) + \Sigma\{t_b b \mid b \in B'\} \in \zeta(KG\alpha)$ and $c, c' \in B'$ are conjugate then $t_c = t_{c'}$. Indeed, let $g^{-1}cg = c'$. We have

$$I(\alpha) = A - g^{-1}Ag = I(\alpha) + (t_{c'} - t_c)c' + \Sigma\{s_b b \mid b \in B', b \neq c'\}.$$

Again by Theorem 1, $t_{c'} = t_c$. It follows that if $c \in B'$ has an infinite conjugacy class, then $t_c = 0$ since only finitely many $t_b \neq 0$. Thus every element of $\zeta(KG\alpha)$ is uniquely writable as

$$I(\alpha) + \Sigma\{t_b b \mid b \in B', \mathrm{Cl}(b) \text{ is finite}\}$$

where $t_b \neq 0$ only for a finite number of $b \in B'$ and $t_b = t_c$ if b is a conjugate of c. Conversely, every element of this form belongs to $\zeta(KG\alpha)$ since the subset of B' of all elements with finitely many conjugates is invariant under conjugation. Thus $\zeta(KG\alpha)$ is spanned, as a K-module, by

$$\{I(\alpha) + \Sigma\,\mathrm{Cl}(b) \mid b \in B, \mathrm{Cl}(b) \text{ is finite and } T(b) = \{1\}\}.$$

Actually, the center of $KG\alpha$ is spanned, as a K-module, by

$$\{I(\alpha) + \Sigma\,\mathrm{Cl}(d) | d \in D, \mathrm{Cl}(d) \text{ is finite and } T(d) = \{1\}\}.$$

This is true since if $b \in B$, $b \in B(d)$ and $T(b) = \{1\}$, then $B(d) = \mathrm{Cl}(d) = \mathrm{Cl}(b)$. As $\mathrm{Cl}(d)$ and $\mathrm{Cl}(d')$ are disjoint subsets of B if $d \neq d'$ and $T(d) = T(d') = \{1\}$, the set

$$\{I(\alpha) + \Sigma\,\mathrm{Cl}(d) | d \in D, \mathrm{Cl}(d) \text{ is finite and } T(d) = \{1\}\}$$

is K-linearly independent, and so, it is a basis for $\zeta(KG\alpha)$ as a K-module.

Remarks. From the proof of Theorem 7, it is clear that the conclusions of the theorem remain valid if K is a commutative ring with 1 such that $1 - \alpha(n)$ is not a zero divisor in K for every $1 \neq n \in N$. Also $\zeta(K[G])$, as a K-module, is the direct sum of $I(\alpha) \cap \zeta(K[G])$ and the K-submodule of $\zeta(K[G])$ spanned by $\{\Sigma\,\mathrm{Cl}(d) | d \in D, \mathrm{Cl}(d) \text{ is finite and } T(d) = \{1\}\}$. Thus

$$\zeta(KG\alpha) = \zeta(K[G])/(I(\alpha) \cap \zeta(K[G])).$$

The K-submodule $I(\alpha) \cap \zeta(K[G])$ is spanned by

$$\{\Sigma\,\mathrm{Cl}(d) | d \in D, \mathrm{Cl}(d) \text{ is finite and } T(d) \neq \{1\}\}.$$

If $g \in G$, $\mathrm{Cl}(g)$ is finite and $T(g) \neq \{1\}$, then $T(g)$ is a finite subgroup of N and so $T(g)$ is isomorphic to a finite subgroup of $K^{\times}$. If K is an integral domain, $T(g)$ is cyclic. Let $T(g) = \{1, a, \dots, a^{r-1}\}$, $r > 1$. Then

$$(1 - \alpha(a))\left(1 + \alpha(a) + \cdots + \alpha(a)^{r-1}\right) = 1 - \alpha(a)^r = 0.$$

Hence $1 + \alpha(a) + \cdots + \alpha(a)^{r-1} = 0$ in K, since $a \neq 1$. But

$$\Sigma\,\mathrm{Cl}(g) = (\Sigma T(g))(g_1 + \cdots + g_n),$$

where $\{g_1, \dots, g_n\}$ is a transversal for $\mathrm{Cl}(g)$ modulo N. Hence

$$\begin{aligned}\Sigma\,\mathrm{Cl}(g) &= \left(\Sigma T(g) - \Sigma\left\{\alpha(a)^i | 0 \leq i < r\right\}\right)(g_1 + \cdots + g_n)\\ &= \left(\Sigma\left\{(a^i - \alpha(a^i)) | 0 \leq i < r\right\}\right)(g_1 + \cdots + g_n) \in I(\alpha).\end{aligned}$$

3. An FC-group is a group in which every element has only finitely many conjugates. If G is a group, then $\Delta(G)$ is the set of all elements of G with only finitely many conjugates. The set $\Delta(G)$ is a characteristic subgroup of G and is called the FC-center of G (cf. [11, page 115], [12, page 121], [13, page 424]). A group G will be called *completely central* if for any $g \in \Delta(G) - \zeta(G)$, there is $h \in G$ such that $1 \neq [g, h] \in \zeta(G)$. In other words, for any $g \in \Delta(G) - \zeta(G)$, there is $1 \neq n \in \zeta(G)$ such that g is a conjugate of ng.

THEOREM 8. *Suppose K is an integral domain, G is a group, N is a central subgroup of G and α is an injective homomorphism of N into $K^{\times}$. Then the center of $KG\alpha$ is isomorphic to K iff $N = \zeta(G)$ and G is a completely central group.*

Proof. Suppose $N = \zeta(G)$ and G is a completely central group and $D \subseteq G$ satisfies the conditions of Theorem 7. If $T(d) = \zeta(G) \cap [d, G] = \{1\}$ and $\mathrm{Cl}(d)$ is finite, then $d \in \zeta(G)$. But $D \cap N = D \cap \zeta(G)$ is a singleton set. We can assume that $D \cap N = \{1\}$. By Theorem 7, $I(\alpha) + 1$ is a basis for $\zeta(KG\alpha)$ as a K-module. Thus $\zeta(KG\alpha) \cong K$.

Conversely, let $\zeta(KG\alpha) \cong K$. Since K is an integral domain, $\zeta(KG\alpha)$ is a free K-module with basis

$$\{I(\alpha) + \Sigma\, \mathrm{Cl}(d) | d \in D, d \in \Delta(G), T(d) = \{1\}\}.$$

Over non-trivial commutative rings with 1, the dimension of a free module is invariant (cf. [3, page 273]). Thus

$$|\{d | d \in D \cap \Delta(G), T(d) = \{1\}\}| = 1.$$

Hence, if $g \in G - N$ and $g \in \Delta(G)$, $T(g) \neq \{1\}$. Indeed, there is $n \in N$, $d \in D$ such that $ng \in \mathrm{Cl}(d)$. Thus $T(ng) = T(d) = T(g)$. Since $g \notin N$, $d \notin N$. Also, as $g \in \Delta(G)$ and $\mathrm{Cl}(ng) = n\,\mathrm{Cl}(g)$, $d \in \Delta(G)$. Hence $T(d) \neq \{1\}$; otherwise, $\zeta(KG\alpha)$ would have a basis containing 2 elements: $I(\alpha) + 1$ and $I(\alpha) + \Sigma\, \mathrm{Cl}(d)$. Thus $g \notin \zeta(G)$; i.e., $N = \zeta(G)$, and also G is completely central.

THEOREM 9. *Suppose K is a field, G is a group, N is a central subgroup of G and α is an injective homomorphism of N into $K^{\times}$. If $[G : N]$ is finite and $[G : N] \neq 0$ in K, then $KG\alpha$ is a central simple K-algebra iff $N = \zeta(G)$ and G is a completely central group.*

Proof. By Theorem 6, $KG\alpha$ is semisimple. Then $KG\alpha$ is a central simple K-algebra iff $\zeta(KG\alpha) \cong K$. Indeed, if $KG\alpha$ is not simple, then $KG\alpha \cong A \oplus B$, $A \neq 0$, $B \neq 0$. Also $\zeta(KG\alpha) = \zeta(A) \oplus \zeta(B) \not\cong K$. If $KG\alpha$ is simple, then $KG\alpha \cong M_n(D)$ is the ring of $n \times n$-matrices over a division ring D and $\zeta(KG\alpha) \cong \zeta(D) \cong K$ iff $M_n(D)$ is a central simple K-algebra. Theorem 9 then follows from Theorem 8.

For any positive integer n and any division ring D, the ring $M_n(D)$ of all $n \times n$-matrices over D is additively spanned by the group of units of $M_n(D)$. For finite dimensional simple algebras we have a sort of converse to Theorem 9.

THEOREM 10. *Suppose K is a field and A is a central simple K-algebra. Let G be a subgroup of the group of units of A spanning A as a K-space. If $[G : \zeta(G)]$ is finite and $[G : \zeta(G)] \neq 0$ in K, then A is a direct sumand of $KG\alpha$, where α is the inclusion mapping of $\zeta(G)$ into $K^{\times}$. Furthermore, $A \cong KG\alpha$ iff G is completely central.*

Proof. As G spans A as a K-space, $\zeta(G) \subseteq \zeta(A) = K1$. Thus α is an injective homomorphism of $\zeta(G)$ into $K^{\times}$. By Theorem 6, $KG\alpha$ is semisimple. Then $KG\alpha$ is the direct sum of its simple components. One of these components is isomorphic to A. Indeed, the inclusion mapping of G into A extends to a homomorphism of $KG\alpha$ onto A. Thus there is an ideal B of $KG\alpha$ such that $A \cong KG\alpha/B$. But $KG\alpha$ is semisimple. Hence there is an ideal A' of $KG\alpha$ such that $KG\alpha = B \oplus A'$. Hence $A \cong A'$. Furthermore $KG\alpha \cong A$ iff $KG\alpha$ is a central simple K-algebra. By Theorem 9, this is the case precisely when G is completely central.

4. If G is a finite group, its center is isomorphic to a subgroup of $K^{\times}$ for some integral domain K iff the center is cyclic. From Theorems 8 and 9, the class of completely central groups is interesting. If G is the group of quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$ and $N = \{1, -1\} = \zeta(G)$ and K is any field of characteristic not 2 and $\alpha(1) = 1$, $\alpha(-1) = -1$ is the injective homomorphism of N into $K^{\times}$, then $KG\alpha$ is a central simple K-algebra; $KG\alpha$ is a division ring iff K is formally real. Over fields K that are not formally real, $KG\alpha \cong M_2(K)$. The group of quaternions is a completely central group. The dihedral group D of order 8 is also completely central. If $\zeta(D) = \{1, a\}$ and K is a field of characteristic not 2 and $\alpha(1) = 1$, $\alpha(a) = -1$ is the injective homomorphism of $\zeta(D)$ into $K^{\times}$, then $KD\alpha \cong M_2(K)$.

That all nonabelian groups of order 8 are completely central may be derived from Theorem 10, directly, or from the following:

THEOREM 11. *If G is a nilpotent class 2-group, then G is a completely central group.*

Proof. Let $g \in G - \zeta(G)$. Then there is $h \in G$ such that $[g, h] \neq 1$. But $G/\zeta(G)$ is abelian. Hence $[g, h] \in \zeta(G)$ and so G is completely central.

If G is a group such that $\Delta(G) = \zeta(G)$, then G is completely central. Thus any torsion free nilpotent group is completely central since for torsion free nilpotent groups, the center and the FC-center coincide (cf. [12, p. 130]). It is not true that all nilpotent groups are completely central. The dihedral group of order 16 is an example of a nilpotent class 3-group that is not completely central. Indeed, this group can be presented as

$$D = \left\langle \{a, b\};\, a^8 = b^2 = 1, ba = a^{-1}b \right\rangle.$$

Since $[a, b] = a^{-1}bab = a^{-2}b^2 = a^{-2} = a^6$ and $[a, D] = \{1, a^6\}$ and a^6 is not a central element in D, the group D is not completely central. The dihedral group of order 16 is the nilpotent group of least order that is not completely central.

Clearly a finite group with a trivial center is not completely central. Thus all symmetric groups S_n, $n \geq 3$ and all alternating groups A_n, $n \geq 4$ are not completely central groups. However, the following proposition will show that many groups are completely central.

PROPOSITION 12. *Let G_i, $i \in I$ be a family of groups and let H be a subgroup of G_i for every $i \in I$. Suppose $[G_i : H] > 1$ for at least 3 different indices or for some $i \neq j$, $[G_i : H] > 1$ and $[G_j : H] > 2$. Then the FC-center of the free product of G_i, $i \in I$ with the amalgamated subgroup H is contained in H.*

Proof. Let $1 \in A_i$ be a right transversal of G_i modulo H, $i \in I$. Let G be the free product of G_i, $i \in I$, with the amalgamated subgroup H. Then every element in G has a unique normal form (cf. [13, page 179]):

If $g \in G$, $g \in H$, then $l(g) = 0$. If $g \notin H$, then $g = ha_1a_2 \dots a_n$, $1 \neq a_k \in A_{i_k}$, $1 \leq k \leq n$, $i_k \neq i_{k+1}$, $1 \leq k < n$, $h \in H$, $l(g) = n$.

If $i_1 = i_n$, let $c \in A_j$, $c \neq 1$, $j \neq i_1$. Then

$$(ca_1)^{-m}g(ca_1)^m = a_1^{-1}c^{-1} \dots a_1^{-1}c^{-1}ha_1 \dots a_nca_1 \dots ca_1$$

has length $4m + n$. Thus $\mathrm{Cl}(g)$ is infinite.

Let $i_1 \neq i_n$. If there are three distinct groups G_i such that $[G_i : H] > 1$, let $c \in A_j$, $c \neq 1$, $j \neq i_1$, $j \neq i_n$. Again, $(ca_1)^{-m}g(ca_1)^m$ has length $4m + n$ and $\mathrm{Cl}(g)$ is infinite. If $i_1 \neq i_n$ and one of A_{i_1}, A_{i_n} has more than 2 elements say $|A_{i_n}| > 2$, let $c \in A_{i_n}$, $c \neq 1$, $c \neq a_n$. Then $c^{-1} \notin H$, $a_nc^{-1} \notin H$ and

$$\left(c^{-1}a_1\right)^{-m}g\left(c^{-1}a_1\right)^m = a_1^{-1}c \cdots a_1^{-1}cha_1 \cdots \left(a_nc^{-1}\right)a_1c^{-1} \cdots a_1c^{-1}a_1$$

has length $4m + n - 1$ and $\mathrm{Cl}(g)$ is infinite. Thus $\Delta(G) \subseteq H$.

The restrictions of Proposition 12 are necessary. The free product of $\{1, a\}$ and $\{1, b\}$ (two cyclic groups of order 2) has a trivial center. But the FC-center is infinite cyclic; it is $\{(ab)^k | k \in \mathbf{Z}\}$. Also, $\mathrm{Cl}((ab)^k) = \{(ab)^k, (ba)^k\}$.

From Proposition 12 it follows that all free groups are completely central. It also follows that the class of completely central groups is not closed under subgroups and is not closed under homomorphic images. Under the conditions of Proposition 12, if H is central in every G_i, $i \in I$, then $H = \zeta(G) = \Delta(G)$ and the free product of G_i, $i \in I$ with amalgamated central subgroup H is completely central.

PROPOSITION 13. *Let G_i, $i \in I$, be a family of groups and let $G = \mathrm{Cr}\{G_i | i \in I\}$ be the Cartesian product of G_i, $i \in I$, and let $H = \mathrm{Dr}\{G_i | i \in I\}$ be the direct sum of G_i, $i \in I$. Then $\zeta(G)$ is the Cartesian product of $\zeta(G_i)$, $i \in I$, and $\Delta(G)/\zeta(G)$ is the direct sum of $\Delta(G_i)/\zeta(G_i)$, $i \in I$, and $\Delta(H)$ is the direct sum of $\Delta(G_i)$, $i \in I$.*

Proof. $\mathrm{Cl}((\dots, g_i, \dots))$ is the Cartesian product of $\mathrm{Cl}(g_i)$, $i \in I$ (in G). This is finite iff $\mathrm{Cl}(g_i)$ is finite for every $i \in I$ and only a finite number of them is different from $\{g_i\}$; i.e., $\mathrm{Cl}((\dots, g_i, \dots))$ is finite iff $g_i \in \zeta(G_i)$ for all but a finite number of $i \in I$ and $g_i \in \Delta(G_i)$ for all $i \in I$.

Thus the class of completely central groups is closed under Cartesian products and direct sums. In fact, $\mathrm{Cr}\{G_i | i \in I\}$ is completely central iff $\mathrm{Dr}\{G_i | i \in I\}$ is completely central iff every G_i, $i \in I$, is completely central. Thus the class of completely central groups is closed under direct summands.

5. Let G_i, $i \in I$, be a family of groups and let N be a central subgroup of G_i for every $i \in I$. The direct sum of G_i, $i \in I$, with the amalgamated subgroup N will be denoted by $\times_N \{G_i | i \in I\}$. It is the quotient of the free product of G_i, $i \in I$, with the amalgamated subgroup N by the normal subgroup generated by the commutators $[G_i, G_j]$, $i, j \in I$, $i \neq j$. If $N = \zeta(G_i)$ for every $i \in I$, $\times_N \{G_i | i \in I\}$ is called the central product of G_i, $i \in I$ (cf. [13, page 141]).

We will need the following proposition whose proof is routine:

PROPOSITION 14. *Let G_i, $i \in I$, be a family of groups and let N be a central subgroup of G_i for every $i \in I$. Then*

$$\zeta\big(\times_N\{G_i | i \in I\}\big) = \times_N\{\zeta(G_i) | i \in I\}$$

and

$$\Delta\big(\times_N\{G_i | i \in I\}\big) = \times_N\{\Delta(G_i) | i \in I\}.$$

Furthermore, $\times_N \{G_i | i \in I\}$ is completely central iff G_i is completely central for every $i \in I$.

THEOREM 15. *Suppose K is a commutative ring with 1, G_i, $i \in I$, is a family of groups and N is a central subgroup of G_i for every $i \in I$. If α is a homomorphism of N into $K^\times$, then*

$$K\big(\times_N\{G_i | i \in I\}\big)\alpha \cong \otimes_K\{KG_i\alpha | i \in I\}.$$

Proof. Let $1 \in B_i$ be a transversal of G_i modulo N, $i \in I$. Then every element of $\otimes_K \{KG_i\alpha | i \in I\}$ is a K-linear combination of

$$\otimes \{B_i | i \in I\} = \{\otimes \{b_i | i \in I\} | b_i \neq 1 \text{ for only a finite number of } i \in I\}$$

(cf. [4, page 56], [10, page 143]). The mapping $\otimes \{b_i | i \in I\} \to \Pi\{b_i | i \in I\}$ extends to a K-module homomorphism of

$$\otimes_K \{KG_i\alpha | i \in I\}$$

onto

$$K(\times_N \{G_i | i \in I\})\alpha.$$

This mapping is also injective and preserves multiplication.

COROLLARY 16. *Suppose K is a field, G_1, G_2 are groups and N is a central subgroup of G_1 and of G_2. If $[G_i : N]$ is finite, $[G_i : N] \neq 0$ in K, $i = 1, 2$ and α is an injective homomorphism of N into $K^\times$, then $K(G_1 \times_N G_2)\alpha$ is a central simple K-algebra iff $KG_i\alpha$ is a central simple K-algebra, $i = 1, 2$.*

Proof. By Theorem 9, $KG\alpha$ is central simple iff $N = \zeta(G)$ and G is completely central. The corollary then follows from Proposition 14 and Theorem 15. It also follows from Theorem 15 and Azumaya and Nakayama's Theorem (cf. [4, page 363], [10, page 219]).

The following theorem connects the centrality of $KG\alpha$ where α is a homomorphism not necessarily injective with complete centrality of a quotient of G.

THEOREM 17. *Let K be an integral domain and let G be a group. Then the following conditions are equivalent:*

(i) *There is a homomorphism α of $\zeta(G)$ into $K^\times$ such that $\zeta(KG\alpha) \cong K$;*

(ii) *There is a central subgroup M of G such that $\zeta(G/M) = \zeta(G)/M$ is embeddable into $K^\times$ and G/M is completely central.*

Proof. Suppose (i) holds and M is the kernel of α. Let α' denote the natural injective homomorphism of $\zeta(G)/M$ into $K^\times$ induced by α. Then, by Proposition 3, $KG\alpha \cong K(G/M)\alpha'$. By Theorem 8, $\zeta(G)/M = \zeta(G/M)$ and G/M is completely central.

Conversely, if γ is an embedding of $\zeta(G)/M = \zeta(G/M)$ into $K^\times$, let α be the composition of the natural homomorphism of $\zeta(G)$ onto $\zeta(G)/M$ and γ. As $\zeta(G)/M = \zeta(G/M)$, by Theorem 8, $K(G/M)\gamma$ is a central K-algebra if G/M is completely central. But, by Proposition 3 $KG\alpha \cong K(G/M)\gamma$. Hence $\zeta(KG\alpha) \cong K$.

6. We will discuss connections between completely central groups and groups of central type. A finite group G is called of *central type* if it has an irreducible complex representation of degree $[G:\zeta(G)]^{1/2}$ (cf. [6], [8]). If M is a central subgroup of a group G, G will be called *M-completely central* if for every $g \in \Delta(G) - \zeta(G)$, there is $h \in G$ such that $[g, h] \in \zeta(G) - M$. Thus complete centrality coincides with $\{1\}$-complete centrality. Also, if L, M are central subgroups of G and G is M-completely central and $L \subseteq M$, then G is L-completely central.

The next theorem describes the relation between M-complete centrality and complete centrality for FC-groups.

THEOREM 18. *Let M be a central subgroup of an FC-group G. Then G is M-completely central iff $\zeta(G)/M = \zeta(G/M)$ and G/M is completely central.*

Proof. Since G is an FC-group, G/M is also an FC-group. Let G be M-completely central and $g \in \Delta(G) - \zeta(G) = G - \zeta(G)$. Then there is $h \in G$ such that $[g, h] \in \zeta(G) - M$. Hence $Mg \notin \zeta(G/M)$; i.e., $\zeta(G/M) \subseteq \zeta(G)/M$, and $[Mg, Mh] \in \zeta(G/M) - \{M\}$; i.e., G/M is completely central.

Conversely, let G/M be completely central and $\zeta(G)/M = \zeta(G/M)$. If $g \in G - \zeta(G)$, then $Mg \in G/M - \zeta(G)/M = G/M - \zeta(G/M) = \Delta(G/M) - \zeta(G/M)$. Hence, there is $h \in G$ such that $[Mg, Mh] \in \zeta(G/M) - \{M\} = \zeta(G)/M - \{M\}$; i.e., $[g, h] \in \zeta(G) - M$ and G is M-completely central.

If G is a group, by G^0 we denote the opposite group; i.e., the group with the same set of elements as G and $x \circ y = yx$. The next theorem characterizes groups of central type in the class of finite completely central groups.

THEOREM 19. *Let G be a finite group. Then the following conditions on G are equivalent*:

(i) *G is of central type.*

(ii) *There is a central subgroup M of G such that G is M-completely central and $\zeta(G)/M$ is cyclic.*

(iii) *For some integral domain K, there is a homomorphism α of $\zeta(G)$ into $K^{\times}$ such that $\zeta(KG\alpha) \cong K$.*

(iv) *If K is a field such that $|G| \neq 0$ in K, then there is a finite field extension L of K and a homomorphism α of $\zeta(G)$ into $L^{\times}$ such that $LG\alpha$ is a central simple L-algebra.*

(v) *If K is a field such that $|G| \neq 0$ in K and there is a homomorphism α of $\zeta(G)$ into $K^{\times}$, then $K(G \times_{\zeta(G)} G^0)\alpha \cong M_n(K)$, where $n = [G:\zeta(G)]$.*

Proof. Let G be a group of central type and let $T: G \to \mathrm{End}_{\mathbf{C}}(V)$ be its irreducible complex representation where V is a complex vector space of dimension $[G:\zeta(G)]^{1/2}$. Since T is an irreducible representation, $T(\zeta(G)) \subseteq \zeta(\mathrm{End}_{\mathbf{C}}(V)) \cong \mathbf{C}$. Thus every $z \in \zeta(G)$ acts as a scalar multiplication in V. Thus $T(z)v = \alpha(z)v$ for all $v \in V$, $z \in \zeta(G)$, where $\alpha(z) \in \mathbf{C}$. Thus α is a homomorphism of $\zeta(G)$ into $\mathbf{C}^{\times}$. Since $T(z - \alpha(z)1)v = 0$ for all $z \in \zeta(G)$, $v \in V$, T can be viewed as a representation of $\mathbf{C}[G]/I(\alpha) = \mathbf{C}G\alpha$. Thus T can be considered as a homomorphism of $\mathbf{C}G\alpha$ onto $\mathrm{End}_{\mathbf{C}}(V)$. Since $\mathbf{C}G\alpha$ is a $[G:\zeta(G)]$ dimensional $\mathbf{C}$-algebra and $\mathrm{End}_{\mathbf{C}}(V)$ is of dimension $[G:\zeta(G)]$, $\mathbf{C}G\alpha \cong \mathrm{End}_{\mathbf{C}}(V)$; i.e., $\mathbf{C}G\alpha$ is a central simple $\mathbf{C}$-algebra. Thus (i) implies (iii).

Suppose (iii) holds. Then, by Theorem 17, there is a central subgroup M of G such that $\zeta(G)/M = \zeta(G/M)$ is embeddable into $K^{\times}$ and G/M is completely central. Since $\zeta(G)/M$ is finite, $\zeta(G)/M$ is cyclic. Thus (iii) implies (ii).

Suppose (ii) holds and K is a field such that $|G| \neq 0$ in K. Let $\zeta(G)/M$ be of order n. Then $n \neq 0$ in K. Let L be the splitting extension of K for $x^n - 1$. Then $L^{\times}$ has a subgroup of order n. Let α be the composition of the natural mapping $\zeta(G) \to \zeta(G)/M$ and an isomorphism of $\zeta(G)/M$ with the cyclic subgroup of $L^{\times}$ of order n. By Proposition 3 and Theorem 9, $LG\alpha$ is a central simple L-algebra. Thus (ii) implies (iv).

Suppose (iv) holds. Then there is a homomorphism α of $\zeta(G)$ into $\mathbf{C}^{\times}$ such that $\mathbf{C}G\alpha$ is a central simple $\mathbf{C}$-algebra since $\mathbf{C}$ is algebraically closed. Again, since $\mathbf{C}$ is algebraically closed, $\mathbf{C}G\alpha \cong \mathrm{End}_{\mathbf{C}}V$ where V is a complex vector space of dimension equal to the square root of the dimension of $\mathbf{C}G\alpha$; i.e., V is of dimension $[G:\zeta(G)]^{1/2}$. Thus, G has a complex irreducible representation of degree $[G:\zeta(G)]^{1/2}$; i.e., G is of central type. Thus (iv) implies (i).

We will be through if we show that (v) is equivalent to (ii). Suppose (v) holds. Then $K(G \times_{\zeta(G)} G^0)\alpha$ is a central simple K-algebra. By Proposition 3 and Theorem 9, $(G \times_{\zeta(G)} G^0)/\mathrm{Ker}\,\alpha$ is completely central and

$$\zeta\Big(\big(G \times_{\zeta(G)} G^0\big)/\mathrm{Ker}\,\alpha\Big) = \zeta\big(G \times_{\zeta(G)} G^0\big)/\mathrm{Ker}\,\alpha = \zeta(G)/\mathrm{Ker}\,\alpha.$$

By Proposition 14, $G/\mathrm{Ker}\,\alpha$ is completely central. Since

$$\zeta(G)/\mathrm{Ker}\,\alpha = \zeta\Big(\big(G \times_{\zeta(G)} G^0\big)/\mathrm{Ker}\,\alpha\Big) \geq \zeta(G/\mathrm{Ker}\,\alpha),$$

G is $(\mathrm{Ker}\,\alpha)$-completely central. Also, since $\zeta(G)/\mathrm{Ker}\,\alpha$ is isomorphic to a subgroup of $K^{\times}$, $\zeta(G)/\mathrm{Ker}\,\alpha$ is cyclic. Thus (v) implies (ii).

Suppose (ii) holds, K is a field, $|G| \neq 0$ in K and α is a homomorphism of $\zeta(G)$ into $K^{\times}$. Then $KG\alpha$ is a central simple K-algebra by Proposition 3 and Theorem 9. By Theorem 15,

$$K\big(G \times_{\zeta(G)} G^0\big)\alpha \cong KG\alpha \otimes_K KG^0\alpha.$$

Since $KG^0\alpha$ is the opposite of $KG\alpha$, a central simple K-algebra, this implies that $K(G \times_{\zeta(G)} G^0)\alpha \cong KG\alpha \otimes_K KG^0\alpha \cong M_n(K)$, the algebra of $n \times n$-

matrices over K where n is the dimension of $KG\alpha$ over $K = [G : \zeta(G)]$. Thus (ii) implies (v).

There are completely central finite groups which are not of central type. The nilpotent class 2-group G whose presentation in the variety of nilpotent class 2-groups is

$$G = \left\langle \{a, b, c\};\, a^2 = b^2 = c^2 = [a, b]^2 = [b, c]^2 = [c, a]^2 = 1 \right\rangle$$

is completely central by Theorem 11. G is not of central type. Indeed, G is of order 64 and $G/\zeta(G) \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. If G were of central type, then by Lemma 2 of [6, page 150], $G/\zeta(G) \cong H \times H$ for some abelian group H since $G/\zeta(G)$ is abelian. However, $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \not\cong H \times H$ for any abelian group H.

Every group of central type is solvable [8]. This can be generalized to groups whose centers are of finite index.

THEOREM 20. *Let G be a group such that $[G : \zeta(G)]$ is finite. If there is a central subgroup M of G such that G is M-completely central and $\zeta(G)/M$ is embeddable into $\mathbf{C}^\times$, then G is solvable.*

Proof. Let α be the composition of the natural homomorphism $\zeta(G) \to \zeta(G)/M$ and an embedding of $\zeta(G)/M$ into $\mathbf{C}^\times$. As G is an FC-group, by Theorem 18, $\zeta(G)/M = \zeta(G/M)$ and G/M is completely central. Thus, $\mathbf{C}G\alpha$ is a central simple $\mathbf{C}$-algebra by Proposition 3 and Theorem 9. By Theorem 1, $\mathbf{C}G\alpha \cong \mathbf{C}^t[G/\zeta(G)]$. By Theorem 1 of [6, page 146], there is a group H of central type such that $H/\zeta(H) \cong G/\zeta(G)$. Thus $G/\zeta(G)$ is solvable, and so G is solvable.

The conclusion of Theorem 20 remains valid if the field of complex numbers is replaced by any field of characteristic 0.

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