ILLINOIS JOURNAL OF MATHEMATICS
Volume 30, Number 4, Winter 1986

A BACKWARD HARNACK INEQUALITY AND FATOU THEOREM FOR NONNEGATIVE SOLUTIONS OF PARABOLIC EQUATIONS

BY

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Introduction

It is not an uncommon happening in the development of elliptic and parabolic p.d.e. that resolution of a problem first appears in the elliptic case and shortly after there is an attempt to adapt the techniques to the corresponding parabolic problem. In the majority of cases the adaptation succeeds with relative ease; but when it does not succeed so readily, or even not at all, a new and hopefully interesting insight into solutions of the parabolic problem is needed.

Such is the case in the study of the classical Fatou theorem for solutions, $u(x, t)$, of a parabolic partial differential equation of the form

$$Lu(x, t) \equiv \sum_{i, j=1}^{n} D_{x_i}\big(a_{ij}(x, t) D_{x_j} u(x, t)\big) - D_t u(x, t) = 0.$$

In particular, we consider solutions, u, defined in the cylinder $D_+ \equiv Dx(0, \infty)$, $D \subset \mathbf{R}^n$, which are nonnegative there and we want to study their pointwise boundary behavior, especially at points on the lateral boundary, $S_+ \equiv \partial Dx(0, \infty)$.

The assumptions on the operator L and domain $D \subset \mathbf{R}^n$ are as follows:

(i) The matrix $(a_{ij}(x, t))$ is bounded, measurable, symmetric, and uniformly positive definite, i.e., there exists $\lambda > 0$ such that for all $x \in \mathbf{R}^n$, $\xi \in \mathbf{R}^n$ and $t > 0$,

$$\lambda|\xi|^2 \le \sum_{i, j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \le (1/\lambda)|\xi|^2.$$

(ii) D is a bounded Lipschitz domain in $\mathbf{R}^n$.

Received October 5, 1983
[1]Partially supported by a grant from the National Science Foundation
[2]Partially supported by the Italian C.N.R., G.N.A.F.A.

In [2] the question of boundary values for nonnegative solutions of elliptic equations in a Lipschitz domain was studied. It was shown that if

$$Lu \equiv \sum_{i,j=1}^{n} D_{x_i}\big(a_{ij}(x)D_{x_j}u(x)\big) = 0 \quad \text{in } D$$

and $u \geq 0$ there, then at points $P \in \partial D$ which belong to the complement of a "small" exceptional set the pointwise limit of $u(x)$ exists provided x converges to P within a truncated cone contained in the domain and with vertex at P, i.e., $\lim_{x \to P} u(x)$ exists provided $x \to P$ nontangentially. The assumptions on the matrix, $(a_{ij}(x))$ were, as above, the boundedness, measurability symmetry, and uniform positive definiteness. The "small" exceptional set, E, of boundary points at which the nontangential limits fail to exist is a set of L-harmonic measure zero; i.e., for each $x \in D$, $\omega^x(E) = 0$ where $\omega^x(dP)$ is the unique finite Borel measure on ∂D such that for all $\varphi \in C(\partial D)$ the potential

$$u(x) = \int_{\partial D} \varphi(P)\,\omega^x(dP)$$

is the solution to the Dirichlet problem $Lu = 0$ in D, $u|_{\partial D} = \varphi$.

When we attempted to adapt the techniques in [2] to the parabolic problem an interesting difficulty occurred. Essential to the proof of the Fatou theorem in the elliptic case was the "doubling" property of L-harmonic measure, ω^{x_0}, with x_0 fixed inside D. This means that the measure of a surface ball, $\Delta_r(P)$, of radius r and center P and the measure of its concentric double, $\Delta_{2r}(P)$, are equivalent, or, more precisely,

$$\omega^{x_0}\big(\Delta_{2r}(P)\big) \leq C\omega^{x_0}\big(\Delta_r(P)\big)$$

with C independent of r and P. The corresponding doubling property for the L-caloric measure, ω^{x_0, T_0} in the parabolic case (see Section 0) seems difficult to establish and it is, in fact, equivalent to the existence of a "backward" Harnack inequality for nonnegative solutions of parabolic equations which vanish on the entire lateral surface, $S_+ = \partial D x(0, \infty)$. (See Theorem 2.4 and the remark following it.)

The normal Harnack inequality for nonnegative solutions, $u(x, t)$, of $Lu = 0$ in D_+ states that values of u inside D and at time t_1 are controlled or bounded by the value of u at any fixed point inside D and at a later time $t_2 > t_1$. This bound can be taken to be independent of u. (See Theorem 0.2. It is also assumed that t_1 stays at some positive distance from the initial time, $t = 0$.) We point out in Section 1 that when the nonnegative solution, u, vanishes on the entire lateral boundary of D_+ then this forward Harnack inequality can be reversed, i.e., values of u inside D and at time t_2 are controlled or bounded by the value of u at any fixed point inside D and at an earlier time t_1. Once again the bound does not depend on u, and t_1 is assumed to stay at a positive distance from $t = 0$. Another way of expressing this interior backward Harnack inequality is the statement

$$\sup_K u \leq c_K \inf_K u$$

where K is any compact subset of D_+. We emphasize that this Harnack is valid uniformly in u only when u belongs to the class of nonnegative solutions which vanish on $S_+ = \partial D x(0, \infty)$. It is not true for arbitrary nonnegative solutions and, interestingly, it requires also the boundedness of the domain, D. (See Theorem 1.3 and the remark following it.)

As we have already indicated, a backward Harnack inequality is closely related to the doubling property of L-caloric measure. However, for the doubling property a form of the backward Harnack inequality stronger than the interior one described above is required; namely, one must be able to compare values of a solution at points near the boundary. Specifically one needs to prove that in the class of solutions, $u \geq 0$, which vanish on the lateral boundary, S_+,

$$\sup_{K_r} u \leq c \inf_{K_r} u$$

where $K_r = \{(x, t)\colon \ |x - x_0| < r, \ |t - t_0| < r^2\}$ is contained in $D_\delta \equiv Dx(\delta, \infty)$, $\delta > 0$, and $\operatorname{dist}(\{|x - x_0| < r\}, \partial D)$ is equivalent to r. Here c must be found independent of u and K_r. This "backward Harnack at the boundary" and the ensuing Fatou Theorem for general nonnegative solutions of $Lu = 0$ in D_+ are shown in Section 2 to hold in the special case of parabolic operators with time independent coefficients. These results in the general case remain an open problem.

0. Definitions and known results

In this section we set up the notation and recall some known results that will be used throughout the paper.

Our basic domain is a cylinder $D_T = Dx(0, T)$ with Lipschitzian cross-section D. We call a bounded domain $D \subset \mathbf{R}^n$ a *Lipschitz domain* if for each $Q \in \partial D$ there exists a ball, B_{r_0}, centered at Q and a coordinate system of $\mathbf{R}^n$ such that in these coordinates,

$$B_{r_0} \cap D = B_{r_0} \cap \left\{(x', x_n) | x' \in \mathbf{R}^{n-1}, x_n > \varphi(x') \text{ where } \|\nabla\varphi\|_{L^\infty} \leq m\right\}$$

and

$$B_{r_0} \cap \partial D = B_{r_0} \cap \left\{(x', \varphi(x')) | x' \in \mathbf{R}^{n-1}\right\}.$$

We will assume the radius of the ball, B_{r_0}, and the constant m independent of $Q \in \partial D$. These two numbers, r_0 and m, determine what is called the *Lipschitz character* of D.

With S_T we indicate the lateral surface of the cylinder D_T, i.e., $S_T = \partial D \times (0, T)$. The parabolic boundary of D_T is $\partial_P D_T = S_T \cup (Dx\{0\})$. Analogously we set $D_+ = D \times (0, +\infty)$, $S_+ = \partial D \times (0, +\infty)$ and $\partial_p D_+ = S_+ \cup (D \times \{0\})$.

For $(Q, s) \in \partial_p D_T$ and r positive we define

$$\Psi_r(Q, s) = \{(x, t) | 0 < t < T, |x - Q| < r, |t - s| < r^2\},$$
$$\Delta_r(Q, s) = \partial_p D_T \cap \overline{\Psi}_r(Q, s),$$

and call $\Delta_r(Q, s)$ a *parabolic surface box* with radius r and center at (Q, s).

If $Q \in \partial D$ is represented by $(x_0, \varphi(x_0))$ in the above mentioned local coordinates we set

$$\overline{A}_r(Q, s) = (x_0', \varphi(x_0') + r, s + 2r^2),$$
$$\underline{A}_r(Q, s) = (x_0', \varphi(x_0') + r, s - 2r^2).$$

THEOREM 0.1 (Energy estimate, see [1]). *Let u be a nonnegative sub-solution of L in the cylinder $B_{2r}(x_0) \times (t_0 - 4r^2, t_0 + 4r^2)$. Then*

$$\max_{|t - t_0| < r^2} \int_{|x - x_0| < r} u^2(x, t)\, dx + \int_{t_0 - r^2}^{t_0 + r^2} \int_{|x - x_0| < r} |\nabla_x u(x, t)|^2\, dx\, dt$$
$$\leq \frac{C}{r^2} \int_{t_0 - 4r^2}^{t_0 + 4r^2} \int_{|x - x_0| < 2r} u^2(x, t)\, dx\, dt$$

where C depends only on λ, n.

THEOREM 0.2 (Harnack Principle [1]). *Let u be a nonnegative solution of $Lu = 0$ in D_T, and let D' be a convex sub-domain of D such that* $\operatorname{dist}(D', \partial D) = \delta > 0$. *Then for all $x, y \in D$ and $0 < s < t \leq T$ we have*

$$u(y, s) \leq u(x, t) \exp\left[c\left(\frac{|x - y|^2}{t - s} + \frac{t - s}{R} + 1\right)\right],$$

*where $C = C(\lambda, n)$ and $R =$ $\min(1, s, \delta^2)$.

THEOREM 0.3 (Carleson Estimate [5]). *Let $(Q, s) \in \partial_p D_T$, $s < T$, and u be a nonnegative solution of $Lu = 0$ in D_T which continuously vanishes on $\Delta_{2r}(Q, s)$. Then there exists a constant $C = C(\lambda, n, m, r_0)$ such that for $r \leq r_0$ and*

$(x, t) \in \Psi_r(Q, s)$,

$$u(x, t) \leq Cu(\bar{A}_r(Q, s)).$$

For a $\varphi \in C(\partial_p D_T)$ we can uniquely solve the boundary value problem

$$\text{(DP)} \qquad Lu = 0 \quad \text{in } D_T, \quad u|_{\partial_p D_T} = \varphi.$$

For each $(x, t) \in D_T$ the L-caloric measure $\omega^{(x,t)}$ is the unique probability Borel measure on $\partial_p D_T$ with the property that the function

$$u(x, t) = \int_{\partial_p D_T} \varphi(Q, s)\, d\omega^{(x,t)}(Q, s)$$

is the unique solution of (DP). Observe that Theorem 0.2 implies that for $x, y \in D$ and $0 < s < t \leq T$, $\omega^{(y,s)} \ll \omega^{(x,t)}$.

By the results in [1] there exists a unique Green's function $G(x, t; \zeta, \tau)$ for the problem

$$\text{(0.1)} \qquad Lu = f \quad \text{in } D_T, \quad u|_{\partial_p D_T} = 0.$$

Thus for $f \in L^q(0, T; L^p(D))$, and suitable q, p,

$$\text{(0.2)} \qquad u(x, t) = \int_0^T \int_D G(x, t; \zeta, \tau) f(\zeta, \tau)\, d\zeta\, d\tau$$

represents the unique solution of (0.1). Moreover Aronson's estimates (see [1]) imply that if $\Gamma(x, t; \zeta, \tau)$ is the fundamental solution of $Lu = 0$ in the whole space, then there are constants $\alpha_1, \alpha_2, C_1, C_2$ depending only on λ, n such that for all $x, \xi \in \mathbf{R}^n$ and $t > \tau$,

$$\text{(0.3)} \qquad C_1\gamma_1(x - \xi; t - \tau) \leq \Gamma(x, t; \xi, \tau) \leq C_2\gamma_2(x - \xi; t - \tau),$$

where γ_i is the fundamental solution of $L_i = D_t - \alpha_i\Delta$. The same estimates hold for the Green's function $G(x, t; \xi, \tau)$ for a bounded cylinder D_T; what is different in this case, however, is that α_1, C_1 depend in general on the distances of x, ξ from ∂D and on T while α_2, C_2 do not contain such dependencies.

1. Estimates for the L-caloric measure and comparison theorems for nonnegative solutions

It is known that to get information on the boundary behavior of nonnegative solutions of second-order elliptic equations which vanish on a part of the boundary one is led to study the corresponding elliptic measure and its regularity properties in a neighborhood of such a boundary zone. In this context it turns out that the fact that "all nonnegative solutions which are zero on a part of the boundary actually vanish at the same rate" is equivalent to the so-called doubling condition. This is a regularity property satisfied by the elliptic measure and can be stated as follows: "The elliptic measure of a surface box of radius $2r$ is equivalent to the elliptic measure of a box of radius r". To prove this property one has to make explicit the relation between elliptic measure and Green's function, and the main tools to get this are a boundary form of the Harnack Principle and estimates on the Green's function.

For parabolic equations the situation is much more complicated, essentially due to the evolution nature of the latter which is reflected in a time-lag in the Harnack Principle and non self-adjointness of the operator. As a consequence the relation between caloric measure and Green's function is weaker than the elliptic analogue and presents a backward time-lag.

In this section we establish this relation together with comparison results for nonnegative solutions vanishing on a part of the parabolic boundary. In Section 2, when dealing with time-independent operators, we will be able to overcome the above mentioned difficulties establishing the doubling condition. This turns out to be equivalent to an elliptic-type form of the Harnack Principle at the boundary for the Green's function.

We begin with stating a useful consequence of Theorem 0.3.

THEOREM 1.1. *Let $(Q, s) \in \partial_p D_T$ and let u be a nonnegative solution of $Lu = 0$ in D_T that continuously vanishes on $\partial_p D_T \setminus \Delta_{r/2}(Q, s)$. Then there exists a constant $C = C(\lambda, n, m, r_0)$ such that for r sufficiently small, depending on $T - s$ and for each $(x, t) \in D_T \setminus \Psi_r(Q, s)$ we have*

$$u(x, t) \leq Cu(\bar{A}_r(Q, s)). \tag{1.1}$$

Proof. We provide the proof only for the case $s > 0$. The case $s = 0$ is treated in the same way and we leave the details to the reader. By the maximum principle it suffices to prove (1.1) when $(x, t) \in \partial\Psi_r(Q, s)$ and $t > s - \frac{1}{4}r^2$. Fix $\delta \in (0, 1)$ small enough depending on the Lipschitz character of D so that for each $(\bar{Q}, \bar{s}) \in \partial\Psi_r(Q, s) \cap S_T$, $\Psi_{2\delta r}(\bar{Q}, \bar{s}) \cap \Psi_{r/2}(Q, s) = \varnothing$ and $\bar{s} + 2\delta^2 r^2 < s + 2r^2$. By Theorem 0.3, for each such $(\bar{Q}, \bar{s})$ we have

$$u(x, t) \leq Cu(\bar{A}_{\delta r}(\bar{Q}, \bar{s})) \tag{1.2}$$

for each $(x, t) \in \Psi_{\delta r}(\bar{Q}, \bar{s})$, where c depends only on λ, n, m, r_0. Harnack's Principle provides a constant C depending on λ and n such that for

$$(\bar{Q}, \bar{s}) \in \partial\Psi_r(Q, s) \cap \partial_p D_T,$$

we have

$$u(\bar{A}_{\delta r}(\bar{Q},\bar{s})) \le Cu(\bar{A}_r(Q,s)). \tag{1.3}$$

(1.2), (1.3) and a covering argument imply that (1.1) holds on

$$\partial_p\Psi_r(Q,s) \cap \{(x,t)\colon \operatorname{dist}(x,\partial D) \le cr\}$$

where $c > 0$ and depends only on the Lipschitz character of D. We use again the Harnack inequality to get (1.1) on the remaining part of $\partial\Psi_r(Q,s)$ which lies strictly inside D_T. Q.E.D.

COROLLARY 1.2. *With the hypothesis of Theorem* 1.1 *there is a constant C depending on λ, n, m, r_0 such that for r small enough (say $r < \frac{1}{2}\sqrt{T-s}$ and $r < r_0$) and $(x,t) \in D_T \setminus \Psi_r(Q,s)$,*

$$u(x,t) \le Cu(\bar{A}_r(Q,s))\,\omega^{(x,t)}(\Delta_{2r}(Q,s)). \tag{1.4}$$

Proof. As in the proof of Theorem 1.1 it is enough to get the bound

$$\omega^{(x,t)}(\Delta_{2r}(Q,s)) \ge C \tag{1.5}$$

for each $(x,t) \in \partial\Psi_r(Q,s)$, with C having the above dependence. (1.5) is a consequence of uniform Hölder continuity at the boundary of nonnegative solutions of $Lu = 0$ vanishing at the boundary and the fact that

$$\omega^{(x,t)}(\Delta_{2r}(Q,s)) \equiv 1 \quad \text{on } \Delta_{2r}(Q,s).$$ Q.E.D.

Theorem 1.1 implies an elliptic-type Harnack inequality which holds inside D_T and that we may formulate in the following way.

THEOREM 1.3. *Let u be a nonnegative solution of $Lu = 0$ in D_+ which continuously vanishes on S_+, and for $\delta \in (0, \min(\frac{1}{2}r_0, \frac{1}{2}\sqrt{T}))$ set*

$$D_\delta = \{x \in D \mid \operatorname{dist}(x,\partial D) > \delta\},\ D_{\delta,T} = D_\delta \times (\delta^2, T).$$

There exists a positive constant $C = C(\lambda, n, m, r_0, \delta, \operatorname{diam} D, T)$ such that

$$\max_{\bar{D}_{\delta,T}} u \le C \min_{\bar{D}_{\delta,T}} u. \tag{1.6}$$

Proof. Since $u \in C(\bar{D}_{\delta,T})$ there exist (X_0,T_0) and (X_1,T_1) belonging to $\bar{D}_{\delta,T}$ such that $u(X_0,T_0) = \min_{\bar{D}_{\delta,T}} u$, $u(X_1,T_1) = \max_{\bar{D}_{\delta,T}} u$. Let $D^*_{\delta,T}$ denote the cylinder $D \times (\frac{1}{2}\delta^2, T]$. It is clear that $D_{\delta,T} \subset\subset D^*_{\delta,T}$. If $(Q,s) \in S_T$ and

$s = \frac{1}{2}\delta^2$ then $D_T \cap \Psi_{\delta/2}(Q, s) \subset D^*_{\delta/2, T} \setminus \overline{D}_{\delta, T}$, and also $s + \frac{1}{4}\delta^2 = \frac{3}{4}\delta^2$. By the Carleson estimate (Theorem 0.3), applied to the box $D_T \cap \Psi_{\delta/2}(Q, s)$, we get for all $(x, t) \in D_T \cap \Psi_{\delta/4}(Q, s)$,

$$u(x, t) \leq C_1 u\big(\overline{A}_{\delta/4}(Q, s)\big), \tag{1.7}$$

where C_1 depends on λ, n, m, and r_0. The Harnack inequality provides a constant $C_2 = C_2(\lambda, n, \delta, \operatorname{diam} D, T)$ such that for all $(Q, s) \in S_T$ with $s = \frac{1}{2}\delta^2$,

$$u\big(\overline{A}_{\delta/4}(Q, s)\big) \leq C_2 u(X_0, T_0). \tag{1.8}$$

By (1.7), (1.8) we get

$$u\big(x, \tfrac{1}{2}\delta^2\big) \leq C_3 u(X_0, T_0) \tag{1.9}$$

for all $x \in D$ such that $\operatorname{dist}(x, \partial D) \leq \delta/4$. Again by the Harnack inequality we find a constant $C_4 = C_4(\lambda, n, \delta, \operatorname{diam} D, T)$ such that

$$\max_{D_{\delta/4} \times \{\delta^2/2\}} u \leq C_4 u(X_0, T_0). \tag{1.10}$$

Since $u \equiv 0$ on S_T by (1.9), (1.10) and the maximum principle we get

$$\max_{D^*_{\delta, T}} u \leq C u(X_0, T_0) \tag{1.11}$$

with $C = \max(C_3, C_4)$. To conclude the proof observe that $u(X_1, T_1) \leq \max_{D^*_{\delta, T}} u$. Q.E.D.

Remark. Theorem 1.3 may fail to hold if one drops either the boundedness of the base D of D_+, or the fact that u vanishes identically on S_+. In fact, in the first case if $D = \mathbf{R}^n$ for example, and

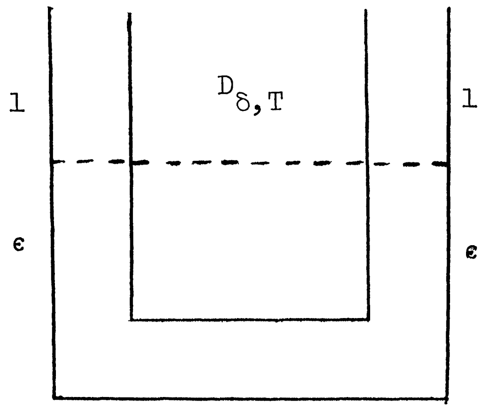
$$u(x, t) = \frac{e^{-|x + x_0|^2/4t}}{(4\pi t)^{n/2}},$$

then u is a solution of $Lu = \Delta u - u_t = 0$ in $\mathbf{R}^n \times (0, T)$, $T > 1$. If $x = (x_1, \ldots, x_n)$ is fixed so that $x_i > 0$ for each $i = 1, \ldots, n$, taking $x_0 = (x_{01}, x_2, \ldots, x_n)$ we get

$$\frac{u(0, 1)}{u(x, 1)} = e^{|x|^2/4} e^{\langle x, x_0 \rangle/2} \to 0,$$

as $x_{01} \to -\infty$. This shows that the boundedness of D is necessary.

To see that the "cooling" condition $u = 0$ on S_+ is necessary too, one can consider, in the case $n = 1$, the situation typified in the diagram, where $D_{\delta, T}$ is as in the statement of Theorem 1.3.



For each $\varepsilon > 0$, we let u_ε be the solution of $Lu = 0$ in D_T corresponding to the boundary values assigned as in the diagram. Since the maximum of u over $D_{\delta, T}$ is strictly bigger than a positive constant independent of ε while the minimum there is less than or equal to ε, (1.6) cannot hold uniformly in ε.

We now establish the main relation between the L-caloric measure and the Green's function.

THEOREM 1.4. *Let $(Q, s) \in S_T$, then for r sufficiently small, say*

$$r < \min\left(\tfrac{1}{2}r_0, \tfrac{1}{2}\sqrt{s}, \tfrac{1}{2}\sqrt{T - s}\right),$$

and each $(x, t) \in D_T$ with $s + 4r^2 \leq t \leq T$ we have

$$(1.12) \quad C^{-1}r^n G(x, t; \bar{A}_r(Q, s)) \leq \omega^{(x,t)}(\Delta_r(Q, s)) \leq Cr^n G(x, t; \underline{A}_r(Q, s))$$

where C is a constant which depends solely on λ, n, r_0, m and T.

Proof. Pick $\varphi \in C_0^\infty(\mathbf{R}^{n+1})$ such that $\varphi \geq 0$ and

$$\varphi = \begin{cases} 1 & \text{in } \Psi_r(Q, s) \\ 0 & \text{outside } \Psi_{6r/5}(Q, s). \end{cases}$$

For $(x, t) \in D_T$ with $t \geq s + 4r^2$ we have

$$(1.13)\quad \begin{aligned} &\omega^{(x,t)}(\Delta_r(Q, s)) \\ &\leq \int_{\partial_p D_T} \varphi(\bar{Q}, \bar{s})\, d\omega^{(x,t)}(\bar{Q}, \bar{s}) - \varphi(x, t) \\ &= \iint_{D_T}\Big[\sum_{i,j} a_{ij}(\xi, \tau) D_{\xi_i} G(x, t; \xi, \tau) D_{\xi_j}\varphi(\xi, \tau) \\ &\qquad + G(x, t; \xi, \tau) D_\tau \varphi(\xi, \tau)\Big]\, d\xi\, d\tau. \end{aligned}$$

Observing that $|D_{\xi_j}\varphi| \leq c/r$, $|D_\tau\varphi| \leq c/r^2$, by Schwarz's inequality we get

$$(1.14)\quad \begin{aligned} \omega^{(x,t)}(\Delta_r(Q, s)) \leq{}& Cr^{n/2}\left(\iint_{\Psi_{6r/5}(Q,s)\cap D_T} |\nabla_\xi G(x, t; \xi, \tau)|^2\, d\xi\, d\tau\right)^{1/2} \\ &+ cr^{-2}\iint_{\Psi_{6r/5}(Q,s)} G(x, t; \xi, \tau)\, d\xi\, d\tau \end{aligned}$$

with C depending only on λ, n. Theorem 0.1 gives

$$(1.15)\quad \begin{aligned} &\left(\iint_{\Psi_{6r/5}(Q,s)} |\nabla_\xi G(x, t; \xi, \tau)|^2\, d\xi\, d\tau\right)^{1/2} \\ &\leq \frac{C}{r}\left(\iint_{\Psi_{5r/4}(Q,s)} G(x, t; \xi, \tau)^2\, d\xi\, d\tau\right)^{1/2}, \end{aligned}$$

after having extended $G(x, t; \cdot, \cdot)$ to be zero outside D_T, which makes it a sub-solution of $L^* = \sum_{i,j=1}^n D_{\xi_j}(a_{ij} D_{\xi_i}) + D_\tau$. Using the analogue of Theorem 0.3 for nonnegative solutions of $L^* v = 0$, for each $(x, t) \in D_T$ with $t \geq s + 4r^2$ and $(\xi, \tau) \in \Psi_{5r/4}(Q, s)$ we get

$$(1.16)\quad G(x, t; \xi, \tau) \leq CG(x, t; \underline{A}_r(Q, s)),$$

where $C = C(\lambda, n, r_0, m, T)$. (1.14), (1.15) and (1.16) give the right-hand side of (1.12).

For the left-hand side of (1.12) we first note that

$$G(x, t; \xi, \tau) \leq C_2\gamma_2(x - \xi; t - \tau),$$

where C_2 and γ_2 are defined in Section 0. Now choose δ depending on m (the

Lipschitz constant) such that if (Q_r, s_r) represents the point $\bar{A}_r(Q, s)$, then the cylinder

$$\Phi_r = \{(x, t) |\ |x - Q_r| < \delta_r, s_r < t < s_r + \delta^2 r^2\}$$

is contained in D_T and $s_r + \delta^2 r^2 < s + 4r^2$. Using the above estimate on G, for $(x, t) \in \partial \Phi_r$ and $t > s_r$ we get

$$r^n G(x, t; \bar{A}_r(Q, s)) \leq C. \tag{1.17}$$

On the other hand (see Lemma 4.2 in [5] for example) for all such points we get

$$\omega^{(x, t)}(\Delta_r(Q, s)) \geq C. \tag{1.18}$$

By the maximum principle, observing that $G(x, t; \bar{A}_r(Q, s)) = 0$ if $t = s_r$ and $x \neq Q_r$, and (1.17), (1.18) we get the left-hand side of (1.12). Q.E.D.

Before stating the next result we need to introduce some notation. For a point $(Q, s) \in S_T$ and r small enough let $\alpha_r(Q, s)$ and $\beta_r(Q, s)$ be the sets

$$\alpha_r(Q, s) = \partial_p(\Psi_r(Q, s) \cap D_T) \setminus \partial_p D_T,$$
$$\beta_r(Q, s) = \{(y, t) = (y', y_n, t) \in \partial_p \Psi_r(Q, s) | y_n \geq \varphi(y') + br\}$$

where $b \in (0, 1)$ is fixed and φ is the function which describes ∂D around Q. Observe that $\operatorname{dist}(\beta_r(Q, s), S_T)$ is equivalent to br.

With ω_r and G_r we indicate the L-caloric measure and the Green's function relative to the domain $\Psi_r(Q, s) \cap D_T$.

LEMMA 1.5. *Let $(Q, s) \in S_T$ and $r < \min(\frac{1}{2}r_0, \frac{1}{2}\sqrt{s}, \frac{1}{2}\sqrt{T - s})$. Then there exists a positive constant $C = C(\lambda, n, r_0, m)$ such that*

$$\omega_r^{(x, t)}(\alpha_r) \leq C \omega_r^{(x, t)}(\beta_r) \tag{1.19}$$

for each $(x, t) \in \Psi_{r/8}(Q, s) \cap D_T$.

Proof. Set $U_r = (\Psi_{r/2}(Q, s) \setminus \Psi_{r/4}(Q, s)) \cap D_T$ and pick $\varphi \in C^\infty(\mathbf{R}^{n+1})$ such that $\varphi \equiv 1$ outside $\Psi_{r/2}(Q, s)$ and $\varphi \equiv 0$ inside $\Psi_{r/4}(Q, s)$. As in Theorem 1.4, for $(x, t) \in \Psi_{r/8}(Q, s) \cap D_T$ we have

$$\begin{aligned} \omega_r^{(x, t)}(\alpha_r) &\leq \int_{\partial_p(\Psi_r(Q, s \cap D_T))} \varphi(y, s)\, d\omega_r^{(x, t)}(y, s) - \varphi(x, t) \\ &= \iint_{U_r} [a_{ij}(\xi, \tau) D_{\xi_i} G_r(x, t); \xi, \tau) D_{\xi_j} \varphi(\xi, \tau) \\ &\qquad + G_r(x, t; \xi, \tau) D_\tau \varphi(\xi, \tau)]\, d\xi\, d\tau. \end{aligned} \tag{1.20}$$

Following the same argument as in Theorem 1.4, for $(x, t) \in \Psi_{r/8}(Q, s)$ we get

$$\omega_r^{(x,t)}(\alpha_r) \leq CG_r\big((x, t); \underline{A}_{r/2}(Q, s)\big)r^n. \tag{1.21}$$

Now, for each $(x, t) \in \Psi_{r/8}(Q, s) \cap D_T$, a maximum principle argument similar to that used for proving (1.12) gives

$$r^n G_r\big(x, t; \underline{A}_{r/2}(Q, s)\big) \leq C\omega_r^{(x,t)}(\beta_r). \tag{1.22}$$

(1.21) and (1.22) imply (1.19). Q.E.D.

THEOREM 1.6 (Local comparison theorem). *Let $(Q, s) \in S_T$ and u, v be two positive solutions of $Lu = 0$ in $\Psi_{2r}(Q, s) \cap D_T$ vanishing continuously on $\Delta_{2r}(Q, s)$. Then there exists a constant $C = C(\lambda, n, r_0, m)$ such that for r sufficiently small, say $r < \min(\frac{1}{2}r_0, \frac{1}{2}\sqrt{s}, \frac{1}{2}\sqrt{T-s})$, and $(x, t) \in \Psi_{r/8}(Q, s) \cap D_T$ we have*

$$\frac{u(x, t)}{v(x, t)} \leq C\frac{u(\bar{A}_r(Q, s))}{v(\underline{A}_r(Q, s))}. \tag{1.23}$$

Proof. By Theorem 0.3,

$$u(x, t) \leq Cu(\bar{A}_r(Q, s)) \tag{1.24}$$

for each $(x, t) \in \Psi_r(Q, s) \cap D_T$; hence, by the maximum principle for each such (x, t)

$$u(x, t) \leq Cu(\bar{A}_r(Q, s))\,\omega_r^{(x,t)}(\alpha_r), \tag{1.25}$$

where $\omega_r, \alpha_r, \beta_r$ have the same meaning as in Lemma 1.5. If $(x, t) \in \beta_r$, Harnack's inequality implies

$$v(x, t) \geq Cv(\underline{A}_r(Q, s)), \tag{1.26}$$

and using the maximum principle again we have

$$v(x, t) \geq Cv(\underline{A}_r(Q, s))\,\omega_r^{(x,t)}(\beta_r) \tag{1.27}$$

for each $(x, t) \in \Psi_r(Q, s) \cap D_T$. From (1.25), (1.27) and Lemma 1.5 we get (1.23). Q.E.D.

Remark. Exchanging the roles of u and v in 1.23, we obtain

$$\frac{1}{C}\frac{u(\underline{A}_r(Q, s))}{v(\bar{A}_r(Q, s))} \leq \frac{u(x, t)}{v(x, t)} \leq C\frac{u(\bar{A}_r(Q, s))}{v(\underline{A}_r(Q, s))} \tag{1.28}$$

for $(x, t) \in \Psi_{r/8}(Q, s) \cap D_T$. (1.28) gives a precise control on the quotient of two positive solutions vanishing on a portion of the lateral boundary. Information of this kind cannot be obtained for positive solutions which vanish on a portion of the base of D_T. As the following counterexample shows, one cannot hope to decide that two nonnegative solutions vanishing on a part of the base actually go to zero at an equivalent rate as $t \to 0^+$. Let $D = B_1(0)$, the unit ball in $\mathbf{R}^n$, and assume $a_{ij} \in C^\infty(\mathbf{R}^{n+1})$; then the solution of the problem

$$(1.29) \qquad Lu = 0 \text{ in } D_+, \qquad v|_{S_+} = g,\ v(x,0) = 0,\ x \in D,$$

is represented by the potential

$$u(x,t) = \int_0^t \int_{\partial D} K(x,t;Q,s) g(Q,s)\, dQ\, ds$$

where $K = \partial G/\partial N_Q$, the conormal derivative of the Green's function, i.e., $N_Q = A(Q)n_Q$ and n_Q is the inward pointing normal to ∂D at Q. Let u_α and u_β denote the solutions of (1.29) corresponding to the lateral data $g = s^\alpha$ and $g = s^\beta$ respectively. Assume $\alpha < \beta$; then

$$(1.30) \qquad \begin{aligned} u_\alpha(x,t) &= \int_0^t \int_{\partial D} K(x,t;Q,s) s^\alpha\, dQ\, ds \\ &> \frac{1}{t^{\beta-\alpha}} \int_0^t \int_{\partial D} K(x,t;Q,s) s^\beta\, dQ\, ds \\ &= \frac{u_\beta(x,t)}{t^{\beta-\alpha}} \end{aligned}$$

for each $(x, t) \in D_+$. (1.30) implies

$$(1.31) \qquad \frac{u_\alpha(x,t)}{u_\beta(x,t)} > \frac{1}{t^{\beta-\alpha}} \to +\infty \quad \text{as } t \to 0$$

for each fixed x in D, which proves the remark.

As a by-product of Theorems 1.3 and 1.6 we get the following:

THEOREM 1.7 (Global comparison theorem). *Let u, v be two nonnegative solutions of $Lu = 0$ in D_+ which continuously vanish on S_+, and for*

$$\delta \in \left(0, \min\left(\tfrac{1}{2}r_0, \tfrac{1}{2}\sqrt{T}\right)\right)$$

define

$$D^*_{\delta,T} = D \times (2\delta^2, T - \delta^2).$$

Then there exists a positive constant $C = C(\lambda, n, m, r_0, \delta, \operatorname{diam} D, T)$ *such that*

$$v(X_0, T)u(x, t) \leq Cu(X_0, T)v(x, t) \tag{1.32}$$

for all $(x, t) \in D^*_{\delta, T}$, *where* $X_0 \in D$ *is fixed.*

Proof. It is clear that for each $(Q, s) \in \partial D \times (2\delta^2, T - \delta^2)$,

$$\psi_{\delta/2}(Q, s) \cap D_T \subset D \times \left(\delta^2, T - \tfrac{3}{4}\delta^2\right).$$

We now use a covering argument similar to that of the proof of Theorem 1.3. There is a finite number of points $(Q_j, s_j) \in \partial D \times (2\delta^2, T - \delta^2)$, $j = 1, \ldots, p$, such that the family of boxes $\Psi_{\delta 2}(Q_j, s_j) \cap D_T$ covers $\partial D \times (2\delta^2, T - \delta^2)$. Apply Theorem 1.6 to each of these boxes to get

$$v\left(\underline{A}_{\delta/4}(Q_j, s_j)\right)u(x, t) \leq C_1 u\left(\overline{A}_{\delta/4}(Q_j, s_j)\right)v(x, t) \tag{1.33}$$

for all $(x, t) \in \Psi_{\delta/32}(Q_j, s_j) \cap D_T$, $j = 1, \ldots, p$, where C_1 depends on λ, n, m, r_0. The Harnack Principle provides constants C_2 and C_3, depending on λ, n, δ, diam D and T, such that

$$u\left(\overline{A}_{\delta/4}(Q_j, s_j)\right) \leq C_2 u(X_0, T), \quad v\left(\underline{A}_{\delta/4}(Q_j, s_j)\right) \geq C_3 v\left(X_0, \delta^2\right) \tag{1.34}$$

for each $j = 1, \ldots, p$. By (1.33) and (1.34) we get

$$v\left(X_0, \delta^2\right)u(x, t) \leq C_4 u(X_0, T)v(x, t) \tag{1.35}$$

for all $(x, t) \in D^*_{\delta, T}$ such that $\operatorname{dist}(x, \partial D) \leq \delta/32$. Using a Harnack inequality again we obtain

$$u(x, t) \leq C_5 u(X_0, T), \quad v(x, t) \geq C_6 v\left(X_0, \delta^2\right), \tag{1.36}$$

for all $(x, t) \in D^*_{\delta, T}$ with $\operatorname{dist}(x, \partial D) > \delta/32$. To complete the proof observe that by Theorem 1.3 there exists $C_7 = C_7(\lambda, n, m, r_0, \delta, \operatorname{diam} D, T)$ such that

$$v\left(X_0, \delta^2\right) \geq C_7 v(X_0, T). \qquad \text{Q.E.D.}$$

2. Time-independent operators: Boundary backward Harnack principle and non-tangential limits

In this section we specialize the results of Section 1 to the study of time-independent operators. Using a simple time-shifting argument and the results previously achieved we are able to get what we call a boundary backward Harnack Principle (Corollary 2.2) for nonnegative solutions which vanish on the lateral boundary. This in turn implies the doubling condition (Theorem 2.4), and is actually equivalent to it. Afterwards we establish an estimate (Theorem 2.5), which is suitable to control the Radon-Nikodym derivative of an L-caloric measure with respect to another, i.e. to control the *kernel function*, which is introduced at this point. As observed by Kemper [4] the notion of kernel function in intimately linked to the principle of positive singularities stated by Emile Picard: "*Given a differential operator L, a domain $\Omega \subseteq \mathbf{R}^n$ and a point $Q_0 \in \partial\Omega$, there is a non-trivial nonnegative solution u of $Lu = 0$, continuously vanishing on $\partial\Omega \setminus \{Q_0\}$. Such u is uniquely determined up to a constant multiple*". In our case the existence and uniqueness of the solution called for in the above principle amounts to an equivalent statement for the kernel function, Theorem 2.7. This theorem can also be viewed as an answer to the problem of determining the Martin boundary of a Lipschitz cylinder D_T with respect to the class of parabolic operators L we deal with. We can then say: "*The Martin boundary of D_T with respect to L is (homeomorphic to) the Euclidean parabolic boundary $\partial_p D_T$ of D_T*".

Finally we establish the representation result Theorem 2.11, and use it to study non-tangential limits along the lines of the classical theorem of Fatou.

THEOREM 2.1 (Backward Harnack Principle). *Let u be a nonnegative solution of $Lu = 0$ in D_+ continuously vanishing on S_+, and for*

$$\delta \in \left(0, \min\left(\tfrac{1}{2}r_0, \tfrac{1}{2}\sqrt{T}\right)\right)$$

let $D^*_{\delta,T}$ be defined as in Theorem 1.7. Then there exists a positive constant $C = C(\lambda, n, m, r_0, \delta, \operatorname{diam} D, T)$ such that for $0 < r \le 1$ and all $(x,t) \in D^*_{\delta,T}$ we have

$$u(x, t + 4r^2) \le Cu(x,t). \tag{2.1}$$

Proof. By Theorem 1.7, for all (x,t) in $D^*_{\delta,T}$ we get

$$u(X_0, T)v(x,t) \le Cv(X_0,T)u(x,t), \tag{2.2}$$

where u, v are two nonnegative solutions satisfying the hypothesis of the theorem. Now, let u be the function in the statement of Theorem 2.1 and define $v(x,t) = u(x, t+4r^2)$. For $(x,t) \in D^*_{\delta,T}$, (2.2) implies

$$u(X_0,T)u(x,t+4r^2) \le Cu\left(X_0, T+4r^2\right)u(x,t). \tag{2.3}$$

Without loss of generality we may assume $u(X_0,T), u(X_0, T+4r^2) \neq 0$. By Theorem 1.3 we find a constant C depending on the above mentioned parameters such that for $r \leq 1$,

$$\frac{1}{C} \leq \frac{u(X_0, T + 4r^2)}{u(X_0, T)} \leq C. \tag{2.4}$$

Then (2.3) and (2.4) give (2.1). Q.E.D.

COROLLARY 2.2 (Boundary Backward Harnack Principle). *Let u be a nonnegative solution of $Lu = 0$ in D_+ continuously vanishing on S_+. Choose δ as in Theorem* 2.1, *and let $(Q, s) \in S_+$ with $2\delta^2 < s < T - \delta^2$. Then there exists $C = C(\lambda, n, m, r_0, \delta, \operatorname{diam} D, T)$ such that for $r \leq \delta/4$,*

$$u(\bar{A}_r(Q, s)) \leq C u(\underline{A}_r(Q, s)). \tag{2.5}$$

Proof. Immediate consequence of Theorem 2.1. Q.E.D.

Theorem 2.1 and Corollary 2.2 have, of course, an adjoint companion if one considers $L^* = \sum_{i,j=1}^n D_{\xi_j}(a_{ij}(\xi) D_{\xi_i}) + D_\tau$ instead of L. From the adjoint version of Corollary 2.2 we get the following result for the Green's function G for L and D_T.

COROLLARY 2.3. *There exists a constant $C = C(\lambda, n, m, r_0, \operatorname{diam} D, T)$ such that for all $(Q, s) \in S_+$ with $0 < s < T - \delta^2$, and all $r \leq \delta/4$,*

$$\frac{1}{C} \leq \frac{G(X_0, T; \underline{A}_r(Q, s))}{G(X_0, T; \bar{A}_r(Q, s))} \leq C. \tag{2.6}$$

Corollary 2.3 together with Theorem 1.4 have as a consequence the so-called doubling condition for the L-caloric measure which we may state as follows.

THEOREM 2.4 (Doubling Condition). *Let $\delta \in (0, \min(\frac{1}{2}r_0, \frac{1}{2}\sqrt{T}))$. There exists a positive constant $C = C(\lambda, n, m, r_0, \delta, \operatorname{diam} D, T)$ such that for all $(Q, s) \in \partial_p D_T$ with $0 \leq s \leq T - \delta^2$ and for all $r \leq \delta/4$,*

$$\omega^{(X_0, T)}(\Delta_{2r}(Q, s)) \leq C \omega^{(X_0, T)}(\Delta_r(Q, s)). \tag{2.7}$$

Proof. For points $(Q, s) \in S_T$ with $0 < s < T - \delta^2$, the proof is an immediate consequence of (1.12) and (2.6). For points $(Q, 0) \in \bar{D} \times \{0\}$, the proof requires some technical adjustment, but essentially goes through in the same way. We give only an outline leaving the details to the reader. Let $\tilde{G}$ be the Green's function for the cylinder $\tilde{D}_T = D \times (-1, T)$. Reasoning as in the proof of Theorem 1.4 one can bound $\omega^{(X_0, T)}(\Delta_r(Q, 0))$ from above by an integral on a horse-shoe shaped domain involving $G(X_0, T; \xi, \tau)$, the Green's function with pole at (X_0, T) for L and D_T. By the maximum principle (applied to the adjoint variables (ξ, τ)), $\tilde{G}(X_0, T; \xi, \tau)$ coincides with $G(X_0, T; \xi, \tau)$ in D_T, therefore we may substitute G with $\tilde{G}$ in the above mentioned integral. By using the Carleson estimate or the Harnack Principle, depending on the distance of Q from ∂D, we get, as for (1.12),

$$\omega^{(X_0,T)}(\Delta_r(Q,0)) \leq Cr^n \tilde{G}(X_0, T; \underline{A}_r(Q,0))$$

where C is independent of r and $\underline{A}_r(Q,0)$ lies below $\Delta_r(Q,0)$. By (2.6) or (1.6), again depending on dist$(Q, \partial D)$, we get

$$\begin{aligned}\omega^{(X_0,T)}(\Delta_r(Q,0)) &\leq Cr^n \tilde{G}(X_0, T; \bar{A}_r(Q,0)) \\ &= Cr^n G(X_0 T; \bar{A}_r(Q,0)).\end{aligned}$$

Now, as in the proof of Theorem 1.4, using the estimates on G, we get the bound from below,

$$\omega^{(X_0,T)}(\Delta_r(Q,0)) \geq \frac{1}{C} r^n G(X_0, T; \bar{A}_r(Q,0)),$$

and this completes the proof. Q.E.D.

Remark. Observe that, by virtue of (1.12), (2.7) is actually equivalent to (2.6) for points $(Q, s) \in S_T$. (2.6), in turn, implies (2.5) if one uses the representation of a solution vanishing on S_T as the integral over a cross-section of its values against the Green's function. Precisely, for $r \leq \delta/4$,

$$\begin{aligned}u(\bar{A}_r(Q,s)) &= \int_D G(\bar{A}_r(Q,s); \xi, \delta^2) u(\xi, \delta^2)\, d\xi \\ &\leq C \int_D G(\underline{A}_r(Q,s); \xi, 2\delta^2) u(\xi, \delta^2)\, d\xi \\ &= Cu(\underline{A}_r(Q,s)).\end{aligned} \tag{2.8}$$

Therefore we can conlude that: "The doubling condition on the lateral boundary is equivalent to the backward Harnack Principle at the boundary (2.5)".

THEOREM 2.5. *Let $\delta \in (0, \min(\frac{1}{2}r_0, \frac{1}{2}\sqrt{T}))$, $(Q, s) \in \partial_p D_T$, with $0 \leq s < T - \delta^2$, and u, v be two nonnegative solutions of $Lu = 0$ in D_+ continuously vanishing on $\partial_p D_+ \setminus \Delta_{r/2}(Q, s)$. Then there exists a positive constant*

$$C = C(\lambda, n, m, r_0, \delta, \operatorname{diam} D, T)$$

such that for all $r \le \delta/4$,

$$u(X_0, T)v(\bar{A}_r(Q,s)) \le Cv(X_0,T)u(\bar{A}_r(Q,s)), \tag{2.9}$$

where $X_0 \in D$ is fixed.

Proof. Since by the maximum principle, $u(x,t) = v(x,t) = 0$ for $0 \le t \le s - \frac{1}{4}r^2$, we may fix the initial time at $t = s + \frac{1}{4}r^2$. Therefore, without loss of generality, we only consider the case of $(Q,s) \in \partial_p D_T$ with either $s = \frac{1}{4}r^2$ or $s = 0$. In both cases, by Corollary 1.2 there is $C = C(\lambda, n, m, r_0)$ such that for $r \le \delta/4$,

$$u(x,t) \le Cu(\bar{A}_r(Q,s))\omega^{(x,t)}(\Delta_r(Q,s)), \tag{2.10}$$

for each $(x,t) \in D_T \setminus \Psi_r(Q,s)$. Now let $v_r(x,t) = v(x,t+4r^2)$. v_r is a nonnegative solution of $Lu = 0$ in D_{T-4r^2} which is continuous in D_{T-4r^2}. We consider the case $s = \frac{1}{4}r^2$; the case $s = 0$ is easier and we leave the details to the reader. If $Q = (Q', Q_n)$, let $B_r = (Q', Q_n + r, 0)$, and choose $\alpha > 0$ depending on m such that if

$$\tilde{\Delta}_{\alpha r}(B_r) = \{x \in D \mid |x - B_r| < \alpha r\}$$

then

$$\tilde{\Delta}_{\alpha r}(B_r) \subset D \quad \text{and} \quad \operatorname{dist}(\tilde{\Delta}_{\alpha r}, \partial D) \cong r.$$

Then we have

$$\begin{aligned} v_r(x,t) &= \int_{\partial_p D_{T-4r^2}} v_r(\bar{Q},\bar{s})\, d\omega^{(x,t)}(\bar{Q},\bar{s}) \\ &\ge \inf_{\tilde{\Delta}_{\alpha r}(B_r)} v_r \cdot \omega^{(x,t)}(\tilde{\Delta}_{\alpha r}(B_r)). \end{aligned} \tag{2.11}$$

By the Harnack inequality there exists C such that

$$\inf_{\tilde{\Delta}_{\alpha r}(B_r)} v_r \ge Cv(\bar{A}_r); \tag{2.12}$$

therefore for each $(x,t) \in D_{T-4r^2}$, (2.11) and (2.12) give

$$v_r(x,t) \ge Cv(\bar{A}_r(Q,s))\omega^{(x,t)}(\tilde{\Delta}_r(B_r)). \tag{2.13}$$

By (1.12), (2.6) and the maximum principle we get

$$\begin{aligned} \omega^{(X_0,T)}(\Delta_r(Q,s)) &\le Cr^n G(X_0, T; \underline{A}_r(Q,s)) \\ &\le Cr^n G(X_0, T; \bar{A}_r(Q,s)) \\ &\le C\omega^{(X_0,T)}(\tilde{\Delta}_{\alpha r}(B_r)). \end{aligned} \tag{2.14}$$

If we take $(x, t) = (X_0, T)$ in (2.10) and (2.13), and use (2.14), we obtain

$$(2.15) \qquad u(X_0, T)v(\bar{A}_r(Q, s)) \le Cv(X_0, T + 4r^2)u(\bar{A}_r(Q, s)).$$

(2.9) now follows from (2.4) and (2.15). Q.E.D.

Remark. As a particular case of Theorem 2.5, we have the doubling condition (2.7). To see this, take

$$u(x, t) = \omega^{(x,t)}(\Delta_{2r}(Q, s)) \quad \text{and} \quad v(x, t) = \omega^{(x,t)}(\Delta_r(Q, s)),$$

and use Lemma 4.2 in [5].

We now introduce the notion of kernel function associated to a parabolic operator $L = \sum_{i,j=1}^n D_{x_i}(a_{ij}(x)D_{x_j}) - D_t$ and a Lipschitz cylinder D_+. Let $(X_0, T) \in D_+$ be fixed.

DEFINITION 2.6. We say that a function $K\colon D_+ \times \partial_p D_+ \to \mathbf{R}^+ \cup \{+\infty\}$ is a kernel function at $(Q, s) \in \partial_p D_+$ (for L and D_+) normalized at (X_0, T) if the following conditions are fulfilled:

(i) $K(x, t; Q, s) \ge 0$ for each $(x, t) \in D_+$ and $K(X_0, T; Q, s) = 1$;
(ii) $K(\cdot, \cdot; Q, s)$ is a (weak) solution of $Lu = 0$ in D_+;
(iii) $K(\cdot, \cdot; Q, s) \in C(\bar{D}_+ \setminus \{(Q, s)\})$ and

$$\lim_{(x,t)\to(Q_0,s_0)} K(x, t; Q, s) = 0 \text{ if } (Q_0, s_0) \in \partial_p D_+ \setminus \{Q, s\}.$$

If $s \ge T$, $K(x, t; Q, s)$ will be taken identically equal to zero.

For domains in $\mathbf{R}^{n+1}$ whose boundary is locally given by the graph of a function that is Lipschitz continuous in space and $\frac{1}{2}$-Hölder in time, and $L = \Delta - D_t$, the kernel function has been introduced by Kemper [3], who established its existence and uniqueness. The next theorem extends this result to our setting. We emphasize that our proof of existence and uniqueness of the kernel function is applicable to time-dependent operators, once the doubling condition is available. Before stating the main theorem we need to introduce some notation. If $(Q, s) \in \partial_p D_T$ and $r > 0$ we define

$$\Phi_r(Q, s) = \{(x, t) | \, |x - Q| < r, s - r^2 < t < s + 4r^2\}.$$

We set $D_T^r = \{(x, t) \in D_T | t > s\} \setminus \Phi_r(Q, s)$. Notice that the last definition makes sense even if $s = 0$. Now for points $(Q, s) \in \bar{S}_T$ fix $b > 0$, depending on the Lipschitz constant m, and define

$$(2.16) \qquad \beta_r(Q, s) = \partial\Phi_r(Q, s) \cap \{(x, t) \in \partial D_T^r | \text{ if } x = (x', x_n) \text{ then } x_n \ge \varphi(x') + br\},$$

where φ is the Lipschitz function that locally describes ∂D around Q. Observe that $\operatorname{dist}(\beta_r(Q, s), S_T) \cong br$. If, instead, $(Q, 0) \in D \times \{0\}$, then for r sufficiently small we define $\beta_r(Q, 0)$ be the top face of $\partial \Phi_r(Q, 0)$, i.e.,

$$\beta_r(Q,0) = \partial \Phi_r(Q,0) \cap \left\{(x, t) \in \partial D_T^r | t = 4r^2\right\}. \tag{2.17}$$

We wish to emphasize that the set β_r is suitably defined for applications of Harnack inequality. The reader should be aware that in the proof below we have sometimes preferred, for the sake of readability, to avoid writing cumbersome, but straightforward, details.

THEOREM 2.7. *There exists a unique kernel function (for L and D_+) at $(Q, s) \in \partial_p D_+, 0 \leq s < T - \delta^2$, normalized at (X_0, T).*

Proof. The existence part is standard and similar to that given in [3]. The geometry, however, is different. For $r > 0$ we let ω_r be the L-caloric measure for the domain D_T^r. For each $n \in \mathbf{N}$ we set: $D_T^n = D_T^{2^{-n}}$, $\omega_n = \omega_{2^{-n}}$, $\beta_n(Q, s) = \beta_{2^{-n}}(Q, s)$, and we define for $(x, t) \in D_T^n$

$$K_n(x, t) = \frac{\omega_n^{(x,t)}(\beta_n(Q, s))}{\omega_n^{(X_0, T)}(\beta_n(Q, s))}. \tag{2.18}$$

We clearly have $K_n \geq 0$, $LK_n = 0$ in D_T^n and $K_n(X_0, T) = 1$, for each n large enough. Since $D_T^n \nearrow D_T$ as $n \to \infty$, the Harnack Principle implies that the sequence $\{K_n\}$ is uniformly bounded and equicontinuous on compact subsets of D_+. Thus we can find a subsequence, still denoted by $\{K_n\}$, that converges on compact subsets of D_+ to a nonnegative solution $\tilde{K}$ of $Lu = 0$. $\tilde{K}(X_0, T) = 1$. Now, let $\alpha_n(Q, s)$ be the set $\partial D_T^n \cap \partial \Phi_{2^{-n}}(Q, s)$, and $\bar{A}_n(Q, s) = \bar{A}_{2^{-n}}(Q, s)$. By Theorem 1.1 and the maximum principle, for each $(x, t) \in D_T^n$ and for n sufficiently large, say $n \geq n_0$, we get

$$K_n(x, t) \leq CK_n(\bar{A}_n(Q, s))\,\omega_n^{(x,t)}(\alpha_n(Q, s)). \tag{2.19}$$

Now, if $(Q_0, s_0) \notin \Delta_{2^{-n_0+1}}(Q, s)$ and (x, t) is near (Q_0, s_0), letting $n \to \infty$ in (2.19) we get (iii). This proves that $\tilde{K}$ is a kernel function.

We now proceed to prove uniqueness. The strategy is to show that if v is another kernel function at (Q, s) normalized at (X_0, T), then there is a positive constant $C = C(\lambda, n, m, r_0, \delta, \dim D, T)$ such that for all $(x, t) \in D_+$,

$$v(x, t) \geq C\tilde{K}(x, t; Q, s). \tag{2.20}$$

From (2.20), the uniqueness of $\tilde{K}$ follows along the lines of Kemper [3].

To prove (2.20), let $v_r(x, t) = v(x, t + 8r^2)$. For each $r > 0$, $v_r \in C(D_T^r)$ so we get

$$(2.21)\quad \begin{aligned} v_r(x, t) &= \int_{\partial_p D_T^r} v_r(\overline{Q}, \bar{s})\, d\omega_r^{(x,t)}(Q, s) \\ &\geq \int_{\beta_r(Q,s)} v_r(\overline{Q}, \bar{s})\, d\omega_r^{(x,t)}(\overline{Q}, \bar{s}) \geq \inf_{\beta_r(Q,s)} v_r \cdot \omega_r^{(x,t)}(\beta_r(Q, s)). \end{aligned}$$

Harnack's inequality provides a constant C_1 such that for r sufficiently small,

$$(2.22)\qquad \inf_{\beta_r(Q,s)} v_r \geq C_1 v(\overline{A}_r(Q, s)).$$

On the other hand, by Theorem 1.1 and the maximum principle, for all $(x, t) \in D_T^r$ we have

$$(2.23)\qquad v(x, t) \leq C_2 v(\overline{A}_r(Q, s))\, \omega_r^{(x,t)}(\alpha_r(Q, s))$$

where, as before, $\alpha_r(Q, s) = \partial D_T^r \cap \partial \Phi_r(Q, s)$. From (2.23) we obtain

$$(2.24)\qquad 1 = v(X_0, T) \leq C_2 v(\overline{A}_r(Q, s))\, \omega_r^{(X_0,T)}(\alpha_r(Q, s)),$$

which gives

$$(2.25)\qquad v(\overline{A}_r(Q, s)) \geq \frac{1}{C_2} \frac{1}{\omega_r^{(X_0,T)}(\alpha_r(Q, s))}.$$

To complete the proof of (2.20) we need the doubling condition for the L-caloric measures ω_r. Suitably modifying the geometrical details of the proofs of Theorems 1.4 and 2.4 we get the existence of a constant C_3, depending on λ, n, m, r_0, δ, diam D and T, but not on r, such that for r sufficiently small,

$$(2.26)\qquad \omega_r^{(X_0,T)}(\alpha_r(Q, s)) \leq C_3 \omega_r^{(X_0,T)}(\beta_r(Q, s)).$$

Putting together (2.21), (2.22), (2.23), (2.25) and (2.26), for all $(x, t) \in D_T^r$, we get

$$(2.27)\qquad v_r(x, t) \geq C \frac{\omega_r^{(x,t)}(\beta_r(Q, s))}{\omega_r^{(X_0,T)}(\beta_r(Q, s))}.$$

Letting $r \to 0^+$ in (2.27), we obtain (2.20).

By (2.20), the proof of uniqueness follows along the lines of the analogous proof for the case of the heat operator; see [3]. Q.E.D.

Remark. In the proof of Theorem 2.7 we used the function

$$\tilde{K}(x, t; Q, s) = \lim_{n\to\infty} K_n(x, t)$$

where K_n is defined, as in (2.18), as a kernel function at (Q, s) with normalization at (X_0, T). We now define

$$K(x, t; Q, s) = \lim_{\delta\to 0^+} \frac{\omega^{(x,t)}(\Delta_\delta(Q, s))}{\omega^{(X_0,T)}(\Delta_\delta(Q, s))}.$$

It is easy to check that K is a kernel function at (Q, s) normalized at (X_0, T). By Theorem 2.7, $K = \tilde{K}$ in D_+; therefore from now on we will use K instead of $\tilde{K}$. Also, to avoid cumbersome details, when dealing with the cylinder D_T we will always assume K to be normalized at the point $(X_0, T_1) = (X_0, T + 1)$. In this way we avoid the limitation $0 \le s < T - \delta^2$ in Theorem 2.7.

COROLLARY 2.8. *For fixed $(x, t) \in D_T$, the function $(Q, s) \to K(x, t; Q, s)$ is continuous on $\partial_p D_T$.*

Proof. Let $(Q_n, s_n) \in \partial_p D_T$ with $(Q_n, s_n) \to (Q, s)$ as $n \to \infty$, and set

$$v_n(x, t) = K(x, t; Q_n, s_n).$$

The sequence $\{v_n\}$ is equicontinuous and equibounded in each compact subset of D_+; therefore it has a subsequence converging uniformly to a function v on compact subdomains of D_+. Since v is a kernel function at (Q, s) normalized at (X_0, T_1) we deduce $v(x, t) = K(x, t; Q, s)$. Q.E.D.

LEMMA 2.9. *Let $(Q_0, s_0) \in \partial_p D_T$. For r sufficiently small we have*

$$\lim_{\substack{(x,t)\to(Q_0,s_0)\\(x,t)\in D_T}} \sup\{K(x, t; Q, s) | (Q, s) \in \partial_p D_T \setminus \Delta_r(Q_0, s_0)\} = 0. \tag{2.28}$$

Proof. We confine ourselves to the case where the point (Q_0, s_0) belongs to the lateral boundary S_T, leaving to the reader the easier consideration of the case $(Q_0, 0) \in D \times \{0\}$. Let Γ be a cone in $\mathbf{R}^n$ with vertex at $Q_0 \in \partial D$ and exterior to D, and set $\Gamma_T = \Gamma \times (0, T)$. Define

$$\Sigma_r = \{(x, t) | \, |x - Q_0| < r/2, |t - s_0| < r^2/4\} \setminus \Gamma_T,$$

and let h_r be the L-caloric measure for Σ_r of the set $D_T \cap \partial_p \Sigma_r$. By Theorem 0.3, the maximum principle and a Harnack inequality, we have

$$\sup\{K(x, t; Q, s) | (Q, s) \in \partial_p D_T \setminus \Delta_r(Q_0, s_0)\} \le C h_r(x, t). \tag{2.29}$$

for $(x, t) \in \Sigma_r \cap D_T$. Now, (2.28) follows from the Hölder continuity of h_r and the fact that $h_r(Q_0, s_0) = 0$. Q.E.D.

Our next result is a theorem of representation for nonnegative solutions of $Lu = 0$ in D_T, where the basic domain D is assumed to be starlike with respect to a point $X_0 \in D$.

THEOREM 2.10. *Let D_T be a Lipschitz cylinder and suppose that D is starlike with respect to X_0. If u is a nonnegative solution of $Lu = 0$ in D_+, there exists a Borel measure ν on $\partial_p D_T$ (depending on u) such that for each $(x, t) \in D_T$,*

$$u(x, t) = \int_{\partial_p D_T} K(x, t; Q, s)\, d\nu(Q, s) \tag{2.30}$$

where K is the kernel function for L and D_+, normalized at (X_0, T_1).

Proof. Set

$$u_r(x, t) = u(x_r, t_r)$$

where

$$x_r = X_0 + (1 - r)(x - X_0)$$

and

$$t_r = (2r - r^2)T_1 + (1 - r)^2 t$$

for $0 < r < 1$. Then u_r is in $C(\overline{D_T})$ and is a solution of $L^r u_r = 0$, where

$$L^r = \sum_{i,j=1}^{n} D_{x_i}\left(a_{ij}(x_r) D_{x_j}\right) - D_t.$$

Therefore if ω_r is the L^r-caloric measure for D_+ and K_r is the kernel function for L^r and D_+, normalized at (X_0, T_1), we have

$$u_r(x, t) = \int_{\partial_p D_T} K_r(x, t; Q, s) u(Q_r, s_r)\, d\omega_r^{(X_0, T_1)}(Q, s). \tag{2.31}$$

Notice that, since

$$\int_{\partial_p D_+} u_r(Q, s)\, d\omega_r^{(X_0, T_1)}(Q, s) = u(X_0, T_1), \tag{2.32}$$

the family of measures $d\nu_r = u_r\, d\omega_r^{(X_0, T_1)}$, $0 < r < 1$, has finite total mass equal to $u(X_0, T_1)$. Therefore there exists a sequence $r_j \to 1$ such that $u_{r_j}\, d\omega_{r_j}^{(X_0, T_1)}$ converges weakly to a measure $d\nu$. Now consider $K_j = K_{r_j}$. For fixed $(Q, s) \in \partial_p D_T$, there is a subsequence, which we still call K_j, converging uniformly on compact subdomains of D_+ to a function which is easily seen to be the kernel function K for L and D_+ at (Q, s). Therefore for each $(Q, s) \in \partial_p D_T$ and $(x, t) \in D_T$ we have

$$K_j(x, t; Q, s) \to K(x, t; Q, s).$$

We now claim that if $\Omega \subset\subset D_T$, then

$$\sup_{\substack{(x,t)\in\Omega \\ (Q,s)\in\partial_p D_T}} |K_j(x, t; Q, s) - K(x, t; Q, s)| \to 0 \quad \text{as } j \to \infty. \tag{2.33}$$

Suppose (2.33) holds; then for $(x, t) \in D_T$ fixed we have

$$K_j(x, t; Q, s) \to K(x, t; Q, s) \text{ as } j \to \infty,$$

uniformly in $(Q, s) \in \partial_p D_T$, and hence

$$\begin{aligned} u(x, t) &= \lim_{j\to\infty} u_{r_j}(x, t) = \lim_{j\to\infty} \int_{\partial_p D_T} K_j(x, t; Q, s)\, d\nu_{r_j}(Q, s) \\ &= \int_{\partial_p D_T} K(x, t; Q, s)\, d\nu(Q, s), \end{aligned}$$

which would complete the proof. We are therefore left with proving (2.33). Assume it is false; then we can find $\Omega \subset\subset D_T$, $\varepsilon_0 > 0$ and two sequences $(x_m, t_m) \in \Omega$, $(Q_m, s_m) \in \partial_p D_T$, such that

$$(x_m, t_m) \to (x, t) \in \overline{\Omega}, (Q_m, s_m) \to (Q, s) \in \partial_p D_T \quad \text{as } m \to \infty,$$

and

$$|K_{j_m}(x_m, t_m; Q_m, s_m) - K(x_m, t_m; Q_m, s_m)| > \varepsilon_0 \tag{2.34}$$

for all $m \in \mathbf{N}$. On the other hand we have

(i) $K_{j_m}(x_m, t_m; Q_m, s_m) - K_{j_m}(x, t; Q_m, s_m) \to 0$ as $m \to \infty$ by the equi-Hölder continuity on Ω of the family of solutions $\{K_{j_m}(\cdot, \cdot; Q_m, s_m)\}$;

(ii) $K_{j_m}(x, t; Q_m, s_m) - K(x, t; Q, s) \to 0$ as $m \to \infty$ since the sequence of solutions $\{K_{j_m}(\cdot, \cdot; Q_m, s_m)\}$ is equibounded and equicontinuous on compact subdomains of D_+, hence converges to the kernel function at (Q, s) for L and D_+;

(iii)
$$\begin{aligned} &K(x_m, t_m; Q_m, s_m) - K(x, t; Q, s) \\ &\quad = \{K(x_m, t_m; Q_m, s_m) - K(x, t; Q_m, s_m)\} \\ &\qquad - \{K(x, t; Q, s) - K(x, t; Q_m, s_m)\} \end{aligned}$$

and each addend in the last sum goes to zero as $m \to \infty$ because of the equi-Hölder continuity of the family of solutions $\{K(\cdot, \cdot; Q_m, s_m)\}$ and the continuity of $K(x, t; \cdot, \cdot)$ on $\partial_p D_T$.

(i), (ii), and (iii) contradict (2.34), hence (2.33) is true. Q.E.D.

The next result is an estimate of the kernel function. Its consequence, Theorem 2.13, constitutes a basic tool when one searches a bound for the non-tangential maximal function in terms of the Hardy-Littlewood maximal function with respect to L-caloric measure.

THEOREM 2.11. *Let $(Q, s) \in \partial_p D_T$. Then there exists a constant*

$$C = C(\lambda, n, m, r_0, \text{diam } D, T)$$

such that for $r \le r(\lambda, n, m, r_0, \text{diam } D, T)$,

$$\sup_{(y,\tau) \in \Delta_r(Q,s)} K(\bar{A}_{2r}(Q, s); y, \tau) \le \frac{C}{\omega^{(X_0, T_1)}(\Delta_r(Q, s))}. \tag{2.35}$$

Proof. For r small, $(y, \tau) \in \Delta_r(Q, s)$ and $\varepsilon > 0$ define

$$u(x, t) = \omega^{(x,t)}(\Delta_r(Q, s)), \quad v(x, t) = \omega^{(x,t)}(\Delta_\varepsilon(y, \tau)).$$

By Theorem 2.5, for ε sufficiently small we have

$$\omega^{(X_0, T_1)}(\Delta_r(Q, s))\,\omega^{\bar{A}_{2r}(Q,s)}(\Delta_\varepsilon(y, \tau)) \le C\omega^{(X_0, T_1)}(\Delta_\varepsilon(y, \tau)). \tag{2.36}$$

Therefore

$$\lim_{\varepsilon \to 0^+} \frac{\omega^{\bar{A}_{2r}(Q,s)}(\Delta_\varepsilon(y, \tau))}{\omega^{(X_0, T_1)}(\Delta_\varepsilon(y, \tau))} = K(\bar{A}_{2r}(Q, s); y, \tau) \le \frac{C}{\omega^{(X_0, T_1)}(\Delta_r(Q, s))}. \tag{2.37}$$

Then (2.35) follows, since $(y, \tau) \in \Delta_r(Q, s)$ is arbitrary. Q.E.D.

Now, for $(Q, s) \in \partial_p D_T$ and r small we set $\Delta_j(Q, s) = \Delta_{2^j r}(Q, s)$, $j \in \mathbf{N} \cup \{0\}$, and $R_0(Q, s) = \Delta_0(Q, s)$, $R_j(Q, s) = \Delta_j(Q, s) \setminus \Delta_{j-1}(Q, s)$. By Theorem 2.11, we have the following result.

THEOREM 2.12. *Let $(Q_0, s_0) \in \partial_p D_T$ and let r, depending on $T - s$, be sufficiently small. There exists a sequence $\{C_j\}$ of positive numbers, independent* *of r and* (Q_0, s_0), *such that* $\Sigma_j C_j < +\infty$ *and*

$$(2.38) \qquad \sup_{(Q,s)\in R_j(Q_0,s_0)} K(\bar{A}_{2r}(Q_0, s_0); Q, s) \le \frac{C_j}{\omega^{(X_0,T_1)}(\Delta_j(Q_0, s_0))}.$$

Proof. Fix $(Q, s) \in R_j(Q_0, s_0)$ and for each $j \in \mathbf{N} \cup \{0\}$ set $\bar{A}_j = \bar{A}_{2^{j+1}r}(Q_0, s_0)$. For $j = 0, 1, 2, \ldots, 8$, say, a Harnack inequality and Theorem 2.11 give

$$(2.39) \qquad \sup_{(Q,s)\in R_j(Q_0,s_0)} K(\bar{A}_{2r}(Q_0, s_0); Q, s) \le \frac{C_j}{\omega^{(X_0,T_1)}(\Delta_j(Q_0, s_0))}.$$

Now, let $j \ge 8$. Using Theorem 2.11 again we obtain

$$(2.40) \qquad \sup_{(Q,s)\in R_j(Q_0,s_0)} K(\bar{A}_{j+1}; Q, s) \le \frac{C}{\omega^{(X_0,T_1)}(\Delta_j(Q_0, s_0))}.$$

Now observe that for $(Q, s) \in R_j(Q_0, s_0)$, $K(\cdot, \cdot; Q, s)$ is a nonnegative solution which vanishes on $\partial_p D_T \setminus \Delta_{2^{j-5}r}(Q, s)$; then by Theorem 1.1 we have

$$(2.41) \qquad K(x, t; Q, s) \le CK(\bar{A}_{2^{j-4}r}(Q, s)) \quad \text{for each } (x, t) \in D_T \setminus \Psi_{2^{j-4}r}(Q, s).$$

Now for $(Q, s) \in R_j(Q_0, s_0)$ let $(Q_r, s_r) = \bar{A}_{2^{j-4}r}(Q, s)$, and

$$(Q_{0_r}, s_{0_r}) = \bar{A}_{j+1} = \bar{A}_{2^{j+1}r}(Q_0, s_0).$$

We have $|Q_{0_r} - Q_r| \sim 2^j r$, while

$$\begin{aligned} s_{0_r} - s_r &\ge 2^{2(j+1)}r^2 - 2^{2(j-1)}r^2 - 2^{2(j-4)}r^2 \\ &= 2^{2j}r^2(2^2 - 2^{-2} - 2^{-8}) \\ &> 2^{2j+1}r^2. \end{aligned}$$

Then by the Harnack Principle we get

$$(2.42) \qquad K(\bar{A}_{2^{j-4}r}(Q, s); Q, s) \le CK(\bar{A}_{j+1}; Q, s).$$

For each $(x, t) \in D_T \setminus \Psi_{2^{j-4}r}(Q, s)$, (2.41), (2.42) imply

$$(2.43) \qquad K(x, t; Q, s) \le CK(\bar{A}_{j+1}; Q, s).$$

Taking $(x, t) = \bar{A}_0 = \bar{A}_{2r}(Q_0, s_0)$ in (2.43) and using (2.40) we obtain

$$(2.44)\qquad \sup_{(Q,s)\in R_j(Q_0,s_0)} K(\bar{A}_{2r}(Q_0, s_0); Q, s) \leq \frac{C}{\omega^{(X_0,T_1)}(\Delta_j(Q_0, s_0))}.$$

To get (2.38) from (2.44) we argue as follows. Let Γ be a fixed closed cone in $\mathbf{R}^n$ exterior to D, having vertex at Q_0 and axis along the x_n-direction in the local coordinates around Q_0. Set $\Gamma_T = \Gamma \times (0, T)$ and let

$$\Sigma_j = \{(x, t) |\, |x - Q_0| < 2^{j-1}r, |t - s_0| < 4^{j-1}r^2\} \setminus \Gamma_T.$$

If h_j is the L-caloric measure of the set $D_T \cap \partial_p \Sigma_j$ with respect to Σ_j, the maximum principle gives

$$(2.45)\qquad \sup_{(Q,s)\in R_j(Q_0,s_0)} K(\bar{A}_{2r}(Q_0, s_0); Q, s) \leq \frac{C h_j(\bar{A}_{2r}(Q_0, s_0))}{\omega^{(X_0,T_1)}(\Delta_j(Q_0, s_0))}.$$

for $(x, t) \in D_T \cap \Sigma_j$. To complete the proof we need to show that

$$(2.46)\qquad \sum_{j=8}^{\infty} h_j(\bar{A}_{2r}(Q_0, s_0)) < +\infty.$$

A rescaling argument and the Hölder continuity of h_j in $D_T \cap \Sigma_j$ give (2.46). Q.E.D.

At this point we have all the tools we need to study non-tangential limits. Since the theory is by now standard we will not give the details of the proofs, but we will limit ourselves to stating the theorems and giving an outline of their proofs (see also [3] for the case of the heat equation).

For $(Q, s) \in \partial_p D_T$ we introduce the definition of parabolic non-tangential cone with vertex at (Q, s). If $(Q, s) \in S_T$ set

$$\Gamma(Q, s) = \{(x, t) | C_1 > x_n - Q_n > C_2|x' - Q'| + C_3|t - s|^{1/2}\}.$$

The constants C_1, C_2 and C_3 are chosen in dependence of the Lipschitz constant m. For u defined in D_T the non-tangential maximal function u^* defined on $\partial_p D_T$ is

$$u^*(Q, s) = \sup\{|u(x, t)|\, |(x, t) \in \Gamma(Q, s)\}.$$

Finally we define the Hardy-Littlewood maximal function of a measure ν on $\partial_p D_T$ with respect to the L-caloric measure $\omega^{(X_0,T_1)}$ as

$$M_\omega(\nu)(Q, s) = \sup_{r>0} \frac{\nu(\Delta_r(Q, s))}{\omega^{(X_0,T_1)}(\Delta_r(Q, s))}.$$

THEOREM 2.13. *If ν is a finite Borel measure on $\partial_p D_T$, with D starlike with respect to X_0, and $u(x,t) = \int_{\partial_p D_T} K(x,t;Q,s)\,d\nu(Q,s)$, then*

$$u^*(Q,s) \le CM_\omega(\nu)(Q,s) \tag{2.47}$$

for each $(Q,s) \in \partial_p D_T$. C depends only on λ, m, r_0, diam *D, T, C_1, C_2, C_3 and $\omega^{(X_0,T_1)}$.*

Proof. By a Harnack inequality, if C_1, C_2, C_3 are fixed suitably, we get

$$u^*(Q,s) \le C \sup_{0<r<\alpha} u(\bar{A}_{2r}(Q,s)), \tag{2.48}$$

where α depends on C_1, C_2, C_3. Theorems 2.10, 2.12 give

$$\begin{aligned} u(\bar{A}_{2r}(Q,s)) &= \int_{\partial_p D_T} K(\bar{A}_{2r}(Q,s);\bar{Q},\bar{s})\,d\nu(\bar{Q},\bar{s}) \\ &= \sum_{j=0}^{N} \int_{R_j(Q,s)} K(\bar{A}_{2r}(Q,s);\bar{Q},\bar{s})\,d\nu(\bar{Q},\bar{s}) \\ &\le \sum_{j=0}^{N} c_j \frac{\nu(\Delta_j(Q,s))}{\omega^{(X_0,T)}(\Delta_j(Q,s))} \\ &\le CM_\omega(\nu)(Q,s). \end{aligned}$$

Notice that N depends on $|D|$, T and r, and that we have used the same notation introduced for Theorem 2.12. Q.E.D.

THEOREM 2.14. *Let u be a nonnegative solution of $Lu = 0$ in D_T; then u has non-tangential limit along the parabolic cone $\Gamma(Q,s)$ for almost every $(d\omega^{(X_0,T_1)})(Q,s) \in \partial_p D_T$.*

Proof. Since D is locally Lipschitz we may assume D starlike with respect to X_0. In this case we write $d\nu = f\,d\omega^{(X_0,T_1)} + d\nu_s$ where $d\nu_s \perp d\omega^{(X_0,T_1)}$ and $f \in L^1(\partial_p D_T, d\omega^{(X_0,T_1)})$. If $F = \operatorname{supp} \nu_s$, Theorem 2.10 gives

$$\begin{aligned} u(x,t) &= \int_{\partial_p D_T} f(Q,s)K(x,t;Q,s)\,d\omega^{(X_0,T_1)}(Q,s) \\ &\quad + \int_F K(x,t;Q,s)\,d\nu_s(Q,s) \\ &= u_a(x,t) + u_s(x,t). \end{aligned}$$

Notice that $\omega^{(X_0,T_1)}(F) = 0$. A standard strategy based on Theorem 2.13

implies $u_a(x,t) \to f(Q,s)$ for a.e. $(d\omega^{(X_0,T_1)})(Q,s) \in \partial_p D_T$ and $(x,t) \in \Gamma(Q,s)$, while for $(Q,s) \notin F$, Lemma 2.9 implies $u_s(x,t) \to 0$ as $(x,t) \to (Q,s)$ along $\Gamma(Q,s)$. Q.E.D.

THEOREM 2.15. *If u is a bounded solution of $Lu = 0$ in D_T, then*

$$u(x,t) = \int_{\partial_p D_T} K(x,t;Q,s) f(Q,s)\, d\omega^{(X_0,T_1)}(Q,s) \tag{2.49}$$

with $f \in L^\infty(\partial_p D_T, d\omega^{(X_0,T_1)})$.

Proof. We may assume $0 \le u \le M$ in D_T. There exists a Borel measure ν on $\partial_p D_T$ such that

$$u(x,t) = \int_{\partial_p D_T} K(x,t;Q,s)\, d\nu(Q,s).$$

Write $d\nu = d\nu_s + f\, d\omega^{(X_0,T_1)}$, where

$$d\nu_s \perp d\omega^{(X_0,T_1)} \quad \text{and} \quad f \in L^1\left(\partial_p D_T, d\omega^{(X_0,T_1)}\right),$$

and let

$$u_a = \int_{\partial_p D_T} Kf\, d\omega^{(X_0,T_1)}, \quad u_s = \int_{\partial_p D_T} K\, d\nu_s.$$

We have $0 \le u_a \le M, 0 \le u_s \le M$. Theorem 2.15 implies $u_a(x,t) \to f(Q,s)$ non-tangentially for a.e. $(d\omega^{(X_0,T_1)})(Q,s) \in \partial_p D_T$. Then

$$f \in L^\infty\left(\partial_p D_T, d\omega^{(X_0,T_1)}\right).$$

We want to show $d\nu_s = 0$. If not, there exists $(Q_0, s_0) \in \partial_p D_T$ such that

$$\lim_{r\to 0} \frac{\nu_s(\Delta_r(Q_0,s_0))}{\omega^{(X_0,T_1)}(\Delta_r(Q_0,s_0))} = +\infty. \tag{2.50}$$

On the other hand we have

$$M \ge u_s(\bar{A}_{2r}(Q_0,s_0)) \ge \int_{\Delta_r(Q_0,s_0)} K(\bar{A}_{2r}(Q_0,s_0);Q,s)\, d\nu_s(Q,s). \tag{2.51}$$

For $(Q,s) \in \Delta_r(Q_0,s_0)$, Corollary 1.2 gives

$$K(\bar{A}_r(Q,s);Q,s) \ge \frac{C}{\omega^{(X_0,T_1)}(\Delta_r(Q,s))}.$$

It is clear that

$$\omega^{(X_0,T_1)}(\Delta_r(Q,s)) \le \omega^{(X_0,T_1)}(\Delta_{2r}(Q_0,s_0)) \le C\omega^{(X_0,T_1)}(\Delta_r(Q_0,s_0));$$

therefore, using Harnack inequality,

$$K(\bar{A}_{2r}(Q_0,s_0);Q,s) \ge \frac{C}{\omega^{(X_0,T_1)}(\Delta_r(Q_0,s_0))}$$

for each $(Q,s) \in \Delta_r(Q_0,s_0)$. This leads to a contradiction since by (2.51),

$$M \ge u_s(\bar{A}_{2r}(Q_0,s_0)) \ge C\frac{\nu_s(\Delta_r(Q_0,s_0))}{\omega^{(X_0,T_1)}(\Delta_r(Q_0,s_0))}.$$

Q.E.D.

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