ON THE HOMOLOGY OF LOCAL RINGS

BY

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Introduction

In §1 we discuss the structure of $\mathrm{Tor}^R(K, K)$ when R is a commutative, noetherian ring, and K is a homomorphic image of R which is a field. If we view K as an R-module, $\mathrm{Tor}^R(K, K)$ is defined [1, Chapter VI], and, because R is commutative, it is a module over R. Since the kernel of the homomorphism yielding K annihilates $\mathrm{Tor}^R(K, K)$, as R-module it is merely a K-module, i.e., a vector space over K. It does, in fact, possess more structure: Two maps,

$$m:\mathrm{Tor}^R(K, K) \otimes_K \mathrm{Tor}^R(K, K) \to \mathrm{Tor}^R(K, K)$$

and

$$M:\mathrm{Tor}^R(K, K) \to \mathrm{Tor}^R(K, K) \otimes_K \mathrm{Tor}^R(K, K),$$

can be defined which equip $\mathrm{Tor}^R(K, K)$ with an algebra and coalgebra structure respectively.

If R is a local ring, then K is the unique homomorphic image of R which is a field; in this case m and M are compatible in the sense that $\mathrm{Tor}^R(K, K)$ becomes a Hopf Algebra. This is the theorem of §1.

As corollaries we prove a result of Serre's [4, Theorem 4] and a result of Tate's [3, Theorem 7]. The characterization of regular local rings as those of finite homological dimension, first announced in [5], has been given three proofs: in [3], [4], and [6]. Each of them uses crucially the structure of $\mathrm{Tor}^R(K, K)$, in particular, Serre's Theorem. Thus the first corollary would yield still another proof of that result.

In §2 we add to the riches of the Koszul complex, a less well-known homological invariant. Defined in a general setting in [1] and exploited by Auslander and Buchsbaum in [7] it is shown to be rich enough to distinguish not only regular local rings but also local complete intersections, i.e., those local rings which are homomorphic images of regular local rings by ideals whose rank is equal to the number of elements needed to generate them.

This paper is the essential contents of §§1 and 3 of [10]. I wish to thank John Tate for his help and advice during the preparation of the thesis. I also wish to thank Professor O. Zariski for his guidance during the final stages of preparation and M. Auslander and D. A. Buchsbaum for the use of the manuscript of [7]. The suggestion that $\mathrm{Tor}^R(K, K)$ should be a Hopf Algebra was made by Eilenberg and Serre.

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1. The natural algebra and coalgebra structure of $\operatorname{Tor}^R(K, K)$

Throughout, we will use the word "ring" in a restricted sense, meaning a nontrivial, commutative, noetherian ring with a unit element. For modules over such rings we demand that the unit element act as the identity automorphism of the underlying abelian group. Thus, if the ring R is a field, the modules over R (more briefly, R-modules) are simply the vector spaces over R. In this case we will use $[X:R]$ to denote the dimension of X as a vector space over R.

We assume that the reader is acquainted with the notions of tensor product and module with differentiation, and their extensions to the graded case; we also assume that he is acquainted with the torsion functor, $\operatorname{Tor}^R = \sum \oplus \operatorname{Tor}^R_p$. A discussion of these matters, and of most others basic to this paper, can be found in that famed repository, [1].

Our gradings will always be of the form $\sum \oplus_{p=0}^{\infty} X_p$. Recall that if $X = \sum \oplus_{p=0}^{\infty} X_p$ and $Y = \sum \oplus_{q=0}^{\infty} Y_q$ are graded R-modules, then the isomorphism expressing the commutativity of the tensor product, $X \otimes_R Y$, is given by $x \otimes y \to (-1)^{pq} y \otimes x$, where $x \in X_p$ and $y \in Y_q$. Our differentiations will all be of degree -1; i.e., $d(X_p) \subseteq X_{p-1}$ for all p. Recall that any module X can be trivially graded (set $X_0 = X$ and $X_p = 0$ for $p > 0$), and that any module X can be trivially made into a module with differentiation (set $d = 0$). Notice that the trivial structures are compatible; indeed, if the grading is trivial, the differentiation must be. If a module comes to us without grading or differentiation, we will always assume it equipped with the trivial grading and differentiation.

1.1 DEFINITION. An *algebra* is a module X together with a module homomorphism $m: X \otimes_R X \to X$ called the "multiplication." It is said to be *graded* if X is a graded module and m is of degree 0 (i.e., $m((X \otimes_R X)_n) \subseteq X_n$). X is said to have a *unit element* if there is an element $1 \in X$ with $m(x \otimes 1) = m(1 \otimes x) = x$ for all $x \in X$. X is said to be *associative* if the following diagram commutes:

$$\begin{array}{ccc} X \otimes_R (X \otimes_R X) & \approx & (X \otimes_R X) \otimes_R X \\ & {\scriptstyle m(1_X \otimes m)} \searrow \quad \swarrow {\scriptstyle m(m \otimes 1_X)} & \\ & X & \end{array}$$

where the isomorphism (expressed by the symbol $\approx$) is the natural one giving the associativity of the tensor product, 1_X is the identity isomorphism of X, and juxtaposition indicates the composing of maps. X is said to be *commutative* if the following diagram commutes:

$$\begin{array}{ccc} X \otimes_R X & \approx & X \otimes_R X \\ & {\scriptstyle m(1_X \otimes m)} \searrow \quad \swarrow {\scriptstyle m(m \otimes 1_X)} & \\ & X & \end{array}$$

where the isomorphism is the natural one giving the commutativity of the tensor product.

Observe that if X has a unit element it is unique, and if X is graded it must be of degree 0; that the usual axioms for algebras can be captured by interpreting the fact that m is a module homomorphism; and that when X is graded, the commutativity involves a sign, due to the sign in the isomorphism giving the commutativity of the tensor product of modules. Throughout we will use the word "algebra" to mean a nontrivial, associative, commutative algebra with a unit element.

We demand of an *algebra homomorphism*, $f:X \to Y$, that it be a module homomorphism, and that, furthermore, the following diagram commute:

$$\begin{array}{ccc} X \otimes_R X & \xrightarrow{m_X} & X \\ {\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ Y \otimes_R Y & \xrightarrow[m_Y]{} & Y. \end{array}$$

Since all our algebras will have unit elements, we require also that $f(1) = 1$.

A *subalgebra*, X', of X is a submodule with $m(x' \otimes x'') \in X'$ whenever x' and $x'' \in X'$. The multiplication $m':X' \otimes_R X' \to X'$ is given by $m' = m(i \otimes i)$ where $i:X' \to X$ is the injection map. Clearly i becomes an algebra homomorphism. A submodule X' of X is an *ideal* if $m(x' \otimes x'') \in X'$ whenever x' or $x'' \in X'$. (Actually, since we are assuming that X is commutative, it would be enough to say "whenever $x' \in X'$.") If X' is an ideal, then X/X', the factor algebra, has as its module the factor module X/X', and as its multiplication the unique module homomorphism

$$\bar{m}:(X/X') \otimes_R (X/X') \to X/X'$$

which satisfies the commutativity relation:

$$\begin{array}{ccc} X \otimes_R X & \xrightarrow{m} & X \\ {\scriptstyle j\otimes j}\downarrow & & \downarrow{\scriptstyle j} \\ (X/X') \otimes_R (X/X') & \xrightarrow[\bar{m}]{} & X/X' \end{array}$$

where j is the natural map of X onto X/X'. Thus this natural map becomes an algebra homomorphism.

Suppose that X and Y are algebras. We want to make $X \otimes_R Y$ into an algebra in a natural way. Let m_X and m_Y be the multiplications in X and Y respectively, and let $T:Y \otimes_R X \to X \otimes_R Y$ be the isomorphism expressing the commutativity of the module tensor product. Then

$$m = (m_X \otimes m_Y)(1_X \otimes T \otimes 1_Y)$$

is a module homomorphism of $(X \otimes_R Y) \otimes_R (X \otimes_R Y) \to X \otimes_R Y$ and furnishes $X \otimes_R Y$ with the unique algebra structure such that

$$m((x \otimes 1) \otimes (1 \otimes y)) = x \otimes y$$

and the maps $x \to x \otimes 1$ and $y \to 1 \otimes y$ are algebra homomorphisms. In the usual notation we have that $(x \otimes y_n)(x_m \otimes y) = (-1)^{mn} x x_m \otimes y_n y$, where $x_m \in X_m$ and $y_n \in Y_n$.

1.2 DEFINITION. A *coalgebra* is a module X together with a module homomorphism $M: X \to X \otimes_R X$ called the "comultiplication." It is said to be *graded* if X is a graded module and M is of degree 0.

The notions of associativity and commutativity are similar to those for algebras; we will make no use of them. The definition of the tensor product is also similar; again, we will make no use of it.

We will want to use simultaneously the notions of graded algebra and module with differentiation, also the notions of graded algebra and coalgebra. We make them compatible as follows:

If a graded algebra X has as its graded module a complex (more accurately, "left complex," i.e., a graded module with differentiation with the restrictions on the grading and differentiation as above) with d satisfying the commutativity relation

$$\begin{array}{ccc} X \otimes_R X & \xrightarrow{m} & X \\ {\scriptstyle d_{X \otimes_R X}} \downarrow & & \downarrow {\scriptstyle d} \\ X \otimes_R X & \xrightarrow[m]{} & X \end{array}$$

where $d_{X \otimes_R X} = d \otimes 1_X + 1_X \otimes d$, then we call it a *differential graded algebra*

We have, thus, a complex

$$\cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \xrightarrow{d_1} X_0 \to 0$$

with an associative and commutative multiplication satisfying

$$d(x_m x_n') = (dx_m)x_n' + (-1)^m x_m(dx_n').$$

Observe that if R contains an inverse of 2, then for m odd we have from the commutativity that $x_m^2 = 0$. We want this property for all our differential graded algebras and hence demand it. They will then be *strictly commutative.* We will, in addition, demand that the complex be finite (i.e., that each X_m be finitely generated) and, in particular, that X_0 be generated by the unit element of the algebra (connectivity). Thus, our differential graded algebras are the objects called R-algebras in [3].

One checks easily that if X and Y are differential graded algebras, and if $X \otimes_R Y$ is equipped with the grading, multiplication, and differentiation of the tensor product, then $X \otimes_R Y$ is a differential graded algebra. Observe that R, in fact any factor ring of R, trivially equipped, is a differential graded algebra.

Observe also that if X is a differential graded algebra, then $Z(X)$ is a graded subalgebra, and $B(X)$ is a graded ideal of $Z(X)$. Thus, $H(X)$ is a graded algebra. It inherits the associativity and strict commutativity of X and possesses a unit element (since $Z_0(X) = X_0$). And it is finite (i.e., $H_m(X)$ is finitely generated for every m). Moreover, algebra homomorphisms which commute with d induce algebra homomorphisms when passing to homology.

If we have a graded algebra which is also a coalgebra, then we require that the coalgebra map be a graded algebra homomorphism. If, in addition, X is a finite complex with X_0 generated by the unit element of the algebra, and, for $x_m \in X_m$ with $m > 0$, the projections of $M(x_m)$ into $X_m \otimes_R X_0$ and $X_0 \otimes_R X_m$ are, respectively,

$$(*) \qquad x_m \otimes 1 \quad \text{and} \quad 1 \otimes x_m ,$$

then we call X a *Hopf Algebra.*

We will be concerned with a situation in which a differential graded algebra X yields a homology algebra $H(X)$ which is a Hopf Algebra. In this case we will, in order to prove that $H(X)$ is a Hopf Algebra, merely have to construct an algebra homomorphism $M: H(X) \to H(X) \otimes_R H(X)$ satisfying (*) since the other requirements for $H(X)$ are inherited from the corresponding properties of X.

Let X and Y be complexes. Then there is a canonical homomorphism $\alpha: H(X) \otimes_R H(Y) \to H(X \otimes_R Y)$ defined as follows: For $A \in H(X)$ and $B \in H(Y)$, pick representatives x and y. Then $x \otimes y$ is an element of $Z(X \otimes_R Y)$, and the homology class of $x \otimes y$ does not depend on the choice of x and y. Thus, $\alpha(A \otimes B) =$ the homology class of $x \otimes y$ defines α. It is a module homomorphism; moreover, if X and Y are differential graded algebras, it is easy to see that α is an algebra homomorphism. One can compute directly, or derive as an immediate consequence of Theorem IV.7.2. of [1], that when R is a field, α is an isomorphism.

We are interested in $\mathrm{Tor}^R(K, L)$ where K and L are factor rings of R viewed as R-modules. It is, of course, an R-module (since R is commutative), but only to the extent that it is an $R/(M + N)$-module, where M and N are, respectively, the kernels of the natural maps of R onto K and L. To see this simply note that $M(K \otimes_R Y) = 0$ and $N(X \otimes_R L) = 0$; thus, M and N annihilate the homology of $K \otimes_R Y$ and $X \otimes_R L$ which means that $M + N$ annihilates $\mathrm{Tor}^R(K, L)$.

Now, $\mathrm{Tor}^R(K, L)$ is naturally an algebra over $R/(M + N)$. To see this let X be a projective resolution of K. Then $X \otimes_R X$ is a projective complex over $K \otimes_R K \approx K$. Hence there is a map of $X \otimes_R X \to X$ over the isomorphism $K \otimes_R K \to K$. This yields a map of $(X \otimes_R X) \otimes_R (L \otimes_R L) \to X \otimes_R L$ and thus a map of $(X \otimes_R L) \otimes_R (X \otimes_R L) \to X \otimes_R L$. Passing to homology we have a map of $H((X \otimes_R L) \otimes_R (X \otimes_R L)) \to H(X \otimes_R L)$ and applying α, we obtain the module homomorphism

$$H(X \otimes_R L) \otimes_R H(X \otimes_R L) \to H(X \otimes_R L)$$

which furnishes $\mathrm{Tor}^R(K, L)$ with a multiplication. (Observe that the middle tensor product on the left might just as well be over $R/(M + N)$.) It is the "$\pitchfork$-product" of [1]. Notice that the only "nonexplicit" map in the above construction is that of $X \otimes_R X \to X$ over $K \otimes_R K \to K$. Tate has shown [3] that there exist resolutions X of K for which this map can be made associative and strictly commutative, that is, resolutions which are themselves differential graded, strictly commutative algebras. This is an efficient way of seeing that $\pitchfork$ is associative and strictly commutative.

Now assume that $M = N$ and that M is a maximal ideal. $K = R/M$ is a field, and we can describe a canonical coalgebra structure for $T = \mathrm{Tor}^R(K, K)$, which, by the above, is an algebra over K.

Let X be a projective resolution of K, $\varepsilon : X \to K$ the "augmentation." Since $X \otimes_R K \approx X/MX \approx K \otimes_R X$, we have the natural map of $X \otimes_R X \to (X \otimes_R K) \otimes_K (K \otimes_R X)$. (Notice that the middle tensor product on the right is taken over K, permissible since as R-modules $X \otimes_R K$ and $K \otimes_R X$ are merely K-modules.) Passing to homology we have a map $H(X \otimes_R X) \to H((X \otimes_R K) \otimes_K (K \otimes_R X))$. Since K is a field,

$$\alpha : H(X \otimes_R K) \otimes_K H(K \otimes_R X) \to H((X \otimes_R K) \otimes_K (K \otimes_R X))$$

is an isomorphism, and hence applying α^{-1} yields a map

$$H(X \otimes_R X) \to H(X \otimes_R K) \otimes_K H(K \otimes_R X).$$

Using the canonical isomorphisms $(1 \otimes \varepsilon)_*^{-1}$ and $(\varepsilon \otimes 1)_*^{-1}$ we have a map $T \to T \otimes_K T$, which equips T with a coalgebra structure.

Observe that if X is a differential graded algebra resolution, then, since all the maps used in the above are algebra homomorphisms, the canonical coalgebra structure of T is compatible with the canonical algebra structure. Even more is true. We have

1.3 THEOREM. *Let R be a local ring, M the maximal ideal, and set $K = R/M$. Then*

$$T = \mathrm{Tor}^R(K, K)$$

is a Hopf Algebra over K.

Proof. What remains to be shown is that for $\tau \in T_n$, $n > 0$, the image of τ under the comultiplication is of the form

$$1 \otimes \tau + \cdots + \tau \otimes 1,$$

where the "middle terms" are contained in

$$\textstyle\sum \otimes_{i=1}^{n-1} T_i \otimes_K T_{n-i}\,.$$

Now, let X be a minimal projective resolution of K. (X is said to be minimal if $dX \subseteq MX$. Minimal resolutions are extremely advantageous since the differentiation in $X \otimes_R K$ is zero and hence $H(X \otimes_R K) = X \otimes_R K$; thus knowing X means knowing $\text{Tor}^R(K, K)$. That minimal projective resolutions always exist is an immediate consequence of the fact that if A is any R-module with a finite minimal base $\{x_1, \cdots, x_n\}$—in the sense that no x_i can be expressed as a linear combination of the remaining x's—then $\sum_i r_i x_i = 0$ implies that $r_i \in M$ for all i, this last fact being Nakayama's Lemma. It is, of course, Axiom 5 of [9], where minimal resolutions are discussed in a more general setting.) Suppose that $\tau \in T = H(X \otimes_R X)$ is represented by

$$t = x_0^{(0)} \otimes y_0^{(n)} + \cdots + x_n^{(n)} \otimes y_n^{(0)},$$

the superscripts indicating the degree of the element and the "middle terms" being contained in

$$\sum \oplus_{i=1}^{n-1} X_i \otimes_R X_{n-i}\,.$$

The first step in computing the image of τ is to take t into

$$(x_0^{(0)} \otimes 1) \otimes (1 \otimes y_0^{(n)}) + \cdots + (x_n^{(n)} \otimes 1) \otimes (1 \otimes y_n^{(0)}).$$

Because of the minimality each term in the sum is a homology class. Thus, we must show that the first term is $(1 \otimes 1) \otimes ((\varepsilon \otimes 1)_*(\tau))$ and the last $((1 \otimes \varepsilon)_*(\tau)) \otimes (1 \otimes 1)$. But $(\varepsilon \otimes 1)_*(\tau)$ and $(1 \otimes \varepsilon)_*(\tau)$ are, respectively, $\varepsilon(x_0^{(0)}) \otimes y_0^{(n)}$ and $x_n^{(n)} \otimes \varepsilon(y_n^{(0)})$, and therefore we have the assertion, since $\varepsilon(x_0^{(0)})$ and $\varepsilon(y_n^{(0)})$ can pass through the tensor signs, dropping the ε in transit. It needs only to be remarked that the comultiplication is "natural," i.e., does not depend on the resolution chosen.

As immediate consequences of the above theorem we will prove a result of Serre's [4, Theorem 4] and a result of Tate's [3, Theorem 7]. That we are able to prove these results is merely a reflection of the fact that they are consequences of the Hopf structure of $\text{Tor}^R(K, K)$. Thus, before proving them, we will state and prove the necessary lemmas concerning Hopf Algebras.

1.4 LEMMA. *Suppose that* $T = \sum \oplus_{p=0}^{\infty} T_p$ *is a Hopf Algebra over a field* K. *Then the subalgebra generated by* T_1 *is an exterior algebra over* K *with* $[T_1 : K]$ *generators.*

Proof. Let $x_1, x_2, \cdots, x_n$ be a base for T_1 as K-module. Then the subalgebra generated by T_1 is the subalgebra generated by the x's, and, moreover, it is clear that it is a homomorphic image of the exterior algebra over K on n generators. To show that this homomorphism is, in fact, an isomorphism, we must show that no nontrivial relation of the form

$$(**) \qquad \sum_{i_1 < \cdots < i_r} a_{i_1, \ldots, i_r}(x_{i_1} x_{i_2} \cdots x_{i_r}) = 0$$

exists, where, of course, the a's are in K. We do this by induction on r. For $r = 1$ the assertion follows from the fact that the x's form a base for T_1. Now, suppose given a relation of the form (**). Observe that if M is the comultiplication, then $M(x_i) = x_i \otimes 1 + 1 \otimes x_i$ for all i. Let P be the projection of $T \otimes_K T$ onto $T_1 \otimes_K T_{r-1}$. Applying PM to (**), remembering that M is an algebra homomorphism, we have that

$$\sum_{\substack{i_1 < \cdots < i_r \\ \nu=1}}^{r} a_{i_1,\ldots,i_r}(x_{i_\nu} \otimes x_{i_1} \cdots \hat{x}_{i_\nu} \cdots x_{i_r}) = 0.$$

But the elements of the form $x_{i_\nu} \otimes x_{i_1} \cdots \hat{x}_{i_\nu} \cdots x_{i_r}$ are linearly independent elements of $T_1 \otimes_K T_{r-1}$ (by the induction hypothesis). Hence all the a's are zero.

1.5 Lemma. *Suppose $T = \sum \oplus_{p=0}^{\infty} T_p$ is a Hopf Algebra over a field K. Then, if L is the subalgebra generated by T_1, T can be viewed as a graded module over the graded algebra L, and, as such, it is free with a homogeneous base.*

Proof. Pick a set $\{v_\alpha\}$ of elements of T which are homogeneous and whose residues form a base for the K-module T/LT. To prove that the v's together with 1 form a free base for T as L-module, it will be enough to show that the elements of the form $x_{i_1} \cdots x_{i_r} v_\alpha$, where $i_1 < \cdots < i_r$ and the v_α may be absent, are linearly independent over K. We do this by induction on the degree. When the degree is equal to 1, the assertion is a consequence of the linear independence over K of the x's. Now suppose that

$$(***) \qquad \sum a_{i_1,\ldots,i_r,\alpha}\, x_{i_1} \cdots x_{i_r} v_\alpha = 0$$

is homogeneous of degree $n > 1$, and that elements of the above form of degree less than n are linearly independent over K. Let $v_{\alpha_1}, \cdots, v_{\alpha_s}$ be the v's (not necessarily distinct) appearing in (***) which are of maximum degree. Now, degree $(v_{\alpha_j}) < n$ because otherwise we have a contradiction of their residues' linear independence over K. We will draw a contradiction by proving that the coefficients of the terms in which the v_{α_j}'s appear are all zero. Apply PM to (***), where P is as in the proof of the above lemma. We have that

$$\sum_{\substack{i_1 < \cdots < i_r \\ \nu=1, j=1}}^{r,s} a_{i_1,\ldots,i_r,\alpha_j}\, x_{i_\nu} \otimes x_{i_1} \cdots \hat{x}_{i_\nu} \cdots x_{i_r} v_{\alpha_j} + \beta = 0,$$

where β is in the subspace of $T_1 \otimes_K T_{n-1}$ generated by elements of the form $x_k \otimes x_{j_1} \cdots x_{j_t} v_\alpha$ where degree $(v_\alpha) <$ degree (v_{α_j}). By the induction hypothesis we can conclude that $a_{i_1,\ldots,i_r,\alpha_j} = 0$ for $j = 1, \cdots, s$.

1.6 Corollary (Serre). *Let R be a local ring and M the maximal ideal. Let $K = R/M$, and set $T = \operatorname{Tor}^R(K, K)$. Then M/M^2 is a finite-dimensional vector space over K, and, if we put $n = [M/M^2 : K]$, we have that*

$$[T_p : K] \geqq \binom{n}{p}$$

for all p. (As is customary, $\binom{n}{p} = 0$ *whenever* $p > n$.)

Proof. By [1, Chapter VI, Exercise 19] we have that T_1 is isomorphic to M/M^2. Thus, $n = [T_1:K]$. By the above theorem T is a Hopf Algebra over K, and the first lemma tells us that T contains a copy of the exterior algebra on n generators; hence the inequality follows.

1.7 COROLLARY (Tate). *Let* R *be a local ring and* M *the maximal ideal. Let* $K = R/M$, *and set* $T = \mathrm{Tor}^R(K, K)$. *Then, if we put* L = *the subalgebra generated by* T_1, T *can be viewed as a graded module over the graded algebra* L, *and, as such, it is free with a homogeneous base.*

Proof. The corollary follows immediately from the theorem and the second lemma.

Remark. The above lemmas display elementary properties of Hopf Algebras. For Hopf Algebras over perfect fields a structure theorem exists; it can be found in [8, p. 137].

2. Homological characterizations of local complete intersections

Here, we will restrict ourselves to local rings which can be obtained as homomorphic images of regular local rings. (The definition of "regular local ring" and, more generally, the elementary facts concerning local rings that we will use are contained in [2].) The notation will be that of [3].

It will be convenient for us to restrict ourselves to those homomorphic images $R = R'/A'$ in which $A' \subseteq M'^2$, M' being the maximal ideal of the regular local ring R'. That this is no loss of generality can be seen as follows: If $A' \nsubseteq M'^2$, choose $a' \in A'$, $a' \notin M'^2$. Form $R'' = R'/(a')$. R'' is a regular local ring, and $R = R''/A''$ where A'' is the image of A' in R'' under the natural map. If $A'' \nsubseteq M''^2$, we repeat this process, and because R' is noetherian we must, after a finite number of steps, be able to express R as the homomorphic image of a regular local ring with the kernel in the square of the maximal ideal.

2.1 DEFINITION. If R is a local ring and $a_1, a_2, \cdots, a_s \in M$, we say that $a_1, a_2, \cdots, a_s$ is an *R-sequence* if a_1 is not a zero divisor in R and a_i is not a zero divisor modulo $(a_1, a_2, \cdots, a_{i-1})$ for $2 \leqq i \leqq s$.

The above definition is a special case of the "E-sequences," where E is an R-module, of [4], [6], and [7].

Since the Krull dimension of $R/(a)$ is one less than the Krull dimension of R when $a \in M$ is not a zero divisor of R, we have that s is less than or equal to the Krull dimension of R for any R-sequence $a_1, a_2, \cdots, a_s$.

2.2 DEFINITION. The *codimension* of R, denoted by codim R, is the number of elements in the "longest" R-sequence. Codim $R = 0$ will mean that every element of M is a zero divisor.

If we denote the Krull dimension of R by dim R, the above remark shows that codim $R \leqq$ dim R for every local ring.

In [7], where Auslander and Buchsbaum exploit the so-called "Koszul complexes," there appear, as special cases of more general results, characterizations of R-sequences and codim R. In the notation of [3] we have, in fact, that $a_1, a_2, \cdots, a_s$ is an R-sequence if and only if

$$H_1(R\langle T_1, T_2, \cdots, T_s\rangle; dT_i = a_i) = 0.$$

And the codimension of R can be computed as follows: Set

$$E = R\langle T_1, T_2, \cdots, T_n\rangle; dT_i = t_i$$

where $t_1, t_2, \cdots, t_n$ minimally generate M; then codim $R = n - q$ if and only if q is the greatest integer such that $H_q(E) \neq 0$ and $H_{q+1}(E) = 0$.

This characterization of R-sequences means that they are merely sets; i.e., if $a_1, a_2, \cdots, a_s$ is an R-sequence, then so is $a_{p(1)}, a_{p(2)}, \cdots, a_{p(s)}$ for any permutation p of $1, 2, \cdots, s$. In fact, if $a_1, a_2, \cdots, a_s$ and $a_1', a_2', \cdots, a_s'$ are minimal generating systems for an ideal A, then $R\langle T_1, T_2, \cdots, T_s\rangle; dT_i = a_i$ and $R\langle T_1', T_2', \cdots, T_s'\rangle; dT_i' = a_i'$ are isomorphic, and hence if the R-sequence $a_1, a_2, \cdots, a_s$ generates the ideal A, then any minimal generating system of A is an R-sequence. These facts for regular local rings are consequences of the homological characterizations of local complete intersections given below.

We proceed now to the main business of this section, the homological characterization of local complete intersections. For completeness we include their definition.

2.3 Definition. $R = R'/A'$ is called a *local complete intersection* if R' is a regular local ring and A' can be generated by an R'-sequence.

Now, let $R = R'/A'$ be a local ring with maximal ideal M, where R' is regular and $A' \subseteq M'^2$, M' being the maximal ideal of R'. Let us choose $t_1, t_2, \cdots, t_n$ as a minimal generating system for M, forming

$$E = R\langle T_1, T_2, \cdots, T_n\rangle; dT_i = t_i .$$

E is an invariant of R as a differential graded algebra [3, §6], and so is $H(E)$. It follows easily from elementary results in [2] that $t_1, t_2, \cdots, t_n$ is an R-sequence if and only if R is a regular local ring, and the Auslander-Buchsbaum characterization of R-sequences gives immediately Eilenberg's characterization of regular local rings: R is a regular local ring if and only if $H_1(E) = 0$. Thus, $H(E)$ is rich enough as an invariant of R to give the codimension of R and to give a criterion for regularity. The theorem below shows that $H(E)$ is capable of distinguishing local complete intersections.

Suppose that $[A'/M'A':K] = m$. It is easy to see that $[H_1(E):K] = m$; in fact, if $A' = (a_1', a_2', \cdots, a_m')$ and we write $a_i' = \sum_{j=1}^{n} c_{ij}' t_j'$, where t_j' is an element of M' whose image under the natural map is t_j, then

$$s_i = \sum_{j=1}^{n} c_{ij} T_j, \qquad 1 \leqq i \leqq m,$$

are a minimal generating system for $Z_1(E)$ modulo $B_1(E)$, the c_{ij} being the images under the natural map of the c'_{ij}.

Construct $F = E\langle S_1, S_2, \cdots, S_m\rangle; dS_i = s_i$. It is slightly more tedious to verify, but F, like E, is an invariant of R as a differential graded algebra, and hence so is $H(F)$. We will give characterizations of local complete intersections in terms of $H(E)$ and $H(F)$. Those in terms of $H(E)$ will be analogous to Eilenberg's characterization of regular local rings mentioned above. Of course, not merely $H_1(E)$ but also $H_2(E)$ will have to play a role in the characterization. In order to examine the connection between $H(E)$ and $H(F)$ we prove a preliminary lemma.

2.4 Lemma. *Suppose X is a differential graded algebra with $H_0(X) = K$ and s represents σ, a nonzero homology class of degree 1. Then, setting*

$$Y = X\langle S\rangle; dS = s,$$

we have that $i_{2,*}:H_2(X) \to H_2(Y)$ is a surjection with kernel $\sigma H_1(X)$, where $i_2:X_2 \to Y_2$ is the injection map. Moreover, if $H_1(Y) = 0$, $i_{2,*}$ is an isomorphism.

Proof. Consider the exact sequence $0 \to X \xrightarrow{i} Y \xrightarrow{j} Y \to 0$, where i is the injection map and j is defined by

$$j(x_0 + x_1 S + x_2 S^{(2)} + \cdots) = x_1 + x_2 S + \cdots.$$

i and j commute with d, and we get the exact homology triangle and, in particular, the exact sequence

$$H_1(Y) \xrightarrow{d_*} H_2(X) \xrightarrow{i_{2,*}} H_2(Y) \to K \xrightarrow{\beta} H_1(X) \to H_1(Y) \to 0.$$

Since σ is a nonzero homology class, β is an injection, and hence $i_{2,*}$ is a surjection with kernel $d_*(H_1(Y))$. Clearly, if $H_1(Y) = 0$, then $i_{2,*}$ is an isomorphism. About the kernel, let $\xi \in H_1(Y)$ and suppose $x \in X_1$ represents ξ. (Observe that $X_1 = Y_1$.) Computing $d_*(\xi)$ we find that it is represented by xs and hence is equal to $\xi\sigma$. Therefore the kernel of $i_{2,*}$ is $\sigma H_1(X)$.

2.5 Proposition. *The map $i_{2,*}:H_2(E) \to H_2(F)$ is a surjection with kernel $(H_1(E))^2$, $i_2:E_2 \to F_2$ being the injection map.*

Proof. We get from E to F by a series of m steps, $X^1, X^2, \cdots, X^m = F$, where $X^i = X^{i-1}\langle S_i\rangle; dS_i = s_i$, $s_1, s_2, \cdots, s_m$ being a minimal generating system for $Z_1(E)$ modulo $B_1(E)$. The $i_{2,*}$ of the proposition is merely the composition of the $i^j_{2,*}$, $1 \leqq j \leqq m$. The lemma applies to each of these. Hence $i_{2,*}$ is a surjection with kernel

$$\sigma_1 \sum_{j=2}^{m} K\sigma_j + \sigma_2 \sum_{j=3}^{m} K\sigma_j + \cdots + K\sigma_{m-1}\sigma_m$$

or $(H_1(E))^2$.

2.6 LEMMA. *Let X be a differential graded algebra with $H_0(X) = K$ and $H_1(X) = 0$. If a is a zero divisor of R, then $X' = X/aX$ is such that $H_2(X') \neq 0$.*

Proof. Set $I = \{x \in R \mid ax = 0\}$. Since a is a zero divisor, $I \neq 0$. Hence $H_1(X/IX) \neq 0$. Consider the exact sequence

$$0 \to X/IX \xrightarrow{a} X \xrightarrow{j} X' \to 0,$$

where "a" is multiplication by a and j is the natural map. a and j both commute with d, and therefore we obtain the exact homology triangle and, in particular, the exact sequence $H_2(X') \to H_1(X/IX) \to H_1(X) = 0$ which yields the result.

We are now in a position to prove the main result of this section.

2.7 THEOREM. *Let $R = R'/A'$, where R' is a regular local ring and $A' \subseteq M'^2$. Let E and F be as above. Then the following are equivalent:*

(a) *R is a local complete intersection,*
(b) *$H(E)$ is the exterior algebra on $H_1(E)$,*
(c) *$H_2(E) = (H_1(E))^2$,*
(d) *$H_2(F) = 0$.*

Proof. (a) $\Rightarrow$ (b) is an immediate consequence of Theorem 6 of [3]. (b) $\Rightarrow$ (c)is obvious. (c) $\Rightarrow$ (d) is an immediate consequence of the proposition above. Hence the theorem will be proved once we show that (d) $\Rightarrow$ (a). Let

$$E' = R'\langle T_1', T_2', \cdots, T_n'\rangle; dT_i' = t_i',$$

where $t_1', t_2', \cdots, t_n'$ minimally generate M'. Then $H_1(E') = 0$ since R' is regular (and, of course, $H_0(E') = K$). Let $A' = (a_1', a_2', \cdots, a_m')$ where $m = [A'/M'A':K]$. Assume that $a_1', a_2', \cdots, a_m'$ is not an R'-sequence. Let i_0 be the maximal i such that a_i' is a zero divisor modulo $(a_1', a_2', \cdots, a_{i-1}')$. Form

$$X' = (E'/(a_1', a_2', \cdots, a_{i_0-1}')E')\langle S_1', S_2', \cdots, S_{i_0-1}'\rangle; dS_i' = s_i'$$

where $s_i' = \sum_{j=1}^n c_{ij} T_j$, $a_i' = \sum_{j=1}^n c_{ij}' t_j'$, the c's being the $(a_1', a_2', \cdots, a_{i_0-1}')$-residues of the (c')'s, and similarly for the T's. Forming $X'' = X'/a_{i_0}' X'$ we have by the lemma of 2.6 that $H_2(X'') \neq 0$. Now, $H_1(X'')$ is generated by σ_{i_0}, represented by $s_{i_0}' = \sum_{j=1}^n c_{i_0 j}' T_j$. Consequently $X''' = X''\langle S_{i_0}'\rangle; dS_{i_0}' = s_{i_0}'$ is such that $H_1(X''') = 0$, and hence by the lemma of 2.4, $H_2(X''') = H_2(X'') \neq 0$. But now $a_{i_0+1}', \cdots, a_m'$ is an $R'/(a_1', \cdots, a_{i_0}')$-sequence, and we can apply Theorem 3 of [3] and the lemma of 2.4, successively factoring by a_i', $i > i_0$, and then killing the resulting 1-cycle until we achieve F. We will have that $H_2(F) = H_2(X''') \neq 0$, thus proving the assertion.

2.8 COROLLARY. *Every local ring R with $[M/M^2:K] - 1 \leqq$ codim R is a local complete intersection. Moreover, if the ideal A' of R', the regular local ring of which R is a homomorphic image, is taken to be in M'^2, it is principal.*

Proof. Form E. The requirement assures us that $H_2(E) = 0$. Hence clearly $H_2(E) = (H_1(E))^2$, and part (c) of the theorem yields the first assertion. The second follows from the fact that codim $R \leqq \dim R$, since if $A' \subseteq M'^2$, then $[M/M^2:K] = \dim R'$, and because $\dim R$ equals $\dim R'$ less the number of generators of A', this number must be less than or equal to 1.

2.9 COROLLARY. *Every regular local ring of dimension two is a unique factorization domain.*

Proof. Let R' be a regular local ring of dimension two. It is enough to prove that every minimal prime ideal of R' which is contained in M'^2 is principal. But for any such prime ideal P', R'/P' is an integral domain, and M'/P' is not zero. Thus codim $R'/P' \geqq 1$, and the above corollary yields the result.

Remark. Since $H(E)$, the homology of the Koszul complex of a local ring R, is rich enough as a homological invariant to yield the codimension of R, to give a criterion for regularity, and to distinguish local complete intersections, one might naturally ask whether it could be used to compute the Krull dimension of R. The answer is no; i.e., there are local rings R and S with differing Krull dimension with the property that $H(E_R)$ is isomorphic (as an algebra) to $H(E_S)$.

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