UNIQUENESS OF THE INVARIANT MEAN ON AN ABELIAN SEMIGROUP

BY

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I. Introduction

A semigroup is said to have an invariant mean if there exists a positive linear functional of norm one on the space of all bounded real-valued functions on the semigroup which is invariant under left and right translation operators. It is known that every abelian semigroup has an invariant mean. It is the object of this paper to prove that an abelian semigroup has a unique invariant mean if and only if the semigroup has a finite ideal in it.

We first consider finitely generated abelian semigroups and prove the equivalence of the following three conditions:

(1) The semigroup has a unique invariant mean.

(2) The semigroup has a finite ideal in it.

(3) Every homomorphism of the semigroup into the integers is trivial.

Using this theorem we give a proof of our main result.

II. Definitions and notation

Let Σ be a semigroup, and let $m(\Sigma)$ be the space of bounded real-valued functions x on Σ with $\| x \| = \sup_\sigma | x(\sigma) |$. For each σ in Σ we define the *left translation operator* l_σ carrying $m(\Sigma)$ into itself by $(l_\sigma x)(\tau) = x(\sigma\tau)$, for each x in $m(\Sigma)$ and τ in Σ. Similarly we define the *right translation operator* r_σ by $(r_\sigma x)(\tau) = x(\tau\sigma)$. An element ν in $m(\Sigma)^*$ is called a *mean* if $\| \nu \| = 1 = \nu(e)$, where e is the function which is identically one. An element ν in $m(\Sigma)^*$ is called *left-[right-]invariant* if $l_\sigma^* \nu = \nu$ $[r_\sigma^* \nu = \nu]$ for all σ in Σ. We say that ν is *invariant* if $l_\sigma^* \nu = \nu = r_\sigma^* \nu$ for all σ in Σ. A semigroup is called *amenable* if there exists an invariant mean. It is known that a solvable semigroup is amenable [4].

Let $l_1(\Sigma)$ be the space of all real-valued functions φ on Σ such that $\| \varphi \| = \sum_\sigma | \varphi(\sigma) |$ is finite. An element φ of $l_1(\Sigma)$ is called a finite mean if (i) $\varphi(\sigma) \geqq 0$ for all σ in Σ, (ii) $\{\sigma : \varphi(\sigma) > 0\}$ is a finite subset of Σ, and (iii) $\sum_\sigma \varphi(\sigma) = 1$. We can regard $l_1(\Sigma)$ as a subspace of $m(\Sigma)^*$, but to be more precise we define the mapping Q from $l_1(\Sigma)$ into $m(\Sigma)^*$ by

$$(Q\varphi)(x) = \sum_\sigma \varphi(\sigma)x(\sigma), \qquad x \in m(\Sigma), \quad \varphi \in l_1(\Sigma).$$

The mapping Q preserves all the structure of $l_1(\Sigma)$. If φ and θ are elements

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of $l_1(\Sigma)$, we define their product $\varphi\theta$ by

$$(\varphi\theta)\ (\sigma) = \sum_{\sigma_1\sigma_2=\sigma} \varphi(\sigma_1)\theta(\sigma_2).$$

Equipped with this multiplication $l_1(\Sigma)$ becomes a Banach algebra [5].

For σ in Σ we define $\hat{\sigma}$ in $l_1(\Sigma)$ by

$$\hat{\sigma}(\tau) = \begin{cases} 1 & \text{if } \tau = \sigma \\ 0 & \text{if } \tau \neq \sigma. \end{cases}$$

We can then easily see that the mapping $\sigma \to \hat{\sigma}$ is an isomorphism of the semigroup Σ into the multiplicative semigroup of $l_1(\Sigma)$, so that Σ can be considered as a subsemigroup of the multiplicative semigroup of $l_1(\Sigma)$. We will not cause any confusion if we just write σ for $\hat{\sigma}$.

A net of means μ_λ is said to *converge weakly** (*strongly*) *to left* [*right*] *invariance* if for any σ in Σ, $(l_\sigma^* \mu_\lambda - \mu_\lambda)$ $[(r_\sigma^* \mu_\lambda - \mu_\lambda)]$ converges weakly* (strongly) to zero. We say that a net of means μ_λ *converges weakly** (*strongly*) *to invariance* if it converges weakly* (strongly) to left and right invariance. A net of finite means φ_λ is said to converge to invariance if the net $Q\varphi_\lambda$ converges to invariance. It is known that a net of finite means φ_λ converges weakly* [strongly] to invariance if for each σ in Σ

$$w\text{-}\lim_\lambda (\varphi_\lambda \sigma - \varphi_\lambda) = 0 = w\text{-}\lim_\lambda (\sigma\varphi_\lambda - \varphi_\lambda)$$

$$[\lim_\lambda \| \varphi_\lambda \sigma - \varphi_\lambda \| = 0 = \lim_\lambda \| \sigma\varphi_\lambda - \varphi_\lambda \|].$$

It was recently proved by Day [5] that the following three conditions are equivalent for a semigroup:

(i) There is an invariant mean in $m(\Sigma)^*$,

(ii) There exists a net of finite means converging weakly to invariance,

(iii) There exists a net of finite means converging strongly to invariance.

III. Abelian groups

In this section we shall prove that an abelian group G has a unique invariant mean if and only if G is finite. This result has already been proved by Day [5]. Actually we shall prove a little more than what is contained in this statement.

LEMMA 0. *If an abelian semigroup Σ has a finite ideal in it, then Σ has a unique invariant mean.*

Proof. Since Σ has a finite ideal in it, we can find a finite minimal ideal A in Σ. For each x in $m(\Sigma)$ we let

$$\mu(x) = [\sum_{\sigma \in A} x(\sigma)]/N(A),$$

where $N(A)$ denotes the number of elements in A. It can easily be checked that μ is a mean. Suppose now that τ is any element of Σ. Then τA is an ideal in Σ and being contained in A must be equal to A. Thus:

$$\mu(l_\tau x) = [\textstyle\sum_{\sigma\epsilon A} x(\tau\sigma)]/N(A) = [\sum_{\rho\epsilon A} x(\rho)]/N(A) = \mu(x).$$

This shows that μ is an invariant mean. Let ν be any invariant mean. Let z be a point in $m(\Sigma)$ such that z vanishes on A. If $\sigma \epsilon A$, then $(l_\sigma z)(\tau) = z(\sigma\tau) = 0$ since $\sigma\tau \epsilon A$. Consequently $l_\sigma z = 0$ and so $\nu(z) = \nu(l_\sigma z) = \nu(0) = 0$. Next let e^* in $m(\Sigma)$ be defined by

$$e^*(\sigma) = \begin{cases} 1 & \text{if} \quad \sigma \epsilon A \\ 0 & \text{if} \quad \sigma \not\epsilon A. \end{cases}$$

Then $e - e^*$ vanishes on A, and so $\nu(e - e^*) = 0$. Thus $\nu(e^*) = \nu(e) = 1$. If $\tilde{\sigma}$ is the function which takes the value 1 at σ and zero elsewhere, then by invariance of ν, $\nu(\tilde{\sigma}) = \nu(\tilde{\tau})$ for σ and τ in A. But $e^* = \sum_{\sigma\epsilon A} \tilde{\sigma}$, and therefore $\sum_{\sigma\epsilon A} \nu(\tilde{\sigma}) = \nu(e^*) = 1$. Thus for each σ in A, $\nu(\tilde{\sigma}) = 1/N(A)$. Now let x be any point in $m(\Sigma)$. We can write

$$x = \textstyle\sum_{\sigma\epsilon A} x(\sigma)\tilde{\sigma} + z,$$

where z vanishes on A. Consequently

$$\begin{aligned} \nu(x) &= \textstyle\sum_{\sigma\epsilon A} x(\sigma)\nu(\tilde{\sigma}) + \nu(z) = \sum_{\sigma\epsilon A} x(\sigma)\nu(\tilde{\sigma}) \\ &= [\textstyle\sum_{\sigma\epsilon A} x(\sigma)]/N(A) = \mu(x). \end{aligned}$$

This shows that $\nu = \mu$, and thus our assertion is proved.

We will now state some general results which will be used in later sections. For the proofs see [5].

LEMMA 1. *Let Σ be any semigroup, and let $\{\mu_n\}$ be a net of means w^*-convergent to left invariance. Then every w^*-cluster point of the net $\{\mu_n\}$ is a left-invariant mean* [5, p. 520].

LEMMA 2. *If $\{\mu_n\}$ is a net of means on $m(\Sigma)$ which is not w^*-convergent but is w^*-convergent to left invariance, then there is more than one left-invariant mean* [5, p. 531].

THEOREM 1. *Let Σ be a left-amenable semigroup, and let f be a homomorphism of Σ onto another semigroup Σ'. Let F be the mapping defined from $m(\Sigma')$ into $m(\Sigma)$ by*

$$(Fx')(\sigma) = x'(f\sigma), \qquad x' \epsilon m(\Sigma'), \quad \sigma \epsilon \Sigma.$$

Then F^* carries the set of left-invariant means on $m(\Sigma)$ onto the set of left-invariant means on $m(\Sigma')$ [5, p. 531].

Using Corollary 1 on page 534 of [5] we obtain the following result:

THEOREM 2. *If G is a left-amenable group and H is a subgroup of G such that the diameter of the set of left-invariant means on $M(H)$ is two, then the diameter of the set of left-invariant means on $m(G)$ is also two.*

THEOREM 3. *If a left-amenable group G has either a subgroup or a factor group with more than one left-invariant mean, then G has more than one left-invariant mean* [5, p. 534].

LEMMA 3. *Let H be an infinite group which can be expressed as the union of an ascending sequence of finite subgroups. Then H is amenable, and the diameter of the set of invariant means is two.*

Proof. Let $H = \bigcup_{n=1}^{\infty} H_n$,where H_n is an ascending sequence of finite subgroups of H. Let $N(H_n)$ denote the number of elements in H_n. We can clearly assume, without any loss of generality, that $N(H_n) \geqq 10^n N(H_{n-1})$. Define x in $m(H)$ by

$$x(h) = \begin{cases} 1 & \text{if} \quad h \in H_{2n+1} - H_{2n} \\ -1 & \text{if} \quad h \in H_{2n} - H_{2n-1}, \end{cases}$$

and define finite means φ_n in $l_1(H)$ by

$$\varphi_n(h) = \begin{cases} 1/N(H_n) & \text{if} \quad h \in H_n, \\ 0 & \text{if} \quad h \notin H_n. \end{cases}$$

Then the net $\{\varphi_n : n \geqq 1\}$ converges strongly to invariance, for as soon as $h \in H_n$, $\varphi_n h - \varphi_n = 0 = h\varphi_n - \varphi_n$. Moreover,

$$(Q\varphi_n)\,(x) = \textstyle\sum_h \varphi_n(h)x(h) = [\sum_{h \in H_n} x(h)]/N(H_n).$$

Now $\sum_{h \in H_n} x(h)$ is positive if n is odd and is negative if n is even. Also

$$\begin{aligned} N(H_n) &\geqq |\textstyle\sum_{h \in H_n} x(h)| = |\sum_{h \in H_n - H_{n-1}} x(h) + \sum_{h \in H_{n-1}} x(h)| \\ &\geqq \textstyle\sum_{h \in H_n - H_{n-1}} |x(h)| - |\sum_{h \in H_{n-1}} x(h)| \geqq N(H_n - H_{n-1}) - N(H_{n-1}) \\ &= N(H_n) - 2N(H_{n-1}). \end{aligned}$$

Thus

$$1 - 2 \cdot 10^{-n} \leqq |\textstyle\sum_{h \in H_n} x(h)|/N(H_n) \leqq 1,$$

and consequently

$$\limsup_n (Q\varphi_n)\,(x) = 1, \qquad \liminf_n (Q\varphi_n)\,(x) = -1.$$

So we can get two invariant means μ and ν such that $\mu(x) = 1$ and $\nu(x) = -1$. Since $\| x \| = 1$, our assertion about the diameter follows.

THEOREM 4. *An abelian group G has a unique invariant mean if and only if G is finite. If G is infinite, then the diameter of the set of invariant means is two.*

Proof. If the group G is finite, then it follows from Lemma 0 of this section that G has a unique invariant mean. If G is infinite, then there are two possibilities:

Case 1. There exists an element of infinite order in G. Then G contains an infinite cyclic group A. It follows from Theorem 6 that the diameter of the set of invariant means on $m(A)$ is two. It now follows from Theorem 2 of this section that the diameter of the set of invariant means on $m(G)$ is two. It should be noticed that the proof of Theorem 6 does not depend on what we are trying to prove here. We could give a simpler proof for that theorem if we were concerned only with the infinite cyclic group A.

Case 2. Every element of G is of finite order. In this case we can get in G an expanding sequence of finite groups H_n. Letting $H = \bigcup_n H_n$ we see from Lemma 3 that the diameter of the set of invariant means on $m(H)$ is two. Theorem 2 now gives us our required result.

IV. Finitely generated abelian semigroups and minimal ideals

In this section we shall prove that a finitely generated abelian semigroup Σ has a unique invariant mean if and only if Σ has a finite ideal in it. Moreover it will be proved that any one of these conditions is equivalent to the third condition that there are no nontrivial homomorphisms of Σ into the additive semigroup of integers. This result will be used later on to prove the main theorem.

LEMMA 1. *Let Σ be an abelian semigroup having a minimal ideal A. Then Σ has a unique invariant mean if and only if A is finite. If A is infinite, then the diameter of the set of invariant means on $m(\Sigma)$ is two.*

Proof. First of all we observe that A is a group. In order to prove this, all that we have to do is to show that for every a in A, $aA = A$. But this is clear because aA, being an ideal in Σ and being contained in A, must be equal to A due to the minimal character of A.

If A is finite, then Lemma 0 of Section III says that Σ has a unique invariant mean μ given by

$$\mu(x) = [\textstyle\sum_{\sigma \epsilon A} x(\sigma)]/N(A).$$

Suppose now that A is infinite. Corresponding to each invariant mean μ_0 on $m(A)$ we define μ in $m(\Sigma)^*$ by

$$\mu(x) = \mu_0(x \mid A),$$

where x is in $m(\Sigma)$ and $x \mid A$ is the restriction of x to A. It is clear that μ is a mean. We shall now show that μ is actually invariant.

For each $\sigma \epsilon \Sigma$ and $a \epsilon A$, $(\sigma e)\cdot a = \sigma(ea) = \sigma a$, where e is the identity of the group A. Now take $x \epsilon m(\Sigma)$, $\sigma \epsilon \Sigma$, and $a \epsilon A$. Then

$$(l_\sigma x)\,(a) = x(\sigma a) = x(\sigma e a) = (l_{\sigma e}\, x)\,(a),$$

and so

$$(l_\sigma x) \mid A = l_{\sigma e}(x \mid A).$$

Consequently

$$\mu(l_\sigma x) = \mu_0((l_\sigma(x)) \mid A) = \mu_0(l_{\sigma e}(x \mid A)) = \mu_0(x \mid A) = \mu(x).$$

Thus for each invariant mean μ_0, the element μ of $m(\Sigma)^*$ defined as above is an invariant mean. It is also clear that $\| \mu - \nu \| \geqq \| \mu_0 - \nu_0 \|$, and thus our lemma follows quickly from Theorem 4 of Section III.

The proof of the following lemma is immediate.

LEMMA 2. *Let $\sum_{j=1}^{n} a_{ij} x_j = 0$, i in any index set I, be a system of linear equations. Then we can find n indices in I, say $1, 2, \cdots, n$, such that $(x_1, \cdots, x_n)$ is a solution of $\sum_{j=1}^{n} a_{ij} x_j = 0$, $i \in I$, if and only if $(x_1, \cdots, x_n)$ is a solution of $\sum_{j=1}^{n} a_{ij} x_j = 0$, $1 \leqq i \leqq n$.*

It is clear that if the a_{ij} are all integers and there is some nontrivial solution, then we can assume the solution $(x_1, \cdots, x_n)$ to be integral, and we can further assume that the greatest common divisor of $x_1, \cdots, x_n$ is 1.

LEMMA 3. *Every nonzero semigroup of integers has more than one invariant mean.*

Proof. In the proof we shall require the following lemma:

LEMMA 4. *Let S be a set of positive integers such that if x and y are in S then $x + y$ is in S. Suppose that the greatest common divisor of the elements of S is d. Then S contains all multiples of d from some point onwards.*

This lemma is proved on page 176 of [6].

We now come to the proof of Lemma 3. Let the semigroup be called A. Since division by the greatest common divisor of the elements of A yields an isomorphic semigroup, we may assume, without loss of generality, that the greatest common divisor of the elements of A is 1. Let

$$A^+ = \{x \mid x \in A, x > 0\}, \qquad A^- = \{x \mid x \in A, x < 0\}.$$

Suppose that neither A^+ nor A^- is empty. Let d be the greatest common divisor of the elements of A^+, and e the greatest common divisor of the elements of $-A^-$. Then $(d, e) = 1$. Using Lemma 4, we see that there is an integer N such that if n and m are bigger than or equal to N then nd and $-me$ are in A. Clearly we can pick m and n such that $md \in A$, $-ne \in A$, $md > ne$, and $(md, ne) = 1$. Then $md \in A$, $md - ne \in A$, and $(md, md - ne) = 1$. Since md and $md - ne$ are both positive, it follows from Lemma 4 that they will generate a semigroup B which will contain all positive integers from some point onwards. But $B \subseteqq A$ and so A contains all positive integers from some point onwards. Similarly A contains all negative integers from some point onwards. If n is large enough, then $n, -n, n + 1, -n - 1$ are all in A. Consequently 0, 1, and -1 are in A, and so A consists of all the integers. In this case the existence of many invariant means is already known or can easily be proved. If $A^- = \emptyset$, we prove the existence of many invariant means as follows. The case when $A^+ = \emptyset$ is exactly similar.

Pick $N \geqq 0$ such that $n \geqq N \Rightarrow n \epsilon A$. For each $n > N$, define a finite mean φ_n by

$$\varphi_n(k) = \begin{cases} 0 \text{ if } & k \leqq N \quad \text{or if} \quad k > n, \quad k \epsilon A \\ 1/(n-N) & \text{if} \quad N < k \leqq n. \end{cases}$$

If $p \epsilon A$ and n is sufficiently large, then

$$\| \varphi_n p - \varphi_n \| \leqq 2p(n-N)^{-1},$$

and so the φ_n converge strongly to invariance. We will have finished the proof of our lemma if we can find an x in $m(A)$ such that $(Q\varphi_n)(x)$ does not converge. But this can be done easily if we let x take the values $+1$ and -1 alternately on bigger and bigger blocks of positive integers.

DEFINITION. Let Σ be an abelian semigroup generated by $\sigma_1, \cdots, \sigma_n$. We say that $\sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n} = \sigma_1^{\beta_1} \cdots \sigma_n^{\beta_n}$ is a *relation* if $\alpha_i \geqq 0$, $\beta_i \geqq 0$, and there is at least one i for which $\alpha_i \neq \beta_i$.

LEMMA 5. *Let Σ be an abelian semigroup generated by $\sigma_1, \cdots, \sigma_n$, and suppose that the σ_i satisfy the relations*

$$\sigma_1^{\gamma_{01}} \cdots \sigma_n^{\gamma_{0n}} = \cdots = \sigma_1^{\gamma_{n1}} \cdots \sigma_n^{\gamma_{nn}},$$

where the determinant

$$D = \begin{vmatrix} \gamma_{01} \cdots \gamma_{0n} & 1 \\ \cdots\cdots\cdots\cdots & \cdot \\ \gamma_{n1} \cdots \gamma_{nn} & 1 \end{vmatrix} \neq 0.$$

Then Σ contains a finite ideal.

Proof. Let $\sigma = \sigma_1^{\gamma_{01}} \cdots \sigma_n^{\gamma_{0n}}$. Clearly we can assume that $D > 0$, for if $D < 0$ we just interchange any two rows. Let Γ_{ij} denote the cofactor of γ_{ij} in D. Let A be a sufficiently large positive integer so that all the exponents in the following calculation are positive. This is possible, as will be clear on looking at the calculation. Let i be any fixed integer between 1 and n. Then

$$\sigma^{(n+1)A} = \sigma^{(n+1)A+\Sigma_{j=0}^n \Gamma_{ji}} = \sigma^{\Sigma_{j=0}^n (A+\Gamma_{ji})} = \prod_{j=0}^n (\sigma_1^{\gamma_{j1}} \cdots \sigma_n^{\gamma_{jn}})^{A+\Gamma_{ji}}$$
$$= \prod_{k=1}^n \sigma_k^{\Sigma_{j=0}^n (A+\Gamma_{ji})\gamma_{jk}} = \prod_{k=1}^n \sigma_k^{(\Sigma_{j=0}^n \gamma_{jk})A+\delta_{ik}D} = \sigma^{(n+1)A}\sigma_i^D.$$

It follows that if Σ were a group, σ_i^D would be 1 for each i and so the group Σ would be finite. This fact we will need a little later in this proof. Since $\sigma^{(n+1)A} = \sigma^{(n+1)A}\sigma_i^D$ for each i, it follows that

$$\sigma^{(n+1)A} = \sigma^{(n+1)A}\sigma_1^D \cdots \sigma_n^D.$$

Thus we have a relation of the form

$$\sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n} = \sigma_1^{\beta_1} \cdots \sigma_n^{\beta_n},$$

where $\alpha_i < \beta_i$, $1 \leqq i \leqq n$. Now

$$\prod_i \sigma_i^{2\beta_i - \alpha_i} = \prod_i \sigma_i^{\beta_i} \prod_i \sigma_i^{\beta_i - \alpha_i} = \prod_i \sigma_i^{\alpha_i} \prod_i \sigma_i^{\beta_i - \alpha_i} = \prod_i \sigma_i^{\beta_i} = \prod_i \sigma_i^{\alpha_i}.$$

Continuing this process we obtain

$$\prod_i \sigma_i^{(\lambda+1)\beta_i-\lambda\alpha_i} = \prod_i \sigma_i^{\alpha_i} \qquad \text{for any } \lambda \geqq 1.$$

Clearly we can take λ so large that $(\lambda + 1)\beta_i - \lambda\alpha_i > 2\alpha_i$, and thus we might as well assume from the beginning that we have a relation of the form

$$\prod_i \sigma_i^{\alpha_i} = \prod_i \sigma_i^{\beta_i}, \qquad 2\alpha_i < \beta_i \text{ for } 1 \leqq i \leqq n.$$

If we let $e = \prod_i \sigma_i^{\beta_i-\alpha_i}$, then we can easily see that $e^2 = e$. We remark that in trying to prove that $e^2 = e$ we need the fact that $2\alpha_i < \beta_i$. We now let $\lambda_i = k_0(\beta_i - \alpha_i)$, $1 \leqq i \leqq n$, where k_0 is chosen so large that $\lambda_i > \beta_i$. Let

$$\Sigma' = \left\{\prod_i \sigma_i^{\nu_i} : \nu_i \geqq \lambda_i\right\}.$$

Σ' is clearly an ideal, and using the fact that $e = e^2 = e^3 = \cdots = e^N$ for any N, we can easily see that $e \in \Sigma'$. Next suppose that $\nu_i \geqq \lambda_i$, and suppose further that k is chosen so large that $k(\beta_i - \alpha_i) > \nu_i + \lambda_i$. Then

$$e\prod_i \sigma_i^{\nu_i} = \prod_i \sigma_i^{\nu_i+\beta_i-\alpha_i} = \prod_i \sigma_i^{\beta_i} \cdot \prod_i \sigma_i^{\nu_i-\alpha_i} = \prod_i \sigma_i^{\alpha_i} \prod_i \sigma_i^{\nu_i-\alpha_i} = \prod_i \sigma_i^{\nu_i}$$

and

$$e = e^k = \prod_i \sigma_i^{k(\beta_i-\alpha_i)} = \prod_i \sigma_i^{k(\beta_i-\alpha_i)-\nu_i} \prod_i \sigma_i^{\nu_i}.$$

These calculations show that e acts as an identity on Σ' and that $\prod_i \sigma_i^{k(\beta_i-\alpha_i)-\nu_i}$ is the inverse in Σ' of $\prod_i \sigma_i^{\nu_i}$ with respect to e. Thus Σ' is a group. Now

$$\Sigma' = \left\{\prod_i \sigma_i^{\nu_i} : \nu_i \geqq \lambda_i = k_0(\beta_i - \alpha_i)\right\} = \left\{e\prod_i \sigma_i^{\nu_i} : \nu_i \geqq 0\right\}$$
$$= \left\{\prod_i (e\sigma_i)^{\nu_i} : \nu_i \geqq 0\right\},$$

so that Σ' is generated by $e\sigma_1, \cdots, e\sigma_n$. Since

$$\sigma_1^{\gamma_{01}} \cdots \sigma_n^{\gamma_{0n}} = \cdots = \sigma_1^{\gamma_{n1}} \cdots \sigma_n^{\gamma_{nn}},$$

it follows on multiplication by e that

$$(e\sigma_1)^{\gamma_{01}} \cdots (e\sigma_n)^{\gamma_{0n}} = \cdots = (e\sigma_1)^{\gamma_{n1}} \cdots (e\sigma_n)^{\gamma_{nn}}.$$

Consequently Σ' is finite. Thus we have found a finite ideal Σ' in Σ.

We are now in a position to prove the main result of this section.

THEOREM 5. *Let Σ be a finitely generated abelian semigroup. Then the following three conditions on Σ are equivalent:*

(a) *Σ has a unique invariant mean,*

(b) *Σ has a finite ideal in it,*

(c) *If h is a homomorphism of Σ into the integers, then $h(\sigma) = 0$ for all σ in Σ.*

Proof. The implication (b) $\Rightarrow$ (a) holds for any abelian semigroup.

To prove (a) $\Rightarrow$ (c) we will prove that if (c) does not hold, then (a) does not hold. So let h be a nontrivial homomorphism of Σ into the integers. The image A of Σ under h is a nonzero subsemigroup of the integers, and consequently there are many invariant means on $m(A)$. It now follows from Theorem 1 of Section III that there are many invariant means on $m(\Sigma)$.

We will complete the proof by showing that (c) $\Rightarrow$ (b). Let

$$\sigma_1^{\alpha_{i1}} \cdots \sigma_n^{\alpha_{in}} = \sigma_1^{\beta_{i1}} \cdots \sigma_n^{\beta_{in}}, \qquad i \in I$$

be all the relations in Σ. Let us consider the linear equations

$$\sum_{j=1}^n (\alpha_{ij} - \beta_{ij})x_j = 0, \qquad i \in I.$$

Suppose that this system admits a nontrivial solution. Then since this system is equivalent to a finite system, we may assume that the solution $(x_1, \cdots, x_n)$ is integral and that the greatest common divisor of the x_j is one. Clearly $\sigma_j \to x_j$ will give a nontrivial homomorphism of Σ into the integers. Thus if (c) holds, then the above system of linear equations cannot have any nontrivial solution. It follows from Lemma 2 of this section that there exist indices $i = 1, 2, \cdots, n$ such that the system

$$\sum_{j=1}^n (\alpha_{ij} - \beta_{ij})x_j = 0, \qquad 1 \leqq i \leqq n$$

has no nontrivial solution. We have thus obtained relations

$$\sigma_1^{\alpha_{i1}} \cdots \sigma_n^{\alpha_{in}} = \sigma_1^{\beta_{i1}} \cdots \sigma_n^{\beta_{in}}, \qquad 1 \leqq i \leqq n$$

such that the matrix $(\alpha_{ij} - \beta_{ij})$ is nonsingular. Let

$$\gamma_{0j} = \max_{1 \leqq i \leqq n} \beta_{ij}, \qquad 1 \leqq j \leqq n.$$

Multiplying the above relations by $\sigma_1^{\gamma_{01}-\beta_{i1}} \cdots \sigma_n^{\gamma_{0n}-\beta_{in}}$, we get new relations

$$\sigma_1^{\gamma_{01}} \cdots \sigma_n^{\gamma_{0n}} = \cdots = \sigma_1^{\gamma_{n1}} \cdots \sigma_n^{\gamma_{nn}}.$$

Clearly the matrix

$$\begin{bmatrix} \gamma_{01} - \gamma_{11}, \cdots, \gamma_{0n} - \gamma_{1n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma_{01} - \gamma_{n1}, \cdots, \gamma_{0n} - \gamma_{nn} \end{bmatrix}$$

is nonsingular. An application of Lemma 5 of this section now yields the existence of a finite ideal in Σ. This concludes the proof of our theorem.

It should be noticed that the implications (b) $\Rightarrow$ (a) $\Rightarrow$ (c) hold even when Σ is not finitely generated. However (c) does not imply (a) or (b) if Σ is not finitely generated. An example is afforded by taking Σ to be the additive group of rational numbers. Here every homomorphism of Σ into the integers is trivial, and yet Σ possesses many invariant means, and Σ has no finite ideal in it.

V. A certain class of abelian semigroups

We will adopt the following notation for this section and also for the two following sections: Letters like δ and ∂ will denote finite subsets of the abelian semigroup Σ. For each δ, Σ_δ will denote the semigroup generated by the set δ in Σ. If Σ_δ has a finite minimal ideal in it, it will be denoted by A_δ. The identity of the group A_δ will be denoted by e_δ. If Σ_δ does not have any finite ideal in it, then by Theorem 5 we can find a nontrivial homomorphism of Σ_δ into the integers I. We will call this homomorphism h_δ, and I_δ will be the image of Σ_δ under h_δ. We will further assume that the greatest common divisor of the set $h_\delta(\delta)$ is one. In case this homomorphism h_δ exists, we will denote by H_δ the linear mapping from $m(I_\delta)$ into $m(\Sigma_\delta)$ defined by

$$(H_\delta\, x)\ (\sigma) = x(h_\delta\, \sigma) \qquad \text{for } \sigma \in \Sigma_\delta .$$

We say $\delta_1 \geqq \delta_2$ if $\delta_1 \supseteqq \delta_2$. Under this order the set of all finite subsets of Σ is a directed set.

The object of this section is to prove the following theorem.

THEOREM 6. *Let Σ be an abelian semigroup. Suppose that for a cofinal system of finite sets δ, Σ_δ does not have any finite ideal in it. Then the diameter of the set of invariant means on $m(\Sigma)$ is two.*

We will divide the proof of this theorem into several parts.

DEFINITION. We say that x in $m(\Sigma)$ is *almost convergent* if all invariant means on $m(\Sigma)$ assume the same value at x. If σ and τ are in Σ, then we say that $\sigma \geqq \tau$ if there exists an element ρ of Σ such that $\sigma = \tau\rho$. Under this order Σ becomes a directed set.

LEMMA 1. *Let Σ be an abelian semigroup. For each x in $m(\Sigma)$ we define*

$$p(x) = \inf_{\sigma_1,\cdots,\sigma_n} \lim\, \sup_\sigma\, (1/n) \sum_{1 \leqq i \leqq n} x(\sigma_i\, \sigma),$$

where the inf *is taken over all finite sequences of elements of Σ. Then x is almost convergent if and only if $p(x) = -p(-x)$; hence Σ has a unique invariant mean if and only if $p(x) = -p(-x)$ for all x. In case that is so, p is the unique invariant mean. Moreover if we can find an element x^0 of norm one in $m(\Sigma)$ such that $p(x^0) = p(-x^0) = 1$, then the diameter of the set of invariant means is two.*

Proof.[1] It can easily be seen that p is a positive homogeneous subadditive functional and $-p(-x) \leqq p(x)$.

We shall first prove that if μ is any invariant mean and x is any point in $m(\Sigma)$, then

$$-p(-x) \leqq \mu(x) \leqq p(x).$$

Take any sequence $\sigma_1, \cdots, \sigma_n$ of elements of Σ. Since μ is a mean, $\mu(y) \leqq \sup_\sigma y(\sigma)$ for any y in $m(\Sigma)$. Take any τ in Σ. Then

$$\mu(y) = \mu(l_\tau\, y) \leqq \sup_\sigma\, (l_\tau\, y)\ (\sigma) = \sup_\sigma\, y(\tau\sigma) = \sup_{\sigma \geqq \tau}\, y(\sigma).$$

Taking $y = (1/n) \sum_{i=1}^{n} l_{\sigma_i}\, x$, we see that

$$\mu(x) \leqq \sup_{\sigma \geqq \tau}\, (1/n) \sum_{i=1}^{n} x(\sigma_i\, \sigma).$$

[1] I am grateful to the referee for the simplification of the original proof.

This being true for any τ in Σ and any sequence $\sigma_1, \cdots, \sigma_n$ of elements of Σ, we conclude that $\mu(x) \leqq p(x)$ for all x in $m(\Sigma)$. Consequently

$$-p(-x) \leqq -\mu(-x) = \mu(x) \leqq p(x), \qquad x \text{ in } m(\Sigma).$$

From this it immediately follows that if $p(x) = -p(-x)$ for some x in $m(\Sigma)$, then this x is almost convergent. Also it shows that if $p(x) = -p(-x)$ for all x in $m(\Sigma)$, then p is the unique invariant mean on $m(\Sigma)$. This proves one part of our lemma. To prove the other part we proceed as follows: By an application of the Hahn-Banach extension theorem we can construct a linear functional μ which is below p at each x. It can easily be seen that μ is a mean. To prove that μ is invariant we do the following calculation:

$$p(x - l_\rho x) = \inf_{\sigma_1,\cdots,\sigma_n} \limsup_\sigma (1/n) \textstyle\sum_{i=1}^n [x(\sigma_i \sigma) - x(\rho\sigma_i \sigma)]$$
$$\leqq \limsup_\sigma (1/n) \textstyle\sum_{i=1}^n [x(\rho^i\sigma) - x(\rho^{i+1}\sigma)] \leqq 2\| x \|/n.$$

This being true for all n, $p(x - l_\rho x) \leqq 0$. Similarly $p(l_\rho x - x) \leqq 0$, and consequently $\mu(x - l_\rho x) = 0$, which shows that μ is invariant. Thus any μ under p is an invariant mean. If we look at the proof of the Hahn-Banach theorem we find that the value of μ at x can be taken to be any number between $-p(-x)$ and $p(x)$. Thus if x is almost convergent, $-p(-x)$ must be equal to $p(x)$. Moreover, if for some x_0 of norm one, $-p(-x_0) = -1$ and $p(x_0) = 1$, then we can find two extensions μ and ν such that $\mu x^0 = 1$ and $\nu x^0 = -1$. Since $\| x^0 \| = 1$, it follows that $\| \mu - \nu \| = 2$, so that the diameter of the set of invariant means is two.

This concludes the proof of our lemma.

We now come to the proof of our theorem. Without loss of generality (as will be clear from the argument that follows) we may assume that no Σ_δ has any finite ideal in it. Thus as explained before, we get for each δ, a mapping

$$H_\delta : m(I_\delta) \to m(\Sigma_\delta)$$

defined by

$$(H_\delta(x))(\sigma) = x(h_\delta \sigma), \qquad x \in m(I_\delta), \quad \sigma \in \Sigma_\delta .$$

We divide the integers into two classes C_1 and C_2 as follows:

$$C_1 = \{0\} \cup \textstyle\bigcup_{k=0}^\infty \{n : 10^{2k+1} \leqq |n| < 10^{2k+2}\},$$
$$C_2 = \textstyle\bigcup_{k=0}^\infty \{n : 10^{2k} \leqq |n| < 10^{2k+1}\}.$$

We define x_0 in $m(I)$ by

$$x_0(n) = \begin{cases} -1 & \text{if } n \in C_1 \\ 1 & \text{if } n \in C_2 . \end{cases}$$

Let $y_\delta = H_\delta(x_0 \mid I_\delta)$, so that $y_\delta \in m(\Sigma_\delta)$ and $y_\delta(\sigma) = x_0(h_\delta \sigma)$ for σ in Σ_δ. Let x_δ be an extension of y_δ taking the value one outside Σ_δ. Thus we get a net of elements x_δ of $m(\Sigma)$ all of norm one. Since $m(\Sigma)$ is the conjugate space of $l_1(\Sigma)$, we can get a w^*-convergent subnet of the net $\{x_\delta : \delta\}$. Again without loss of generality (as will be clear from the argument that follows), we may assume that the net $\{x_\delta:\delta\}$ itself converges to a point x^0 in the w^*-topology of $m(\Sigma)$. In particular, $\lim_\delta x_\delta(\sigma) = x^0(\sigma)$ for each σ in Σ. It is clear that $\lim_\delta y_\delta(\sigma) = x^0(\sigma)$ and that the only possible values of $x^0(\sigma)$ are $+1$ and -1. We notice that when we speak of the limit of $y_\delta(\sigma)$ we consider only those Σ_δ which contain the element σ. We let

$$\hat{C}_1 = \{\sigma : x^0(\sigma) = -1\}, \qquad \hat{C}_2 = \{\sigma : x^0(\sigma) = 1\}.$$

Then $\hat{C}_1 \cap \hat{C}_2 = \emptyset$ and $\hat{C}_1 \cup \hat{C}_2 = \Sigma$. Since $y_\delta(\sigma)$ converges to $x^0(\sigma)$, there exists a finite set δ_σ such that $\delta \geqq \delta_\sigma$ implies that $y_\delta(\sigma) = x^0(\sigma)$, that is, $x_0(h_\delta\,\sigma) = x^0(\sigma)$ for $\delta \geqq \delta_\sigma$. This means that $\{h_\delta\,\sigma : \delta \geqq \delta_\sigma\}$ is in the same class C_1 or C_2, that is, this set does not distribute itself over both classes. Thus we see that

$$\hat{C}_1 = \{\sigma: \text{There exists } \delta_\sigma \text{ such that } \delta \geqq \delta_\sigma \Rightarrow h_\delta\,\sigma \in C_1\},$$

and

$$\hat{C}_2 = \{\sigma: \text{There exists } \delta_\sigma \text{ such that } \delta \geqq \delta_\sigma \Rightarrow h_\delta\,\sigma \in C_2\}.$$

This is an important fact and we will use it quite strongly in the proof of the following lemma which is needed for the proof of Theorem 6.

LEMMA 2. *For any finite sequence $\sigma_1, \cdots, \sigma_n$ of elements of Σ and any element τ of Σ, there exist ρ and ρ' in Σ such that $\rho \geqq \tau$, $\rho' \geqq \tau$, $\sigma_i\,\rho \in \hat{C}_1$, and $\sigma_i\,\rho' \in \hat{C}_2$ for $1 \leqq i \leqq n$.*

Proof. We will first prove that for any sequence $\sigma_1, \cdots, \sigma_n$ of elements of Σ, there exist ρ and ρ' such that $\sigma_i\,\rho \in \hat{C}_1$ and $\sigma_i\,\rho' \in \hat{C}_2$ for $1 \leqq i \leqq n$. We shall prove only the existence of the element ρ, the proof of the existence of ρ' being exactly similar.

Clearly we can find a cofinal set Δ' of finite sets δ such that, with a possible change of names,

$$|h_\delta(\sigma_1)| \geqq |h_\delta(\sigma_2)| \geqq \cdots \geqq |h_\delta(\sigma_n)| \qquad \text{for } \delta \text{ in } \Delta'.$$

Either $h_\delta(\sigma_1) \leqq 0$ for all sufficiently large δ in Δ' or for δ in a cofinal $\Delta'' \subseteqq \Delta'$, $h_\delta(\sigma_1) \geqq 0$. The cases $h_\delta(\sigma_1) \leqq 0$ for all sufficiently large δ in Δ' and $h_\delta(\sigma_1) \geqq 0$ for δ in Δ'' are exactly similar and so we assume without any loss of generality that $h_\delta(\sigma_1) \geqq 0$ for all δ in Δ'. Multiplying the σ_i by σ_1 we get

$$h_\delta(\sigma_1^2) \geqq h_\delta(\sigma_1\,\sigma_2) \geqq \cdots \geqq h_\delta(\sigma_1\,\sigma_n) \geqq 0$$

for all δ in Δ'. Since we are interested only in the multiples of the σ_i, we may assume that the $\sigma_1, \cdots, \sigma_n$ are such that

$$h_\delta(\sigma_1) \geqq h_\delta(\sigma_2) \geqq \cdots \geqq h_\delta(\sigma_n) \geqq 0$$

for δ in a cofinal set Δ'. If $h_\delta(\sigma_1) = 0$ for δ in a cofinal set $\Delta'' \subseteqq \Delta'$, then $h_\delta(\sigma_i) = 0$ for these δ, and thus the $\sigma_1, \cdots, \sigma_n$ already lie in $\hat{C}_1$. So we can suppose that $h_\delta(\sigma_1) > 0$ for large enough δ in Δ' or, without loss of generality, for all δ in Δ'. We now let

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$$

and

$$\alpha = \begin{cases} \sigma & \text{if} \quad \sigma \in \hat{C}_2 \\ \sigma^{10} & \text{if} \quad \sigma \in \hat{C}_1 . \end{cases}$$

It is clear that $\alpha \in \hat{C}_2$ and

$$h_\delta(\alpha) \geqq h_\delta(\sigma) \geqq h_\delta(\sigma_i) \qquad \text{for } \delta \text{ in } \Delta' \text{ and } 1 \leqq i \leqq n.$$

We now pick k such that $\alpha^{k-1} \in \hat{C}_2$ and $\alpha^k \in \hat{C}_1$, $2 \leqq k \leqq 10$. Such a k exists since $\alpha^{10} \in \hat{C}_1$. We let $\rho = \alpha^k$, and we assert that this ρ has the required property. In the proof of this assertion which we shall give below, δ always belongs to Δ' and is sufficiently large. Since $\alpha^{k-1} \in \hat{C}_2$ and $\alpha^k \in \hat{C}_1$, therefore $h_\delta(\alpha^{k-1}) \in C_2$ and $h_\delta(\alpha^k) \in C_1$. Thus

$$10^{2k_\delta} \leqq h_\delta(\alpha^{k-1}) < 10^{2k_\delta+1}, \qquad 10^{2k_\delta+1} \leqq h_\delta(\alpha^k) < 10^{2k_\delta+2},$$

where the k_δ are some nonnegative integers. We consider two cases.

Case 1. $k = 2$. In this case

$$h_\delta(\alpha^3) = 3h_\delta(\alpha) < 3\cdot 10^{2k_\delta+1} < 10^{2k_\delta+2}.$$

Thus

$$10^{2k_\delta+1} \leqq h_\delta(\alpha^2) \leqq h_\delta(\alpha^3) < 10^{2k_\delta+2}.$$

Case 2. $k > 2$. In this case

$$h_\delta(\alpha^{k+1}) = h_\delta(\alpha^2) + h_\delta(\alpha^{k-1}) \leqq 2h_\delta(\alpha^{k-1}) < 2\cdot 10^{2k_\delta+1} < 10^{2k_\delta+2}.$$

Thus

$$10^{2k_\delta+1} \leqq h_\delta(\alpha^k) \leqq h_\delta(\alpha^{k+1}) < 10^{2k_\delta+2}.$$

From this we get

$$10^{2k_\delta+1} \leqq h_\delta(\alpha^k) \leqq h_\delta(\alpha^k\sigma_i) \leqq h_\delta(\alpha^{k+1}) < 10^{2k_\delta+2}.$$

Thus for a cofinal system of finite sets δ, $h_\delta(\alpha^k\sigma_i) \in C_1$ for $1 \leqq i \leqq n$. This combined with the fact that for sufficiently large δ, h_δ (any element) lies in the same class C_1 or C_2, proves that $h_\delta(\alpha^k\sigma_i) \in C_1$ for all sufficiently large δ. This shows that $\alpha^k\sigma_i \in \hat{C}_1$ for $1 \leqq i \leqq n$.

If we want our ρ to be bigger than or equal to any preassigned element τ, all that we have to do is to start with $\sigma_1 \tau, \cdots, \sigma_n \tau$ instead of $\sigma_1, \cdots, \sigma_n$ and carry on the above argument.

This concludes the proof of Lemma 2.

Proof of Theorem 6. In view of Lemma 1, it will be enough to prove that for the x^0 we have constructed,

$$p(x^0) = p(-x^0) = 1.$$

We will just verify that $p(x^0) = 1$, the other part being exactly similar.

$$\limsup_\sigma\, (1/n)\textstyle\sum_{i=1}^n x^0(\sigma_i\,\sigma) = \inf_\tau \sup_{\sigma \geqq \tau}\, (1/n)\sum_{i=1}^n x^0(\sigma_i\,\sigma) = \inf_\tau 1 = 1.$$

Consequently $p(x^0) = 1$.

This concludes the proof of our theorem.

VI. Another class of abelian semigroups

The object of this section is to prove the following result.

THEOREM 7. *Let Σ be an abelian semigroup such that there exists a δ_0 such that for all $\delta \geqq \delta_0$, Σ_δ has a finite minimal ideal A_δ in it. Suppose that for each $\delta' \geqq \delta_0$ there is a $\delta'' \geqq \delta'$ such that if $\delta \geqq \delta''$ then $A_\delta \cap A_{\delta'} = \emptyset$. Then the diameter of the set of invariant means is two.*

In the proof of this theorem we will require the following lemma.

LEMMA 1. *If E is a directed system such that for any e in E there is $e' > e$ in E with $e' \neq e$, then there exist two disjoint cofinal subsets of E.*

Proof. We will consider two cases.

Case 1. There is a finite cofinal subset of E. Then there is an e bigger than or equal to all elements of that finite subset, hence e is cofinal in E. But there is $e' > e$ with $e' \neq e$. Thus $\{e'\}$ is another cofinal subset and $\{e\} \cap \{e'\} = \emptyset$.

Case 2. No finite subset of E is cofinal in E. In this case we can easily see that for e in E there exists a sequence $\{e_i\}$ of distinct elements of E with $e_1 = e$ and $e_i < e_{i+1}$. We now let $\mathfrak{A}$ be the set of all triples (A, B, F), where F is a subset of E, and A and B are maps from F into E satisfying the following conditions:

(i) $A(F) \cap B(F) = \emptyset$.

(ii) $A(F)$ and $B(F)$ are cofinal in each other.

(iii) $A(F)$ and $B(F)$ are cofinal in F.

The class $\mathfrak{A}$ is not empty for if we take any sequence $e_1 < e_2 < \cdots < e_n < \cdots$ of distinct elements of E, then we can let $F = \{e_i : i\}$, $Ae_i = e_{2i}$, and $Be_i = e_{2i-1}$. This (A, B, F) clearly belongs to the class $\mathfrak{A}$. We order $\mathfrak{A}$ by saying that $(A, B, F) \leqq (A', B', F')$ if and only if $F \subseteqq F'$ and A and B are respectively the restrictions of A' and B' to F. By an application of Zorn's Lemma we can get a maximal element (A, B, F). We claim that $A(F)$ and $B(F)$ are cofinal in E. First we notice that either both are cofinal in E or neither is cofinal in E. This follows from the fact that $A(F)$ and $B(F)$ are cofinal in each other. So suppose that $A(F)$ and $B(F)$ are not cofinal in E. Then there exists an element e_1 of E such that no successor of e_1 is in $A(F) \cup B(F)$. Thus we can pick a sequence $e_1 < e_2 < \cdots < e_n < \cdots$ of distinct elements of E such that $e_n \notin A(F) \cup B(F)$. Also since $A(F)$ and $B(F)$ are cofinal in F, it follows that $e_n \notin F$. We now let

$$F^* = F \cup \{e_i : i \geqq 1\}$$

and define A^* and B^* on F^* by

$$A^*f = Af \quad \text{if} \quad f \in F \quad \text{and} \quad A^*e_i = e_{2i},$$

$$B^*f = Bf \quad \text{if} \quad f \in F \quad \text{and} \quad B^*e_i = e_{2i-1}.$$

Then $(A^*, B^*, F^*) \in \mathfrak{A}$ and is bigger than and not equal to (A, B, F), contradicting the maximality of (A, B, F). Thus $A(F)$ and $B(F)$ are cofinal in E, and since $(A, B, F) \in \mathfrak{A}$, $A(F) \cap B(F) = \emptyset$. This concludes the proof of our lemma.

We remind the reader that for each $\delta \geqq \delta_0$, Σ_δ has a finite minimal ideal A_δ, and that e_δ is the identity of this group A_δ. We prove the following lemma.

Lemma 2. *Let $\delta \geqq \delta_0$, and suppose that e is an idempotent in Σ such that $e \geqq e_\delta$. Then there exists $\partial \geqq \delta$ such that $e = e_\partial$.*

Proof. First we show that if Σ_{δ_1} and Σ_{δ_2} have respectively the finite minimal ideals A_{δ_1} and A_{δ_2} (we do not assume that $\delta_i \geqq \delta_0$) in them, and if $\Sigma_{\delta_1 \cup \delta_2}$ has the finite minimal ideal $A_{\delta_1 \cup \delta_2}$, then $A_{\delta_1 \cup \delta_2} = A_{\delta_1} A_{\delta_2}$. For

$$(A_{\delta_1} A_{\delta_2}) \Sigma_{\delta_1 \cup \delta_2} \subseteqq A_{\delta_1} A_{\delta_2},$$

and so $A_{\delta_1} A_{\delta_2}$ is an ideal in $\Sigma_{\delta_1 \cup \delta_2}$ and consequently contains the minimal ideal $A_{\delta_1 \cup \delta_2}$. But clearly $A_{\delta_1} A_{\delta_2}$ is a group whose identity is $e_{\delta_1} e_{\delta_2}$, and an ideal in a group being the whole group, we conclude that $A_{\delta_1} A_{\delta_2} = A_{\delta_1 \cup \delta_2}$ and $e_{\delta_1} e_{\delta_2} = e_{\delta_1 \cup \delta_2}$. If we let $\partial = \delta \cup \{e\}$, then this result is applicable, and so $e_\partial = e_\delta \cdot e$. But $e \geqq e_\delta$, and so there exists σ such that $e = e_\delta \sigma$. Then $e_\delta e = e_\delta \cdot e_\delta \sigma = e_\delta^2 \sigma = e_\delta \sigma = e$ and so

$$e_\partial = e_\delta e = e.$$

This concludes the proof of our lemma.

Proof of Theorem 7. Let $E = \{e_\delta : \delta \geqq \delta_0\}$. Under the hypothesis of Theorem 7 the condition on E stated in Lemma 1 is satisfied. So we can divide E into two disjoint parts E_0 and E_1 which are both cofinal in E. Define x^0 in $m(E)$ by

$$x^0(e) = \begin{cases} 1 & \text{if } e \in E_0 \\ -1 & \text{if } e \in E_1 . \end{cases}$$

Define x on $m(\Sigma)$ by

$$x(\sigma) = \begin{cases} x^0(e\delta) & \text{if } \sigma \in A_\delta \text{ for some } \delta \geqq \delta_0 \\ 0 & \text{if } \sigma \notin \bigcup_{\delta \geqq \delta_0} A_\delta . \end{cases}$$

We, of course, have to check that x is consistently defined. To see this, all that we have to do is to prove that if $A_\delta \cap A_\partial \neq \emptyset$, then $e_\delta = e_\partial$. But that is easy, for suppose that $\sigma \in A_\delta \cap A_\partial$, and suppose that the orders of A_δ and A_∂ are i and j respectively. Then $e_\delta = \sigma^{ij} = e_\partial$.

We now define finite means φ_δ in $l_1(\Sigma)$ for each $\delta \geqq \delta_0$. We let

$$\varphi_\delta(\sigma) = \begin{cases} 1/N(A_\delta) & \text{if } \sigma \in A_\delta \\ 0 & \text{if } \sigma \notin A_\delta . \end{cases}$$

Take any σ in Σ, and pick $\delta \geqq \delta_0$ so that $\sigma \in \delta$. Then

$$\varphi_\delta \sigma = [\textstyle\sum_{\tau \in A_\delta} \varphi_\delta(\tau)\tau]\sigma = [\sum_{\tau \in A_\delta} \tau\sigma]/N(A_\delta) = [\sum_{\rho \in A_\delta} \rho]/N(A_\delta) = \varphi_\delta .$$

Thus $\varphi_\delta\,\sigma - \varphi_\delta = 0$ as soon as $\sigma \in \delta$, and so the net $\{\varphi_\delta\}$ converges strongly to invariance.

We next claim that $\lim\sup \varphi_\delta(x) = 1$ and $\lim\inf \varphi_\delta(x) = -1$. Now $\varphi_\delta(x) = \sum_\sigma \varphi_\delta(\sigma)\cdot x(\sigma) = x^0(e_\delta)$, and since $x^0(e_\delta)$ is either $+1$ or -1, it is clear that if $\lim\sup \varphi_\delta(x)$ and $\lim\inf \varphi_\delta(x)$ are different, then the first has to be 1 and the second has to be -1. But if $\lim\sup \varphi_\delta(x) = \lim\inf \varphi_\delta(x)$, it would mean that the net $\{x^0(e_\delta):\delta \geqq \delta_0\}$ converges. In order for this to be true it is necessary that there exist $\delta_1 \geqq \delta_0$ such that the set $\{e_\delta:\delta \geqq \delta_1\}$ is contained entirely in the same set E_0 or E_1, say E_0. But by Lemma 2,

$$\{e_\delta:\delta \geqq \delta_1\} = \{e:e \geqq e_{\delta_1}\},$$

and so we see that $e \geqq e_{\delta_1}$ implies that $e \in E_0$. But this contradicts the cofinality of E_1. Therefore

$$\lim\sup_\delta \varphi_\delta(x) = 1, \qquad \lim\inf_\delta \varphi_\delta(x) = -1.$$

It therefore follows that we can find two invariant means μ and ν such that $\mu(x) = 1$ and $\nu(x) = -1$. This concludes the proof of Theorem 7.

VII. Proof of the main theorem

We are now in a position to prove the main result of our paper.

THEOREM 8. *If Σ is an abelian semigroup, then Σ has a unique invariant mean if and only if Σ contains a finite ideal. In case Σ has many invariant means, the diameter of the set of invariant means is two.*

Proof. If Σ contains a finite ideal, then we have already seen that Σ has a unique invariant mean. Suppose now that Σ has a unique invariant mean. By Theorem 6 there exists δ_0 such that $\delta \geqq \delta_0$ implies that Σ_δ has a finite minimal ideal A_δ in it. Using Theorem 7, we obtain

There exists $\delta' \geqq \delta_0$ such that for all $\delta'' \geqq \delta'$, there exists $\delta \geqq \delta''$ such that $A_\delta \cap A_{\delta'} \neq \emptyset$.

Since Σ_δ contains $\Sigma_{\delta'}$, $A_\delta \cap A_{\delta'}$ is an ideal in $\Sigma_{\delta'}$ and, being contained in $A_{\delta'}$, must be equal to $A_{\delta'}$. Thus $A_{\delta'} \subseteqq A_\delta$. We now let

$$\Delta' = \{\delta:\delta \geqq \delta' \text{ and } A_\delta \supseteqq A_{\delta'}\}, \qquad A = \bigcup_{\delta\in\Delta'} A_\delta .$$

We claim that A is an ideal. To prove this, let $a \in A$ and $\sigma \in \Sigma$. Then there exists $\delta'' \in \Delta'$ such that $a \in A_{\delta''}$. Pick $\delta \in \Delta'$ such that $\delta \supseteqq \delta'' \cup \{\sigma\}$. Then $A_\delta \cap A_{\delta''}$ contains $A_{\delta'}$ and so is not empty. Consequently, since $\delta \geqq \delta''$, we see that $A_\delta \supseteqq A_{\delta''}$, and so $a \in A_\delta$. Since $\sigma \in \delta$, therefore $a\sigma \in A_\delta \subseteqq A$. This proves that A is an ideal in Σ.

Next we claim that A is a group. To see this we observe that if δ_1 and δ_2 are in Δ', then there exists δ in Δ' such that $\delta \supseteqq \delta_1 \cup \delta_2$. Consequently $A_\delta \supseteqq A_{\delta_1} \cup A_{\delta_2}$. This shows immediately that A is a group.

The above two considerations show that A is a minimal ideal in Σ, and since Σ has a unique invariant mean, A must be finite by Lemma 1 of Section IV.

If Σ has many invariant means, then the following situations are possible.

Case 1. For a cofinal system of finite sets δ, Σ_δ has no finite ideal in it. In this case Theorem 6 says that the diameter of the set of invariant means is two.

Case 2. There exists δ_0 such that $\delta \geqq \delta_0$ implies that Σ_δ has a finite minimal ideal A_δ in it. In this case we have the following two situations possible:

(1) For each $\delta' \geqq \delta_0$ there exists $\delta'' \geqq \delta'$ such that $\delta \geqq \delta'' \Rightarrow A_\delta \cap A_{\delta'} = \emptyset$.

(2) There exists $\delta' \geqq \delta_0$ such that for all $\delta'' \geqq \delta'$ there exists $\delta \geqq \delta''$ such that $A_\delta \cap A_{\delta'} \neq \emptyset$.

If (1) holds, then Theorem 7 says that the diameter of the set of invariant means is two. If (2) holds, then as shown above, Σ has a minimal ideal A in it, and since Σ has many invariant means, A must be infinite, and then the assertion about the diameter follows from Lemma 1 of Section IV.

This concludes the proof of our theorem.

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