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ISOMORPHISMS OF INCIDENCE RINGS

BY

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Let R be a ring with identity and let $(X, \leq)$ be a finite preordered set, that is, a finite set X with a relation $\leq$ that is reflexive and transitive but not necessarily antisymmetric. The incidence ring $I(X, R)$ of R over X is a ring with the additive structure of a free R-module with basis $\{f_{xy} \mid x \leq y\}$ and multiplication given by

$$\left(\sum f_{xy} r_{xy}\right)\left(\sum f_{xy} s_{xy}\right) = \sum f_{xy}\left(\sum_{x \leq z \leq y} r_{xz} s_{zy}\right)$$

[2], [5], [13], [16]. Such rings may be considered subrings of $(\text{Card } X) \times (\text{Card } X)$ block upper triangular matrix rings over R [6]. The question of finding classes of rings R such that $I(X, R) \cong I(Y, R)$ implies $X \cong Y$ has been the subject of several studies [2], [3], [11], [14], [16]. Here we consider whether $I(X, R) \cong I(X, S)$ implies $R \cong S$, and in fact show that if R is an indecomposable semiperfect ring and X and Y are finite partially ordered sets, then $I(X, R) \cong I(Y, S)$ if and only if there is a ring T and (necessarily finite) partially ordered sets Z and W with $R \cong I(Z, T)$, $S \cong I(W, T)$, and $X \times Z \cong Y \times W$. It follows that $I(X, R) \cong I(X, S)$ implies $R \cong S$ if R is semiperfect and X is finite. Further, if R is an IR-irreducible ring and X is a finite partially ordered set, then any automorphism of $I(X, R)$ is the composition of an inner automorphism, an automorphism induced by an order automorphism of X, and an automorphism induced by a family of additive maps from R to R satisfying multiplication laws induced by the partial order. (Compare [14, Theorem 2] which characterizes K-algebra automorphisms of $I(X, K)$ for K a field.) Finally, we answer a question left open in [5] by showing that the incidence rings $I(X, R)$ and $I(X, S)$ have Morita duality only if R and S do. Hence $I(X, R)$ is self-dual if and only if R is.

First, let us fix notation and recall certain facts. The symbol $\leq$ will be used for any preorder; the context will make it clear which relation is being considered. Let X denote a finite preordered set throughout. The relation $\sim$ defined on X by $x \sim x'$ iff $x \leq x'$ and $x' \leq x$ is an equivalence relation, and the set $X_0 = X/\sim$ forms a partially ordered set with partial ordering defined on equivalence classes x_0 and x'_0 with representatives x and x' by $x_0 \leq x'_0$ iff $x \leq x'$ in X. The incidence rings $I(X, R)$ and $I(X_0, R)$ are Morita equivalent.

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The product $X \times Y$ of preordered sets X and Y is defined by $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$ for $x, x' \in X$ and $y, y' \in Y$. A result of Lovász ([17]; see also [8]) implies that if X, Y and Z are finite preordered sets, then $X \times Y$ and $X \times Z$ are order isomorphic if and only if Y is order isomorphic to Z. The incidence rings $I(X \times Y, R)$ and $I(X, I(Y, R))$ are naturally isomorphic. A preordered set X is said to be connected if for any elements x and x' of X there is a finite sequence $x = x_1, x_2, \ldots, x_m = x'$ of elements of X with either $x_i \leq x_{i+1}$ or $x_{i+1} \leq x_i$ for $i = 1, \ldots, m-1$. Any finite preordered set can be written uniquely as the sum (disjoint union) of its connected components;

$$X = \sqcup\{Y \mid Y \text{ is a connected component of } X\}.$$

The product distributes over sums. The behavior of the incidence ring construction on sums is given by $I(X \sqcup Y, R) \cong I(X, R) \times I(Y, R)$. An order isomorphism $\theta\colon X \to Y$ induces a ring isomorphism $\bar{\theta}\colon I(X, R) \to I(Y, R)$ via

$$\bar{\theta}\colon \sum f_{xx'} r_{xx'} \mapsto \sum f_{\theta(x)\theta(x')} r_{xx'}.$$

From a semiperfect ring R, we may choose a complete orthogonal set $\bar{R}$ of primitive idempotents $\{e_1, \ldots, e_n\}$. A preordering of $\bar{R}$ may be defined by $e \leq e'$ iff there exists a sequence of idempotents $e = e'_1, \ldots, e'_m = e'$ in $\bar{R}$ with

$$e'_i R e'_{i+1} \neq 0 \quad \text{for } i = 1, \ldots, m-1.$$

With this preordering, $\bar{R}$ is called the associated preordered set of R. R is indecomposable as a ring if and only if $\bar{R}$ is connected [1, Theorem 7.9]. An incidence ring $I(X, R)$ is semiperfect if and only if R is semiperfect and X is finite. A complete orthogonal set of primitive idempotents for $I(X, R)$ is given by $\{f_x e_i \mid x \in X;\ i = 1, \ldots, n\}$ where $f_x = f_{xx}$ for $x \in X$. Here, $f_x e_i \leq f_{x'} e_j$ if and only if $(x, e_i) \leq (x', e_j)$ in $X \times \bar{R}$, so that $I(X, R)$ is order isomorphic to $X \times \bar{R}$ [16, Lemma 4.2]; we will write $X \times \bar{R}$ for $I(X, R)$. If $R \cong R_1 \times R_2$ as rings, then $I(X, R) \cong I(X, R_1) \times I(X, R_2)$. If S is another semiperfect ring, we will use $g_1, \ldots, g_m$ to denote the elements of a complete orthogonal set of idempotents for S; if Y is a preordered set, we will use $h_{yy'}$ (with $h_y = h_{yy}$) for the canonical basis elements of $I(Y, S)$ over S.

We are now ready to begin the analysis of an isomorphism between $I(X, R)$ and $I(Y, S)$.

1. Proposition. *Let R be a semiperfect ring and X a finite preordered set. If $\alpha\colon I(X, R) \to I(Y, S)$ is an isomorphism, then there exists an inner automorphism β of $I(Y, S)$ such that $\beta\alpha(X \times \bar{R}) = Y \times \bar{S}$.*

Proof. Since α is an isomorphism $\alpha(X \times \bar{R})$ is complete orthogonal set of primitive idempotents of the semiperfect ring $I(Y, S)$. Hence there is an inner automorphism of $I(Y, S)$ carrying $\alpha(X \times \bar{R})$ to $Y \times \bar{S}$ [4, Proposition 18.23.5]. ∎

One might initially hope that if $\alpha: I(X, R) \to I(X, R)$ is a ring automorphism with

$$\alpha(X \times \bar{R}) = (X \times \bar{R}),$$

then a copy of R, say $f_x R = f_x I(X, R) f_x$, would be carried onto some $f_{x'} R$. A little reflection shows that this need not be the case even if R is indecomposable and X is connected. For example, let F be a field and $X = \{1, 2\}$ with ordering $1 \leq 1$, $1 \leq 2$, $2 \leq 2$. Then $R = I(X, F)$ is isomorphic to the ring of 2×2 upper triangular matrices over F. Define $\alpha: I(X, R) \to I(X, R)$ by interchanging the 2nd and 3rd rows and columns of the elements of $I(X, R)$ considered as matrices over F:

$$\alpha: \begin{pmatrix} \begin{pmatrix} a & b \\ & c \end{pmatrix} & \begin{pmatrix} x & y \\ & z \end{pmatrix} \\ & \begin{pmatrix} p & q \\ & r \end{pmatrix} \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} a & x \\ & p \end{pmatrix} & \begin{pmatrix} b & y \\ & q \end{pmatrix} \\ & \begin{pmatrix} c & z \\ & r \end{pmatrix} \end{pmatrix}.$$

(The map α corresponds to the twisting order automorphism τ of $X \times X$ given by $\tau(x, x') = (x', x)$.) We do find, however, that $f_x R$ is sent to an incidence ring.

2. Lemma. *Let R be an indecomposable semiperfect ring and let x be any element of a finite preordered set X. Let $f_{x_0} = \sum \{f_{x'} \mid x' \in x_0\}$. Let α be an isomorphism from $I(X, R)$ to $I(Y, S)$ such that $\alpha(X \times \bar{R}) = Y \times \bar{S}$. Then there exist an idempotent u of S and a subset Z of Y such that*

$$\alpha(f_{x_0} I(X, R) f_{x_0}) = \left(\sum \{h_z \mid z \in Z\}\right) I(Y, uSu) \left(\sum \{h_z \mid z \in Z\}\right).$$

Hence $M_m(R) \cong I(Z, uSu)$ where x_0 has m elements and $M_m(R)$ is the $m \times m$-matrix ring over R. The subset Z consists of the first coordinates of the pairs in the set $\alpha(x_0 \times \bar{R})$.

Proof. First, notice that α restricted to $X \times \bar{R}$ is in fact an order isomorphism, since the ordering on $X \times \bar{R}$ is determined by the transitive extension of the ring-theoretic condition $f_x e_i R f_{x'} e_j \neq 0$. Because R is indecomposable, $\bar{R}$ is connected. Hence there are idempotents $g'_1, \ldots, g'_k$ in $\bar{S}$ and a subset Z of Y with

$$\alpha(x_0 \times \bar{R}) = Z \times \{g'_1, \ldots, g'_k\} \quad [16, \text{Lemma } 3.1].$$

Let $u = g'_1 + \cdots + g'_k$. Then

$$\begin{aligned}\alpha(f_{x0}) &= \alpha\left(\sum \{f_{x'}e_i \mid x' \in x_0, i = 1, \ldots, n\}\right) \\ &= \sum \{h_z g'_j \mid z \in Z; j = 1, \ldots, k\} \\ &= \sum \{h_z u \mid z \in Z\}.\end{aligned}$$

Hence

$$\begin{aligned}\alpha(f_{x0} I(X, R) f_{x0}) &= \alpha(f_{x0}) I(Y, S) \alpha(f_{x0}) \\ &= \left(\sum h_z u\right) I(Y, S) \left(\sum h_z u\right) \\ &= \left(\sum h_z\right) I(Y, uSu) \left(\sum h_z\right). \quad \blacksquare\end{aligned}$$

Hence we see that the image of $f_x R$ under an isomorphism α is not contained in some $h_y S$ only if R is itself nontrivially an incidence ring if x_0 is a singleton. Define an indecomposable semiperfect ring R to be *IR-irreducible* if whenever T is a ring and Z is a partially ordered set with $R \cong I(Z, T)$, Z is a singleton (so also $T \cong R$). If R is IR-irreducible and $\alpha: I(X, R) \to I(Y, S)$ is an isomorphism with X partially ordered, then the subset Z of Y in Lemma 2 must be a singleton, so that there is an idempotent u of S with $R \cong uSu$. Furthermore, we have:

3. Proposition. *Let R be an IR-irreducible ring and X and Y finite partially ordered sets with $I(X, R) \cong I(Y, S)$. Then there exist an idempotent u of S with $R \cong uSu$ and a subset W of X such that $S \cong I(W, R)$ and $X \cong Y \times W$.*

Proof. Let $\alpha: I(X, R) \to I(Y, S)$ be an isomorphism with $\alpha(X \times \bar{R}) = Y \times \bar{S}$. First assume that S is idecomposable. If $x \in X$ then from Lemma 2 and the IR-irreducibility of R, there exist an element y of Y and an indempotent u of S with $\alpha(f_x R) = h_y uSu$. Apply Lemma 2 to $I(Y, S)$, y, and α^{-1} to obtain a subset W of X and an idempotent v of R with

$$\alpha^{-1}(h_y S) = \sum \{f_w \mid w \in W\} vRv$$

and $S \cong I(W, vRv)$. Then

$$f_x R = \alpha^{-1}\alpha(f_x R) \subseteq f_x vRv;$$

hence $v = 1$ and $S \cong I(W, R)$. Since

$$I(X, R) \cong I(Y, S) \cong I(Y, I(W, R)) \cong I(Y \times W, R),$$

we see that $X \cong Y \times W$ (16, Theorem 4.3].

If S is not indecomposable, write S as the product $\prod_{j=1}^{p} S_j$ of indecomposable rings S_j and X as the sum $\sqcup_{k=1}^{l} X_k$ of its connected components. Then

$$\prod_{k=1}^{l} I(X_k, R)$$

is the decomposition of $I(X, R)$ into indecomposable rings since for each k, $X_k \times \bar{R}$ is connected. For each factor S_j of S, there are unique components $X_1^j, \ldots, X_t^j$ of X with $I(Y, S_j) = \alpha(I(X^j, R))$, where $X^j = X_1^j \sqcup \cdots \sqcup X_t^j$; this gives a decomposition $\sqcup_{j=1}^{p} X^j$ of X (1, Theorem 7.9]. Hence for each j there is a subset W_j of X^j with $S_j \cong I(W_j, R)$ and $X^j \cong Y \times W_j$. Then

$$S = \prod S_j = \prod I(W_j, R) = I(\sqcup W_j, R),$$

and

$$X = \sqcup X^j \cong \sqcup (Y \times W_j) = Y \times (\sqcup W_j).$$

Since $W_j \subseteq X^j$, the W_j are disjoint so that $W_j \subseteq X$. ▮

4. COROLLARY. *If R and S are IR-irreducible rings with $I(R, X) \cong I(S, Y)$, then $R \cong S$ and $X \cong Y$.*

Proof. The partially ordered set W of Proposition 3 must be a singleton. ▮

Corollary 4 provides the uniqueness of the IR-irreducible decomposition of an indecomposable semiperfect ring R in the following proposition.

5. PROPOSITION. *If R is an indecomposable semiperfect ring, then there exist an IR-irreducible ring T and a finite partially ordered set Z such that $R \cong I(Z, T)$; T and Z are unique up to isomorphism.*

Proof. If R is not IR-irreducible, write $R = I(Y, S)$ with Y not a singleton. Since $\bar{R}$ is finite and $\bar{R} \cong Y \times \bar{S}$, the cardinality of $\bar{S}$ is less than that of $\bar{R}$, providing a bsis for induction. ▮

Consequently, although an incidence ring over an indecomposable semiperfect ring may be isomorphic to a second incidence ring with neither the ground rings nor the partially ordered sets isomorphic, we can decompose the ground rings and obtain isomorphisms.

6. THEOREM. *Let R be an indecomposable ring and X and Y finite partially ordered set. If $I(X, R) \cong I(Y, S)$, then there exist an IR-irreducible ring T, idempotents $v \in R$ and $u \in S$ with $T = vRv \cong uSu$, and preordered sets Z and W such that $R \cong I(Z, T)$, $S \cong I(W, T)$ and $X \times Z \cong Y \times W$.*

Proof. Let T be an IR-irreducible ring and Z a partially ordered set with $I(Z, T) \cong R$. Then

$$I(Y, S) \cong I(X, R) \cong I(X \times Z, T).$$

Proposition 3 yields a subset W of $X \times Z$ such that $S \cong I(W, T)$ and $X \times Z \cong W \times Y$. ∎

Immediately from Proposition 1 and the result of Lovász concerning cancellability of finite preordered sets [7], we may conclude that if R is semiperfect and X is a finite preordered set, then $I(X, R) \cong I(Y, R)$ implies $X \cong Y$. From Theorem 6 and Lovász's result we may conclude that if R is an indecomposable semiperfect ring and X is a finite partially ordered set, then $I(X, R) \cong I(X, S)$ implies $R \cong S$. The indecomposability assumption on R is unnecessary:

7. THEOREM. *If R is a semiperfect ring and X is a finite partially ordered set, then $I(X, R) \cong I(X, S)$ implies $R \cong S$.*

Proof. Let $\alpha: I(X, R) \to I(X, S)$ be an isomorphism with $\alpha(X \times \bar{R}) = X \times \bar{S}$. Write R as a product $\prod_{i=1}^{l} R_i$ of indecomposable semiperfect rings. With each R_i is associated an IR-irreducible ring T_i and a finite partially ordered set Z_i with $R_i \cong I(Z_i, T_i)$. Some (or all) of the T_i may be isomorphic; let $\{T_\gamma \mid \gamma \in A\}$ be a set of representatives of the isomorphism classes of the T_i. Collect the factors of R having associated IR-irreducible rings isomorphic to T_γ and let R_γ be their product. R_γ is isomorphic to $I(Z_\gamma, T_\gamma)$ where Z_γ is the sum of the partially ordered sets associated to the factors of R_γ. Let S_γ be the product of the indecomposable factors of S having T_γ as their associated IR-irreducible rings; we will show first that $\alpha(I(X, R_\gamma)) = I(X, S_\gamma)$. To this end, decompose X as the sum of its connected components $\sqcup X_k$ and $S = \prod S_j$ as the product of indecomposable rings. The factorization of $I(X, S)$ into indecomposable rings is given by $\prod I(X_k, S_j)$. Let $I(X_k, S_j)$ be a factor in the decomposition of $\alpha(I(X, R_\gamma))$. Then there is a component X_p of X and a factor R_i of R_γ with

$$I(X_k, S_j) = \alpha(I(X_p, R_i)) \cong I(X_p \times Z_i, T_\gamma).$$

Hence by Proposition 3 and Lemma 5, T_γ is the associated IR-irreducible ring of S_j. Thus $\alpha(I(X, R_\gamma)) \subseteq I(X, S_\gamma)$. Applying a similar argument to α^{-1} and $I(X, S_\gamma)$ shows that $\alpha^{-1}(I(X, S_\gamma)) \subseteq I(X, R_\gamma)$, so that $\alpha(I(X, R_\gamma)) = I(X, S_\gamma)$. It also is apparent that $S = \prod S_\gamma$. Now

$$I(X, S_\gamma) \cong I(X, R_\gamma) \cong I(X \times Z_\gamma, T_\gamma).$$

Again by Proposition 3, there exists $Y_\gamma \subseteq X \times Z_\gamma$ with

$$S_\gamma \cong I(Y_\gamma, T_\gamma) \quad \text{and} \quad X \times Y_\gamma \cong X \times Z_\gamma.$$

Cancel X; then $Y_\gamma \cong Z_\gamma$ so that $S_\gamma \cong R_\gamma$. Thus $S = \prod S_\gamma \cong \prod R_\gamma = R$. ∎

Of course, if there is an isomorphism α from $I(X, R)$ to $I(Y, S)$, the components R_γ and S_γ still satisfy $\alpha(I(X, R_\gamma)) = I(Y, S_\gamma)$. This would allow us to obtain a conclusion analogous to, albeit more complicated than, that of Theorem 6 in the case of arbitrary semiperfect rings.

Let us now turn to the structure of an automorphism α of a finite incidence ring $I(X, R)$ over an IR-irreducible ring R. First by applying the inverse of an inner automorphism β, we may obtain the automorphism $\beta^{-1}\alpha$ satisfying

$$\beta^{-1}\alpha(X \times \bar{R}) = X \times \bar{R},$$

yielding an order automorphism of $X \times \bar{R}$. Moreover, since R is IR-irreducible, for any $x \in X$ there is an $x' \in X$ with $\beta^{-1}\alpha(\{x\} \times \bar{R}) = \{x'\} \times \bar{R}$, so that $\beta^{-1}\alpha$ induces an order automorphism θ of X as well, via $\theta(x) = x'$. Let $\bar{\theta}$ be the ring automorphism of $I(X, R)$ induced by θ. Finally, let $x \leq y$ in X. Then

$$\bar{\theta}^{-1}\beta^{-1}\alpha(f_{xy}R) = f_{xy}R.$$

Let ϕ_{xy} be the additive map from R to R induced by the restriction of $\bar{\theta}^{-1}\beta^{-1}\alpha$ to $f_{xy}R$. The family $\{\phi_{xy} \mid x \leq y\}$ satisfies $\phi_{xz}(r)\phi_{zy}(s) = \phi_{xy}(rs)$ for $x \leq z \leq y$ and $r, s \in R$, since

$$\bar{\theta}^{-1}\beta^{-1}\alpha(f_{xz}r)\bar{\theta}^{-1}\beta^{-1}\alpha(f_{zy}s) = \bar{\theta}^{-1}\beta^{-1}\alpha(f_{xy}rs)$$

Conversely, any such family $\{\phi_{xy} \mid x \leq y\}$ of additive isomorphisms from R to R yields a ring automorphism ϕ of $I(X, R)$. Thus we have shown:

8. THEOREM. *Let R be an IR-irreducible ring and X a finite partially ordered set. A map α is an automorphism of $I(X, R)$ if and only if there exist an inner automorphism β of $I(X, R)$, an order automorphism θ of X, and a family $\{\phi_{xy} \mid x \leq y\}$ of additive automorphisms of R satisfying $\phi_{xz}(r)\phi_{zy}(s) = \phi_{xy}(rs)$ for $x \leq z \leq y$ and $r, s \in R$, such that $\alpha = \beta\bar{\theta}\phi$.*

This decomposition of α is not unique, for there may exist distinct automorphisms of $I(X, R)$ arising from additive isomorphisms of R satisfying the multiplicative conditions that compose to give an inner automorphism of $I(X, R)$; see the discussion and example in [14].

A ring R is said to be (Morita) dual to a ring S if there is a left R-right S-bimodule C that is a left R-injective cogenerator and a right S-injective cogenerator with $S \cong \text{End}({}_RC)$ and $R \cong \text{End}(C_S)$ canonically [9], [10], [15]. If $R \cong S$, R is said to be self-dual. It is known that if R is dual to S, then $I(X, R)$ is dual to $I(X, S)$ for any finite preordered set X. Until now, only partial results have been available concerning the converse [5]. The converse is true in general; we will now prove it. Since the basic ring of $I(X, R)$ is $I(X_0, R_0)$ where X_0 is the associated partially ordered set of X and R_0 is the basic ring of R, it is sufficient to consider basic rings and partially ordered sets in the proof of the following theorem.

9. THEOREM. *The incidence rings $I(X, R)$ and $I(X, S)$ are dual if and only if R and S are dual. In particular, $I(X, R)$ is self-dual if and only if R is self-dual.*

Proof. Only rings that are semiperfect admit Morita dualities [12], so as noted above it is sufficient to show that if X is a finite partially ordered set and R and S are basic, then $I(X, R)$ is dual to $I(X, S)$ only if R is dual to S. Let C be a minimal left injective cogenerator for R. Since R is a quotient ring of $I(X, R)$. R admits a duality; in fact, R is dual to End $({}_RC)$ [10]. Hence $I(X, R)$ is dual to $I(X, \text{End}\,({}_RC))$ [5, Corollary 3]. Because $I(X, \text{End}\,({}_RC))$ and $I(X, S)$ are each basic and dual to $I(X, R)$, $I(X, \text{End}\,({}_RC))$ and $I(X, S)$ are isomorphic [15, Proposition 1.5], [1, Proposition 27.14]. Then by Theorem 7, End $({}_RC) \cong S$ and R is dual to S. ▮

REFERENCES

1. F. W. ANDERSON and K. R. FULLER, *Rings and categories of modules*, Springer-Verlag, New York, 1974.
2. W. BELDING, *Incidence rings of pre-ordered sets*, Notre Dame J. Formal Logic, vol. 14 (1973), pp. 481–509.
3. P. DOUBILET, G. C. ROTA and R. P. STANLEY, *On the foundations of combinatorial theory (VI): The idea of generating function*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, vol. 2 (1970–71), pp. 267–318.
4. C. FAITH, *Algebra II: Ring theory*, Springer-Verlag, Berlin, 1976.
5. K. R. FULLER and J. K. HAACK, *Duality for semigroup rings*, J. Pure Appl. Algebra, vol. 22 (1981), pp. 113–119.
6. J. K. HAACK, *Incidence rings with self-duality*, Proc. Amer. Math. Soc., vol. 78 (1980), pp. 165–169.
7. L. LOVÁSZ, *Operations with structures*, Acta Math. Acad. Sci. Hungar., vol. 18 (1967), 321–328.
8. R. MCKENZIE, *Cardinal multiplication of structures with a reflexive relation*, Fund. Math., vol. 70 (1971), pp. 59–101.
9. K. MORITA, *Duality of modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, vol. 6 (1958), pp. 85–142.
10. B. J. MÜLLER, *Linear compactness and Morita duality*, J. Algebra, vol. 16 (1970), pp. 60–66.
11. N. A. NACHEV, *On incidence rings*, Vest. Mos. Univ. Mat. (Moscow Univ. Math Bull.), vol. 32 (1977), pp. 29–34.
12. B. L. OSOFSKY, *A generalization of quasi-Frobenius rings*, J. Algebra, vol. 4 (1966), pp. 272–287.
13. G. C. ROTA, *On the foundations of combinatorial theory (I): Theory of Mobius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, vol. 2 (1964), pp. 340–368.
14. R. P. STANLEY, *Structure of incidence algebras and their automorphism groups*, Bull. Amer. Math. Soc., vol. 76 (1970), pp. 1236–1239.
15. P. VÁMOS, *Rings with duality*, Proc. London Math. Soc., vol. 35 (1977), pp. 275–289.
16. E. R. VOSS, *On the isomorphism problem for incidence rings*, Illinois J. Math., vol. 24 (1980), pp. 624–638.

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