ELEMENTARY MAPPINGS INTO IDEALS OF OPERATORS

BY

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $n \geq 1$ and n-tuples of operators

$$A = (A_1, \ldots, A_n) \quad \text{and} \quad B = (B_1, \ldots, B_n),$$

let $R \equiv R(A, B)$ denote the *elementary operator* on $\mathcal{L}(\mathcal{H})$ defined by

$$R(X) = A_1 X B_1 + \cdots + A_n X B_n \quad [18].$$

This prescription includes several special cases of interest, e.g., the inner derivations δ_A ($X \to AX - XA$) [1], the left and right multiplications L_A and R_A ($X \to AX$, $X \to XA$), the generalized derivations $T(A, B)$ ($X \to AX - XB$) [9], and the elementary multiplication operators $S(A, B)$ ($X \to AXB$ [13].

In [11], C. K. Fong and A. R. Sourour described the case when Ran $(R(A, B))$, the range of the elementary operator $R(A, B)$, is contained in either the trivial ideal (0) or the ideal $\mathcal{K}(\mathcal{H})$ of all compact operators on $\mathcal{H}$. In considering the identity $R = 0$, Fong and Sourour reduce to the case when $\{B_1, \ldots, B_n\}$ is linearly independent, and show that in this case $R = 0$ if and only if $A_i = 0$ ($1 \leq i \leq n$) [11, Theorem 1]. Analogously, they show that to study the inclusion Ran $(R(A, B)) \subset \mathcal{K}(\mathcal{H})$, it suffices to consider the case when $\{B_1, \ldots, B_n\}$ is independent modulo $\mathcal{K}(\mathcal{H})$; in this case, Ran $(R(A, B)) \subset \mathcal{K}(\mathcal{H})$ if and only if $A_i \in \mathcal{K}(\mathcal{H})$ ($1 \leq i \leq n$) [11, Th. 3]. In [2], C. Apostol and L. Fialkow studied the problem of characterizing when the range of an elementary operator is contained in an arbitrary (two-sided) ideal of $\mathcal{L}(\mathcal{H})$. It is proved in [2, Theorem 1.1] that if $\{B_1, \ldots, B_n\}$ is independent modulo $\mathcal{K}(\mathcal{H})$ and $\mathcal{I}$ is a proper two-sided ideal of $\mathcal{L}(\mathcal{H})$, then Ran $(R(A, B)) \subset \mathcal{I}$ if and only if $A_i \in \mathcal{I}$ ($1 \leq i \leq n$). (It is not difficult to see that the hypothesis of independence modulo $\mathcal{K}(\mathcal{H})$ cannot be weakened to independence modulo $\mathcal{I}$ [2].)

The range inclusion problem for elementary operators with arbitrary coefficient sequences remains unsolved, but in the sequel we take a first step

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towards considering arbitrary coefficients by solving the problem for generalized derivations and elementary multiplication operators. As we will see below, in dealing with arbitrary coefficients, there is no analogue of the results of [11] and [2]; indeed, our results concerning the inclusion Ran $(S(A, B)) \subset \mathscr{I}$ will be expressed in terms of s-numbers of operators and ideal sets. Moreover, the proofs of our results are completely different in character from those of [11] and [2], which depend on D. Voiculescu's noncommutative Weyl-von Neumann theorem and its consequences [26].

The generalized derivations $T(A, B)$ have been much studied, initially in the framework of linear operator equations [19], and many of their spectral and metric properties are known [6], [7], [8], [22]. We consider some questions arising when the range of $T(A, B)$ is contained in an ideal $\mathscr{I}$. We show in Theorem 2.4 that T maps $\mathscr{L}(\mathscr{H})$ into $\mathscr{I}$ if and only if $A - \lambda$ and $B - \lambda$ belong to $\mathscr{I}$ for some scalar λ. In this case, if $\mathscr{I}$ is a norm ideal with norm $\| \ \|_{\mathscr{I}}$, then T induces an operator $\mathscr{T}_{\mathscr{I}} : \mathscr{L}(\mathscr{H}) \to \mathscr{I}$, defined by $\mathscr{T}_{\mathscr{I}}(X) = AX - XB$, which is bounded since

$$\|AX - XB\|_{\mathscr{I}} = \|(A - \lambda)X - X(B - \lambda)\|_{\mathscr{I}} \le (\|A - \lambda\|_{\mathscr{I}} + \|B - \lambda\|_{\mathscr{I}})\|X\|.$$

In Section 3 we study properties of $\mathscr{T}_{\mathscr{I}}$ that are analogous to properties of generalized derivations studied in [6], [7], [8], [11]. We show that $\mathscr{T}_{\mathscr{I}}(A, B)$ is compact if and only if $\mathscr{T}_{\mathscr{I}} = 0$, i.e., $A = B = \lambda$ (Proposition 3.1). We show that $\mathscr{T}_{\mathscr{I}}$ is neither surjective nor bounded below (Proposition 3.3., Proposition 3.4). For a collection of ideals $\mathscr{I}$ including the Schatten p-ideals $(1 \le p \le \infty)$, we prove that $\mathscr{T}_{\mathscr{I}}(A, B)$ has closed range if and only if $A - \lambda$ and $B - \lambda$ are finite rank operators for some scalar λ (Theorem 3.5). For the Schatten p-ideals we characterize when $\mathscr{T}_p \equiv \mathscr{T}_{C_p}(A, B)$ has dense range (Proposition 3.12).

Beginning in Section 4, we study the multiplication operator $S(A, B)$. In Theorem 5.6 we prove that the range of S is contained in a proper two-sided ideal $\mathscr{I}$ if and only if $s(A)s(B) \in J$, where $s(\cdot)$ denotes the sequence of s-numbers of an operator and J denotes the ideal set of $\mathscr{I}$ (see below for the definitions of these terms). Further, for the norm ideals C_{Φ} of [12], we estimate the norm of the induced operator $\mathscr{S}_{\mathscr{I}} : \mathscr{L}(\mathscr{H}) \to \mathscr{I}$; in particular, we prove that

$$\|\mathscr{S}_{C_p}(A, B)\| = \|s(A)s(B)\|_p \quad \text{(Theorem 5.7)}.$$

We also prove that $\mathscr{S}_p \equiv \mathscr{S}_{C_p}$ is compact if and only if A and B (as above) are compact (Theorem 6.2), and that $\mathscr{S}_I$ is neither surjective nor bounded below (Proposition 6.3, Proposition 6.5). At the end of the paper we discuss some open questions suggested by our results.

We next recall the description of the proper two sided ideals of $\mathscr{L}(\mathscr{H})$; our account is taken from [3] and [5]. Let $\mathscr{I}$ denote a proper two-sided ideal of $\mathscr{L}(\mathscr{H})$; thus $\mathscr{F} \subset \mathscr{I} \subset \mathscr{K}(\mathscr{H})$, where $\mathscr{F}$ denotes the ideal of all finite rank operators in $\mathscr{L}(H)$. If $K \in \mathscr{I}$, then $|K| = (K^*K)^{1/2}$ is diagonalizable; any sequence (indexed by the positive integers) consisting of all of the eigenvalues

of $|K|$, each repeated according to multiplicity, is a *characteristic sequence* for K.

An *ideal set* J is a collection of sequences of nonnegative real numbers (indexed by the positive integers) such that:

(i) if $\{a_n\}_{n=1}^{\infty} \in J$, then $a_n \geq 0$ for each $n \geq 1$, and $\lim_{n \to \infty} a_n = 0$;

(ii) if $\{a_n\} \in J$ and π is any permutation of the positive integers, then $\{a_{\pi(n)}\} \in J$;

(iii) if $\{a_n\}$ and $\{b_n\}$ are in J, then $\{a_n + b_n\}$ is in J;

(iv) if $\{a_n\}$ is in J, and if $0 \leq b_n \leq a_n$ for all n, then $\{b_n\}$ is in J.

If $\mathscr{I}$ is an ideal, then the collection of all characteristic sequences of all operators in $\mathscr{I}$ is an ideal set, called the ideal set of $\mathscr{I}$. Conversely, if J is an ideal set, the collection of all compact operators on $\mathscr{H}$ whose characteristic sequences are in J forms an ideal with ideal set J. This correspondence between two-sided ideals of $\mathscr{L}(\mathscr{H})$ and ideal sets is bijective and respects inclusion. Thus a compact operator K belongs to an ideal $\mathscr{I}$ if and only if some (equivalently, each) characteristic sequence of K belongs to the ideal set of $\mathscr{I}$.

We next define the sequence of s-numbers of an operator. Let

$$\mathscr{A}(\mathscr{H}) = \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$$

denote the Calkin algebra, and for T in $\mathscr{L}(\mathscr{H})$, let $\tilde{T}$ denote the image of T in $\mathscr{A}(\mathscr{H})$. Let $\|T\|_e = \|\tilde{T}\|$, the essential norm of T; thus $\|T\|_e = \||T|\|_e$, where $|T| = (T^*T)^{1/2}$. Let $\sigma(T)$ denote the spectrum of T. If $\alpha \in \sigma(|T|)$ and $\alpha > \|T\|_e$, then α is an isolated eigenvalue of $|T|$ with finite multiplicity. If there is no such point α in $\sigma(|T|)$, we define $s_n(T) = \|T\|_e$ $(n \geq 1)$. If the sequence of all such points is nonempty but finite, say $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_p$ (with each value repeated according to multiplicity), then $s_n(T) = \alpha_n$ $(1 \leq n \leq p)$ and $s_n(T) = \|T\|_e$ for $n > p$. If the sequence is infinite, $\alpha_1 \geq \alpha_2 \geq \cdots$, we define $s_n(T) = \alpha_n$ for $n \geq 1$. The sequence $s(T) = \{s_n(T)\}_{n=1}^{\infty}$ is the sequence of s-*numbers* of T, whose properties are described in detail in [12].

Our criterion for an operator T to belong to an ideal $\mathscr{I}$ is that some characteristic sequence for T belongs to the ideal set of $\mathscr{I}$. Since characteristic sequences of T are not necessarily monotone, they are more difficult to compute with than s-numbers, so we would like a criterion for ideal membership involving $s(T)$. Note that if T is compact, there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathscr{H}$ such that $|T|e_n = s_n(T)e_n$ $(n \geq 1)$ if and only if T is injective or T is a finite rank operator. Thus for T compact, $s(T)$ is a characteristic sequence for T if and only if T is injective or has finite rank. The following result is essentially contained in [5].

LEMMA 1.1. *Let $\mathscr{I}$ be a proper two-sided ideal of $\mathscr{L}(\mathscr{H})$. A compact operator T belongs to $\mathscr{I}$ if and only if $s(T)$ belongs to the ideal set of $\mathscr{I}$.*

Proof. Lemma 1.2 in [5] implies that if $s \in J$ (the ideal set of $\mathscr{I}$), then any sequence obtained from s by inserting finitely or infinitely many zero terms also belongs to J. For T compact, the preceding discussion shows that a characteristic sequence for T may be so produced from $s(T)$. Thus if $s(T) \in J$, then J contains a characteristic sequence for T, so $T \in \mathscr{I}$.

Conversely, Lemma 1.1 in [5] implies that if $s \in J$ has infinitely many positive terms, then the subsequence of s consisting of precisely these terms also belongs to J. Suppose $T \in \mathscr{I}$ is not a finite rank operator and T is not injective. Let $s \in J$ denote a characteristic sequence for T; since $s(T)$ is a permutation of the subsequence of s consisting of positive terms, it follows that $s(T) \in J$. If T is injective or if T is a finite rank operator, $s(T)$ is clearly a characteristic sequence for T, so $s(T) \in J$.

Let $(\mathscr{I}, \| \quad \|_{\mathscr{I}})$ denote a (symmetric) norm ideal of $\mathscr{L}(\mathscr{H})$ in the sense of [12, Ch. 3, page 68]. We recall certain properties of the norm:

   (i) $\|RXS\|_{\mathscr{I}} \leq \|R\| \|X\|_{\mathscr{I}} \|S\|$ for $R, S \in \mathscr{L}(\mathscr{H})$ and $X \in \mathscr{I}$;

  (ii) $\|X\|_{\mathscr{I}} = \|X\|$ if X is a rank one operator;

 (iii) $\|X\|_{\mathscr{I}} = \|X^*\|_{\mathscr{I}}$ for $X \in \mathscr{I}$;

 (iv) $\|X\| \leq \|X\|_{\mathscr{I}}$ for $X \in \mathscr{I}$;

  (v) $\|UXV\|_{\mathscr{I}} = \|X\|_{\mathscr{I}}$ for U, V unitary and $X \in \mathscr{I}$.

For $1 \leq p \leq \infty$, the Schatten p-ideal C_p is the ideal with ideal set l_p; C_p is a norm ideal under the norm $\|K\|_p = \|s(K)\|_p$. Note that $C_1 \subset C_p$ and $\|X\|_p \leq \|X\|_1$ for $p \geq 1$ and $X \in C_1$.

2. Generalized derivations mapping into ideals of $\mathscr{L}(\mathscr{H})$

We begin by describing the case when the range of a generalized derivation is contained in an ideal of $\mathscr{L}(\mathscr{H})$. To this end, note that the operator $T(A, B)$ determines A and B up to a scalar translation: that is, if A_1 and B_1 are operators such that $AX - XB = A_1X - XB_1$ for each X in $\mathscr{L}(\mathscr{H})$, then there exists a (unique) scalar λ such that $A_1 = A + \lambda$ and $B_1 = B + \lambda$ [11, Example 1]. Recall also that every derivation on $\mathscr{L}(\mathscr{H})$ is inner, i.e., if δ is a derivation, then there exists $B \in \mathscr{L}(\mathscr{H})$ such that $\delta(X) = BX - XB$ ($X \in \mathscr{L}(\mathscr{H})$) [16]. The following lemmas appear in [15] and are included here for completeness.

LEMMA 2.1. *An operator $T: \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ is a generalized derivation if and only if there exists an operator M in $\mathscr{L}(\mathscr{H})$ and a derivation δ on $\mathscr{L}(\mathscr{H})$ such that $T = L_M + \delta$.*

Proof. If $T = T(A, B)$, let $M = A - B$ and $\delta = \delta_B$. The converse follows from the above mentioned characterization of derivations on $\mathscr{L}(\mathscr{H})$.

LEMMA 2.2. *Let $\mathscr{I}$ be a two-sided ideal of $\mathscr{L}(\mathscr{H})$ and let $T = L_M + \delta_B$. If Ran $(T) \subset \mathscr{I}$, then Ran $(\delta_B) \subset \mathscr{I}$.*

Proof. If Ran $(T) \subset \mathscr{I}$, then $M = T(1) \in \mathscr{I}$, so Ran $(\delta_B) = $ Ran $(T - L_M) \subset \mathscr{I}$.

LEMMA 2.3. *If $B \in \mathscr{I}$ and Ran $(T(A, B)) \subset \mathscr{I}$, then $A \in \mathscr{I}$.*

Proof. Since $A - B = T(1) \in \mathscr{I}$, then $A = (A - B) + B \in \mathscr{I}$.

THEOREM 2.4. *Let $\mathscr{I}$ denote a proper two-sided ideal of $\mathscr{L}(\mathscr{H})$. Then Ran $(T(A, B)) \subset \mathscr{I}$ if and only if there exists a (unique) scalar λ such that $A - \lambda$ and $B - \lambda$ belong to $\mathscr{I}$.*

Proof. Suppose Ran $(T(A, B)) \subset \mathscr{I}$. Since $T = L_{A-B} + \delta_B$, Lemma 2.2 implies that Ran $(\delta_B) \subset \mathscr{I}$. Thus $\dot{B}$, the image of B in $\mathscr{L}(\mathscr{H})/\mathscr{I}$, is in the center of $\mathscr{L}(\mathscr{H})/\mathscr{I}$, so Theorem 2.9 in [5] implies that there exists a (unique) scalar λ such that $\dot{B} = \lambda$, i.e., $B - \lambda \in \mathscr{I}$. Since $T(A, B) = T(A - \lambda, B - \lambda)$, Lemma 2.3 implies that $A - \lambda \in \mathscr{I}$. The converse is obvious.

COROLLARY 2.5 [5, Theorem 2.9], [3, Theorem 4.3]. Ran $(\delta_B) \subset \mathscr{I}$ *if and only if there exists a scalar λ such that $B - \lambda \in \mathscr{I}$.*

Remark. M. Hoffman [14], working in the context of multiplier ideals and essential commutants, has obtained several refinements of Corollary 2.5 as follows:

(i) If Ran $(\delta_T | C_2) \subset C_1$, then there exists a scalar λ such that $T - \lambda \in C_2$ [14, Corollary 5.8].

(ii) If Ran $(\delta_T | \mathscr{K}(\mathscr{H})) \subset C_p$ for some $p \geq 1$, then there is a scalar λ such that $T - \lambda \in C_p$ [14, Corollary 5.9].

(iii) If $\mathscr{I}$ is an ideal properly containing $\mathscr{F}$ and Ran $(\delta_T | \mathscr{I}) \subset \mathscr{F}$, then there exists a scalar λ such that $T - \lambda \in \mathscr{F}$ [14, Corollary 5.4].

A consequence of the last result, or of Corollary 2.5, is that if Ran $(\delta_T) \subset \mathscr{I}$ for every ideal $\mathscr{I}$ properly containing $\mathscr{F}$, then there exists a scalar λ such that $T - \lambda \in \mathscr{F}$. This is because the intersection of all such ideals $\mathscr{I}$ is precisely $\mathscr{F}$ [4, Corollary 4.7].

3. Properties of the induced operator $\mathscr{T}_{\mathscr{I}}$

We again consider the case when Ran $(T(A, \cdot B)) \subset \mathscr{I}$ (a proper ideal of $\mathscr{L}(\mathscr{H})$). By virtue of Theorem 2.4 we may assume (as we shall do in the sequel) that A and B belong to $\mathscr{I}$. We now proceed to study properties of the induced operator $\mathscr{T}_{\mathscr{I}}: \mathscr{L}(\mathscr{H}) \to \mathscr{I}$ in the case when $\mathscr{I}$ is a norm ideal with norm $\| \ \|_{\mathscr{I}}$. In [11, Theorem 2], Fong and Sourour characterized the compact elementary operators; in particular, they showed that $T(A, B)$ is compact if and only if $T(A, B) = 0$ [11, Example 1].

PROPOSITION 3.1. $\mathscr{T}_{\mathscr{I}}(A, B)$ *is compact if and only if $A = B = 0$.*

Proof. The inclusion mapping $P: (\mathscr{I}, \| \quad \|_{\mathscr{I}}) \to \mathscr{L}(\mathscr{H})$ is continuous since the $\mathscr{I}$-norm dominates the operator norm. If $\mathscr{T}_{\mathscr{I}}$ is compact, it follows that $T(A, B) = \mathscr{P}\mathscr{T}_{\mathscr{I}}(A, B)$ is also compact, and thus $A = B = 0$. The converse is trivial.

COROLLARY 3.2. *$\mathscr{T}_{\mathscr{I}}$ is compact if and only if $\mathscr{T}_{\mathscr{I}} = 0$.*

Proof. The result follows from Proposition 3.1 and [11, Example 1].

For operators A and B in $\mathscr{L}(\mathscr{H})$, let $\sigma_r(A) = \{\lambda \in: A - \lambda$ is not right invertible (surjective)$\}$ and let $\sigma_l(B) = \{\lambda \in: B - \lambda$ is not left invertible (bounded below)$\}$. It is known that $\mathscr{T}(\mathscr{A}, \mathscr{B})$ is surjective if and only if $\sigma_r(A) \cap \sigma_l(B) = \phi$ and that this condition is equivalent to the right invertibility of $T(A, B)$ in $\mathscr{L}(\mathscr{L}(\mathscr{H}))$ [6, Theorem 3.2].

PROPOSITION 3.3. *$\mathscr{T}_{\mathscr{I}}$ is not surjective.*

Proof. Suppose to the contrary that $\mathscr{T}_{\mathscr{I}}$ is surjective; then

$$\mathscr{F} \subset \mathscr{I} = \text{Ran} (\mathscr{T}_{\mathscr{I}}) \subset \text{Ran} (T(A, B)).$$

It follows from [6, Theorem 2.1] that $T(A, B)$ is surjective, which is clearly impossible.

An extension of this result to arbitrary elementary operators is given in Proposition 6.3.

$T(A, B)$ is bounded below if and only if $\sigma_l(A) \cap \sigma_r(B) = \phi$ and this condition is equivalent to the left invertibility of $T(A, B)$ in $\mathscr{L}(\mathscr{L}(\mathscr{H}))$ [6, Theorem, 3.5].

PROPOSITION 3.4. *$\mathscr{T}_{\mathscr{I}}(A, B)$ is not bounded below.*

Proof. Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal basis for $\mathscr{H}$; since A and B are compact, $Ae_n \to 0$ and $B^*e_n \to 0$. For $n \geq 1$, define the operator $X_n \in \mathscr{L}(\mathscr{H})$ by $X_n(v) = (v, e_n)e_n$; X_n is a rank one projection. Since

$$(AX_n - X_n B)(v) = (v, e_n)Ae_n - (v, B^*e_n)e_n,$$

then

$$\|AX_n - X_n B\|_{\mathscr{I}} \leq \|AX_n\|_{\mathscr{I}} + \|X_n B\|_{\mathscr{I}} = \|(\cdot, e_n)Ae_n\|_{\mathscr{I}} + \|(\cdot, B^*e_n)e_n\|_{\mathscr{I}}$$

$$= \|Ae_n\| + \|B^*e_n\| \to 0.$$

Since $\|X_n\| = 1$, it follows that $\mathscr{T}_{\mathscr{I}}$ is not bounded below.

In [1], C. Apostol proved that an inner derivation δ_B has closed range in $\mathscr{L}(\mathscr{H})$ if and only if B is similar to a Jordan model. Some results are known concerning the case when $T(A, B)$ has closed range (beyond the general results cited above for the cases when T is surjective or bounded below). In [8, Theorem 4.6] it is proved that if A and B are compact, then $T(A, B)$ has closed range if and only if A and B are finite rank operators. We next

provide an analogue for the operator $\mathcal{T}_{\mathcal{J}}$: the proof of the following result is modeled on that of [8].

THEOREM 3.5. *Let $\mathcal{J}$ be a norm ideal of $\mathcal{L}(\mathcal{H})$ and let A and B denote operators in $\mathcal{J}$.*

(i) *If A and B are finite rank operators, then $\mathcal{T}_{\mathcal{J}}(A, B)$ has closed range in $\mathcal{J}$.*

(ii) *Suppose $C_1 \subset \mathcal{J}$ and $\|X\|_{\mathcal{J}} \leq \|X\|_{\mathcal{J}}$ for $X \in C_1$. $\mathcal{T}_{\mathcal{J}}(A, B)$ has closed range in $\mathcal{J}$ if and only if A and B are finite rank operators.*

COROLLARY 3.6. *$\mathcal{T}_p$ has closed range if and only if A and B are finite rank operators $(1 \leq p \leq \infty)$.*

Proof. C_p satisfies the hypothesis of Theorem 3.5 (ii).

Before proving Theorem 3.5 we require some additional notation. Let $\mathcal{X}$ and $\mathcal{Y}$ denote Banach spaces and let $R: \mathcal{X} \to \mathcal{Y}$ denote a bounded linear operator. Let $\gamma(R) = \inf \{\|Rx\|: x \in \mathcal{X}, \text{dist}(x, \ker(R)) \geq 1\}$. It is known that R has closed range if and only if $\gamma(R) > 0$ [17, Chapter IV, Theorem 5.2]. In the sequel we consider the case when $\mathcal{X} = \mathcal{L}(\mathcal{H})$, $\mathcal{Y} = \mathcal{J}$ (a norm ideal), and $R = \mathcal{F}_{\mathcal{J}}(A, B)$, where $A, B \in \mathcal{J}$. Let

$$\mathcal{U}(\mathcal{H}) = \{U \in \mathcal{L}(\mathcal{H}): U \text{ is unitary}\}.$$

The proof of Theorem 3.5 will be given via a sequence of lemmas.

The referee observed the following: If $R \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ and S_i is a linear or conjugate-linear isometry of $\mathcal{X}_i$ onto itself, then $\gamma(R) = \gamma(S_2 R S_1)$. The proof is elementary.

LEMMA 3.7. *If $U \in \mathcal{U}(\mathcal{H})$, then $\gamma(\mathcal{T}_{\mathcal{J}}(U^*AU, B)) = \gamma(\mathcal{T}_{\mathcal{J}}(A, B))$.*

Proof. Apply the referee's observation with $S_1 = L_u$, $S_2 = L_{u^*}$.

LEMMA 3.8. *$\gamma(\mathcal{T}_{\mathcal{J}}(A, B)) = \gamma(\mathcal{T}_{\mathcal{J}}(B^*, A^*))$.*

Proof. Apply the referee's observation with $S_1 = S_2 = $ conjugation, $X \to X^*$.

LEMMA 3.9. *If there exists $\{U_n\} \subset \mathcal{U}(\mathcal{H})$ and $A' \in \mathcal{J}$ such that $\|U_n^*AU_n - A'\|_{\mathcal{J}} \to 0$, then $\gamma(\mathcal{T}_{\mathcal{J}}(A', B)) = \gamma(\mathcal{T}_{\mathcal{J}}(A, B))$.*

Proof. For $\gamma \geq 0$, let $\mathcal{L}_\gamma = \{R \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathcal{J}): \gamma(R) \geq \gamma\}$. By a straightforward modification of [1, Lemma 1.9], it can be seen that $\mathcal{L}_\gamma$ is norm-closed

in $\mathscr{L}(\mathscr{L}(\mathscr{H}), \mathscr{I})$. Let $\gamma = \gamma(\mathscr{T}_{\mathscr{I}}(A, B))$; from Lemma 3.7 we see that $\gamma(\mathscr{T}_{\mathscr{I}}(U_n^* A U_n, B)) = \gamma$ for $n \geq 1$. For $X \in \mathscr{L}(\mathscr{H})$,

$$\|\mathscr{T}_{\mathscr{I}}(A', B)(X) - \mathscr{T}_{\mathscr{I}}(U_n^* A U_n, B)(X)\|_{\mathscr{I}} = \|(A' - U_n^* A U_n)X\|_{\mathscr{I}}$$

$$\leq \|A' - U_n^* A U_n\|_{\mathscr{I}} \|X\|,$$

and thus $\mathscr{T}_{\mathscr{I}}(U_n^* A U_n, B) \to \mathscr{T}_{\mathscr{I}}(A', B)$ in $\mathscr{L}(\mathscr{L}(\mathscr{H}), \mathscr{I})$. Since $\mathscr{L}_{\gamma}$ is closed, it follows that $\gamma(\mathscr{T}_{\mathscr{I}}(A', B)) \geq \gamma = \gamma(\mathscr{T}_{\mathscr{I}}(A, B))$. Observe that

$$\|U_n A' U_n^* - A\|_{\mathscr{I}} = \|A' - U_n^* A U_n\|_{\mathscr{I}} \to 0;$$

the above argument thus implies that $\gamma(\mathscr{T}_{\mathscr{I}}(A, B)) \geq \gamma(\mathscr{T}_{\mathscr{I}}(A', B))$, completing the proof.

LEMMA 3.10.   *If A and B are finite rank operators, then $\mathscr{T}_{\mathscr{I}}(A, B)$ has closed range.*

Proof. Suppose $\{X_n\} \subset \mathscr{L}(\mathscr{H})$, $K \in \mathscr{I}$, and $\|AX_n - X_n B - K\|_{\mathscr{I}} \to 0$. Since the $\mathscr{I}$-norm dominates the usual operator norm, K is in the closure of the range of $T(A, B)$. Since A and B are finite rank operators, Corollary 4.3 of [8] implies that $T(A, B)$ has closed range. Thus there exists $X \in \mathscr{L}(\mathscr{H})$ such that $AX - XB = K$, and it follows that $\mathscr{T}_{\mathscr{I}}(A, B)$ has closed range.

LEMMA 3.11.   (i)   *If $\ker(B) \cap \ker(B^*) \neq \{0\}$ and the range of A is not closed, then the range of $\mathscr{T}_{\mathscr{I}}(A, B)$ is not closed.*

(ii)   *If $\ker(A) \cap \ker(A^*) \neq \{0\}$ and the range of B^* is not closed, then the range of $\mathscr{T}_{\mathscr{I}}(A, B)$ is not closed.*

Proof. Observe that (ii) follows from (i) via an application of Lemma 3.8 and the above mentioned fact that $\mathscr{T}_{\mathscr{I}}$ has closed range if and only if $\gamma(\mathscr{T}_{\mathscr{I}}) > 0$. To prove (i), we rely on the proof of Lemma 4.4 in [8]. Under the hypothesis of (i), this proof exhibits a sequence $\{V_n\}$ of rank one operators, and a rank one operator W, such that $\|AV_n - V_n B - W\| \to 0$, $W \notin \text{Ran}(T(A, B))$, and such that $AV_n - V_n B - W$ is a rank one operator. Thus

$$\|\mathscr{T}_{\mathscr{I}}(V_n) - W\|_{\mathscr{I}} = \|AV_n - V_n B - W\| \to 0,$$

and it follows that $\mathscr{T}_{\mathscr{I}}$ does not have closed range.

Proof of Theorem 3.5.   The proof of (i) is the content of Lemma 3.10, and Lemma 3.10 also yields one direction of (ii).

For the converse direction of (ii), suppose first that A is not a finite rank operator. Since $A \in \mathscr{I}$, A is compact and thus A does not have closed range. Since B is also compact, $0 \in R_e(B)$ (the set of reducing essential eigenvalues of

B [20]), and so Theorem 4.6 in [20] implies that there exists $\{U_n\} \subset \mathscr{U}(\mathscr{H})$ and there exists a unitary operator $V: \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ such that

$$U_n^* B U_n - V^*(B \oplus 0)V \in C_1 \quad \text{for } n \geq 1,$$

and such that

$$\|U_n^* B U_n - V^*(B \oplus 0)V\|_1 \to 0.$$

Hence $U_n^* B U_n - V^*(B \oplus 0)V \in \mathscr{I}$ and

$$\|[U_n^* B U_n - V^*(B \oplus 0)V]^*\|_{\mathscr{I}} = \|U_n^* B U_n - V^*(B \oplus 0)V\|_{\mathscr{I}}$$

$$\leq \|U_n^* B U_n - V^*(B \oplus 0)V\|_1 \to 0.$$

Lemmas 3.8 and 3.9 now imply that

$$\gamma(\mathscr{T}_{\mathscr{I}}(A, B)) = \gamma(\mathscr{T}_{\mathscr{I}}(B^*, A^*)) = \gamma(\mathscr{T}_{\mathscr{I}}(V^*(B^* \oplus 0)V, A^*)).$$

Since $B \oplus 0$ has an infinite dimensional reducing kernel

$$(\ker(B \oplus 0) \cap \ker(B^* \oplus 0)),$$

so does $V^*(B^* \oplus 0)V$, and since the range of A is not closed, Lemma 3.11(ii) implies that the range of $\mathscr{T}_{\mathscr{I}}(V^*(B^* \oplus 0)V, A^*)$ is not closed. Thus $\gamma(\mathscr{T}_{\mathscr{I}}(A, B)) = 0$ and it follows that $\mathscr{T}_{\mathscr{I}}(A, B)$ does not have closed range. In the case when B is not a finite rank operator, apply the preceding case and Lemma 3.8 to $\mathscr{T}_{\mathscr{I}}(B^*, A^*)$.

Let $\sigma_{\mathrm{re}}(\cdot)$ and $\sigma_{\mathrm{le}}(\cdot)$ denote right and left essential spectra. In [7, Theorem 1.1] it was shown that $T(A, B)$ has norm dense range $\mathscr{L}(\mathscr{H})$ if and only if $\sigma_{\mathrm{re}}(A) \cap \sigma_{\mathrm{le}}(B) = \phi$ and there exists no nonzero trace class operator K such that $BK = KA$. (A corresponding result for arbitrary elementary operators was recently obtained in [10].) Here we consider the case when $\mathscr{T}_{\mathscr{I}}$ has dense range.

PROPOSITION 3.12. *For* $1 \leq p \leq \infty$, *let* $A, B \in C_p$.

(i) *If* $1 < p < \infty$, $\mathscr{T}_p(A, B)$ *has dense range if and only if there is no nonzero operator* $K \in C_q$ $(1/p + 1/q = 1)$ *such that* $BK = KA$.

(ii) *If* $p = \infty$, $\mathscr{T}_p(A, B)$ *has dense range if and only if there is no nonzero operator* $K \in C_1$ *such that* $BK = KA$.

(iii) *If* $p = 1$, $\mathscr{T}_1(A, B)$ *has dense range if and only if there is no nonzero operator* $K \in \mathscr{L}(\mathscr{H})$ *such that* $BK = KA$.

Proof. (i) The range of $\mathscr{T}_p$ is not dense if and only if

$$\mathscr{T}_p^*: C_p^* \to \mathscr{L}(\mathscr{H})^*$$

is not injective. It is well known that C_p^* is isometrically isomorphic to C_q, where $1/p + 1/q = 1$ [12, Chapter III, Theorem 12.3]; further, for $J \in C_p$ and $K \in C_q$, we have $JK \in C_1$ [12, page 92], and if we define $f_K(J) = \mathrm{tr}\,(JK)$

(where tr $(\cdot)$ denotes the trace function), then $f_K \in C_p^*$ and $\|f_K\| = \|K\|_q$ [12, Theorem 12.3]. Under this identification,

$$\mathscr{T}_p^*(f_K)(X) = f_K(\mathscr{T}_p(X)) = \text{tr} \, ((AX - XB)K) \quad \text{for each } X \in \mathscr{L}(\mathscr{H}).$$

Hence the range of $\mathscr{T}_p$ is not dense in C_p if and only if there exists a nonzero operator $K \in C_q$ such that $\text{tr} \, ((AX - XB)K) = 0$ for every $X \in \mathscr{L}(\mathscr{H})$. Since AX and XB are in C_p and K is in C_q, then AXK and XBK are trace class operators; moreover, since A is in C_p and XK is in C_q, then Theorem 8.2 in [12] implies that $\text{tr} \, (AXK) = \text{tr} \, (XKA)$. Thus

$$0 = \text{tr} \, (AXK - XBK) = \text{tr} \, (AXK) - \text{tr} \, (XBK) = \text{tr} \, (XKA) - \text{tr} \, (XBK)$$

$$= \text{tr} \, (X(KA - BK)) = \text{tr} \, ((KA - BK)X) \quad (X \in \mathscr{L}(\mathscr{H})).$$

Now Lemma 1 in Chapter IV of [21] implies that $KA = BK$, so the proof of (i) is complete.

The proofs of (ii) and (iii) follow similar arguments, using the identifications $C_\infty^* = C_1$ and $C_1^* = \mathscr{L}(\mathscr{H})$ [12], [21].

The preceding result has a close analogue concerning the operators $T_p(A, B) = T(A, B) | C_p \colon C_p \to C_p$ $(1 \le p \le \infty)$.

PROPOSITION 3.13. *Let A and B be in $\mathscr{L}(\mathscr{H})$.*

(i) *For $1 < p \le \infty$, T_p has dense range if and only if there exists no nonzero operator $K \in C_q$, $1/p + 1/q = 1$, such that $BK = KA$.*

(ii) *T_1 has dense range if and only if there is no nonzero operator $K \in \mathscr{L}(\mathscr{H})$ such that $BK = KA$.*

Proof. Calculations similar to those above show that

$$T_p(A, B)^* = -T_q(B, A) \, (1 < p \le \infty) \quad \text{and} \quad T_1(A, B)^* = -T(B, A).$$

Remark. It was shown in [7, Proposition 4.1] that $T_1(A, B)$ has dense range if and only if $T_p(A, B)$ has dense range for every p, $1 \le p \le \infty$. This result has an analogue for the operators $\mathscr{T}_p(A, B)$. Indeed, it follows easily from Proposition 3.12 that if A and B are in C_1, then $\mathscr{T}_1(A, B)$ has dense range if and only if $\mathscr{T}_p(A, B)$ has dense range for each p, $1 \le p \le \infty$. More generally, it follows from Proposition 3.12 that if A and B are in C_p, $p \ge 1$, and $\mathscr{T}_p(A, B)$ has dense range, then $\mathscr{T}_{p'}(A, B)$ has dense range for every $p' \ge p$.

For the norm ideals $\mathscr{I} = C_\phi^{(0)}$ of [12], results analogous to Proposition 3.12 can be obtained using the description of the dual space of $\mathscr{I}$ given in [12, Chapter III, Theorem 12.2]; we leave the formulation of such results to the interested reader. For arbitrary norm ideals we have the following observation.

COROLLARY 3.14. *Let $\mathcal{I}$ be a norm ideal and let A and B be in $\mathcal{I}$. If $\mathcal{T}_{\mathcal{I}}(A, B)$ has dense range, then $\mathcal{T}_{\infty}$ has dense range (equivalently, there exists no nonzero trace class operator K such that $BK = KA$).*

Proof. Let $C \in \mathcal{K}(\mathcal{H})$, and let $\{K_n\}$ be a sequence of finite rank operators such that $\|K_n - C\| \to 0$. Since $\mathcal{T}_{\mathcal{I}}$ has dense range and $\{K_n\} \subset \mathcal{F} \subset \mathcal{I}$, there exists a sequence $\{X_n\} \subset \mathcal{L}(\mathcal{H})$ such that

$$\|AX_n - X_n B - K_n\|_{\mathcal{I}} \to 0,$$

and thus

$$\|AX_n - X_n B - C\| \leq \|AX_n - X_n B - K_n\| + \|K_n - C\|$$

$$\leq \|AX_n - X_n B - K_n\|_{\mathcal{I}} + \|K_n - C\| \to 0.$$

Thus $\mathcal{T}_{\infty}$ has dense range and the result follows from Proposition 3.12 (ii).

We remark that Proposition 3.12 (i) readily implies that the converse of Corollary 3.14 is false.

4. s-numbers of bounded operators

Let $\mathcal{I}$ denote a proper two-sided ideal of $\mathcal{L}(\mathcal{H})$. If A and B are in $\mathcal{L}(\mathcal{H})$ and Ran $(S(A, B)) \subset \mathcal{I}$, we define the linear mapping $\mathcal{S}_{\mathcal{I}}(A, B): \mathcal{L}(\mathcal{H}) \to \mathcal{I}$ by $\mathcal{S}_{\mathcal{I}}(X) = AXB$ $(X \in \mathcal{L}(\mathcal{H}))$. In the case when $\mathcal{I}$ is a norm ideal with norm $\| \quad \|_{\mathcal{I}}$, the boundedness of $\mathcal{S}_{\mathcal{I}}$, considered as an operator from $\mathcal{L}(\mathcal{H})$ to $(\mathcal{I}, \| \quad \|_{\mathcal{I}})$, follows from the Closed Graph Theorem, and we will compute the norm of $\mathcal{S}_{C_p}$ $(1 \leq p \leq \infty)$. Our first result, which sets the stage for the sequel, is a special case of the Fong-Sourour characterization of the inclusion Ran $(R(A, B)) \subset \mathcal{K}(\mathcal{H})$ [11, Theorem 3].

PROPOSITION 4.1. *Ran $(S(A, B)) \subset \mathcal{K}(\mathcal{H})$ if and only if $A \in \mathcal{K}(\mathcal{H})$ or $B \in \mathcal{K}(\mathcal{H})$.*

Proof. Suppose neither A nor B is compact; then there exist closed infinite dimensional subspaces $\mathcal{M}$ and $\mathcal{N}$ in the ranges of A and B respectively. Let V be a partial isometry with initial space $\mathcal{N}$ and final space $\mathcal{H}$; clearly AVB is not compact, since its range contains $\mathcal{M}$. The converse is trivial.

It is perhaps natural to conjecture that Ran $(S(A, B))$ is contained in an ideal $\mathcal{I}$ if and only if A or B belongs to $\mathcal{I}$. This supposition is false, however, for if $A \in C_p$ and $B \in C_q$, where $1/p + 1/q = 1$, then Ran $(S(A, B)) \subset C_1$ without A or B necessarily belonging to C_1 [12, page 92]. We intend to show that this example truely represents the general situation.

In order to establish our results, we must introduce some facts about the s-numbers of bounded (noncompact) operators, which were defined in the introduction. Most of the following results are modifications of results in

[12] concerning s-numbers of compact operators; as suggested in [12, page 64], these results are probably in the literature, but we could not find a reference. We refer the reader to [12, Chapter II] for the basic facts about the s-numbers of bounded operators; in particular we note that the s-numbers of A and A^* coincide, and the s-numbers of A coincide with the s-numbers of $|A|$. In the sequel $s_n(A)$ denotes the nth s-number of an operator A ($1 \leq n < \infty$) and $s_\infty(A) = \lim_{n \to \infty} s_n(A)$; thus $s_\infty(A) = \|A\|_e$.

LEMMA 4.2.   *Let A be in $\mathcal{L}(\mathcal{H})$ and let $\varepsilon > 0$. There exists an operator $B \in \mathcal{L}(\mathcal{H})$ and an orthonormal sequence $\{e_n\}_{n=1}^\infty \subset \mathcal{H}$ such that* (i) *$\|B - A\| \leq \varepsilon$;* (ii) *$B^*B$ is diagonalizable;* (iii) *$s_n(B) = s_n(A)$ for each $n \geq 1$; and* (iv) *$B^*Be_n = s_n(B)^2 e_n$ for all n.*

Proof. If A is compact, let $B = A$; the existence of $\{e_n\}$ satisfying (iv) follows from the diagonalizability of $|A|$. If A is not compact, then $s_\infty(A) = s_\infty(|A|) > 0$. Let $A = U|A|$ denote the polar decomposition of A, and let $E(\cdot)$ denote the spectral measure of $|A|$. Let

$$0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n = s_\infty(A)$$

define a partition of $[0, s_\infty(A)]$ such that $\alpha_{i+1} - \alpha_i < \varepsilon$ for $1 \leq i \leq n - 1$. We set $\mathcal{H}_i = E([\alpha_i, \alpha_{i+1}))\mathcal{H}$ for $1 \leq i \leq n - 2$; $\mathcal{H}_{n-1} = E([\alpha_{n-1}, \alpha_n])$; and $\mathcal{H}_\infty = E((\alpha_n, \infty))\mathcal{H}$.

Define an operator P on $\mathcal{H}$ by $P|\mathcal{H}_1 = 0$; $P|\mathcal{H}_i = \alpha_{i+1}1_{\mathcal{H}_i}$ for $2 \leq i \leq n - 1$; and $P|\mathcal{H}_\infty = |A||\mathcal{H}_\infty$. Let $B = UP$; since $\alpha_{i+1} - \alpha_i < \varepsilon$ for each i, it follows that $\|A - B\| = \|U(|A| - P)\| \leq \||A| - P\| \leq \varepsilon$. By the construction, we have $U^*UP = P$, and $B^*B = P^2$ is diagonalizable since $|A||\mathcal{H}_\infty$ is diagonalizable by definition of $s_\infty(|A|)$. It now follows easily that $s_n(B) = s_n(|B|) = s_n(|A|) = s_n(A)$ for each $n \geq 1$.

If $\dim(\mathcal{H}_\infty) = \infty$, then there exists an infinite orthonormal sequence $\{e_n\}$ in $\mathcal{H}_\infty$ such that $|A|e_n = s_n(A)e_n$, and thus

$$Pe_n = |A|e_n = s_n(A)e_n = s_n(B)e_n,$$

so $B^*Be_n = P^2e_n = s_n(B)^2 e_n$ ($n \geq 1$). If $\dim(\mathcal{H}_\infty) < \infty$, then $\dim(\mathcal{H}_{n-1}) = \infty$; choosing $\{e_n\}$ to be an orthonormal basis of $\mathcal{H}_{n-1}$, then

$$Pe_n = s_\infty(A)e_n = s_n(A)e_n$$

for all sufficiently large n, and the result follows.

LEMMA 4.3.   *Let $A \in \mathcal{L}(\mathcal{H})$ and let $x_1, \ldots, x_n$ be vectors in $\mathcal{H}$. Then*

$$\det \|(Ax_j, Ax_k)\|_{1 \leq j, k \leq n} \leq s_1(A)^2 \cdots s_n(A)^2 \det \|(x_j, x_k)\|_{1 \leq j, k \leq n}.$$

Proof. For A compact, this is [12, Chapter II, Lemma 3.1], originally due to H. Weyl [25]. However, the keys to the proof given in [12] are that (i)

$|A|$ is diagonalizable, and that (ii) there exists an orthonormal basis $\{e_i\}$ of $\mathscr{H}$, consisting of eigenvectors of $|A|$, such that $|A|e_i = s_i(A)e_i$ $(1 \le i \le n)$.

By Lemma 4.2, we can find operators B_k $(k \ge 1)$, such that

$$\|B_k - A\| \to 0, \quad s_i(B_k) = s_i(A) \quad \text{for } 1 \le i \le n \text{ and } k \ge 1,$$

and such that each B_k satisfies (i) and (ii) above. Hence, for each i,

$$\det \|(B_i x_j, B_i x_k)\| \le s_1(A)^2 \cdots s_n(A)^2 \det \|(x_j, x_k)\|,$$

and the result follows from the continuity of the determinant.

LEMMA 4.4. *Let $A \in \mathscr{L}(\mathscr{H})$ and let $\varepsilon > 0$. There exists an orthonormal sequence $\{e_i\}_{i=1}^{\infty} \subset \mathscr{H}$ such that $\|A^*Ae_i - s_i(A)^2 e_i\| \le \varepsilon$ for each $i \ge 1$.*

Proof. We retain the notation used in the proof of Lemma 4.2. Thus $E(\cdot)$ denotes the spectral measure of $|A|$. If dim $(\mathscr{H}_\infty) = \infty$, then there is an orthonormal sequence $\{e_i\}_{i=1}^{\infty} \subset \mathscr{H}_\infty$ such that $|A|e_i = s_i(A)e_i$ and hence $A^*Ae_i = |A|^2 e_i = s_i(A)^2 e_i$ for $i \ge 1$.

If dim $(\mathscr{H}_\infty) = m < \infty$, let $e_1, \ldots, e_m$ be an orthonormal basis for $\mathscr{H}_\infty$ such that $A^*Ae_i = |A|^2 e_i = s_i(A)^2 e_i$ for $1 \le i \le m$. For any $\delta > 0$,

$$\mathscr{H}_\delta \equiv E([s_\infty(A) - \delta, s_\infty(A)])\mathscr{H}$$

is infinite dimensional. Let $\{e_{m+1}, e_{m+2}, \ldots\}$ be an orthonormal basis for $\mathscr{H}_\delta$. Then $\| |A|e_{m+i} - s_{m+i}(A)e_{m+i}\| \le \delta$, and thus

$$\begin{aligned}
\|A^*Ae_{m+i} - s_{m+i}(A)^2 e_{m+i}\| &= \| |A|^2 e_{m+i} - s_{m+i}(A)^2 e_{m+i}\| \\
&\le \| |A|^2 e_{m+i} - |A|s_{m+i}(A)e_{m+i}\| \\
&\quad + \| |A|s_{m+i}(A)e_{m+i} - s_{m+i}(A)^2 e_{m+i}\| \\
&\le \| |A| \|\delta + s_{m+i}(A)\delta \le 2\|A\|\delta < \varepsilon
\end{aligned}$$

for all $i \ge 1$ if δ is sufficiently small.

LEMMA 4.5. *Let $A, B \in \mathscr{L}(\mathscr{H})$ and let $n \ge 1$. Then*

$$\prod_{j=1}^{n} s_j(AB) \le \prod_{j=1}^{n} s_j(A) \prod_{j=1}^{n} s_j(B).$$

Proof. For A and B compact, this is [12, Chapter II, Lemma 4.2]. If

$$\{e_1, \ldots, e_n\}$$

is an orthonormal sequence of n vectors in $\mathscr{H}$, then by Lemma 4.3 we obtain

$$\det \|(ABe_j, ABe_k)\| \leq s_1(A)^2 \cdots s_n(A)^2 \det \|(Be_j, Be_k)\|$$

$$\leq [s_1(A)^2 \cdots s_n(A)^2] s_1(B)^2 \cdots s_n(B)^2 \det \|(e_j, e_k)\|$$

$$= \prod_{j=1}^{n} s_j(A)^2 \prod_{j=1}^{n} s_j(B)^2.$$

To complete the proof, it suffices to show that if $\varepsilon > 0$, then there exists an orthonormal sequence $\{e_1, \ldots, e_n\}$ such that

$$\prod_{j=1}^{n} s_j(AB)^2 - \varepsilon \leq \det \|(ABe_j, ABe_k)\|.$$

Let T denote the $n \times n$ diagonal matrix $\mathrm{diag}\,(s_1(AB)^2, \ldots, s_n(AB)^2)$, thought of as an operator on $\mathbf{C}^n$. Given $\varepsilon > 0$, let $\delta > 0$ be such that if S is an $n \times n$ matrix and $\|S - T\| < \delta$, then $\det(T) - \varepsilon \leq \det(S)$. By Lemma 4.4, there is an orthonormal sequence $e_1, \ldots, e_n$ such that

$$\|(AB)^*(AB)e_j - s_j(AB)^2 e_j\| < \delta/n^2 \quad \text{for } 1 \leq j \leq n.$$

Let S be the matrix whose row j, column k entry is $((AB)^*(AB)e_j, e_k)$. A straightforward calculation shows that $\|T - S\| < \delta$, and thus

$$\prod_{j=1}^{n} s_j(AB)^2 - \varepsilon = \det(T) - \varepsilon \leq \det(S) = \det \|(ABe_j, ABe_k)\|.$$

The proof is now complete.

PROPOSITION 4.6.   *For $A, B \in \mathscr{L}(\mathscr{H})$ and $1 \leq p < \infty$,*

$$\sum_{j=1}^{\infty} s_j(AB)^p \leq \sum_{j=1}^{\infty} s_j(A)^p s_j(B)^p.$$

Proof. For A and B compact, this is a special case of [12, Theorem 4.2], and for $p = 1$ the general case (A and B bounded) appears in [12, page 63].

From Lemma 4.5 we have $\prod_{j=1}^{n} s_j(AB) \leq \prod_{j=1}^{n} s_j(A)s_j(B)$, and hence

$$\sum_{j=1}^{n} \log\,(s_j(AB)) \leq \sum_{j=1}^{n} \log\,(s_j(A)s_j(B)) \quad \text{for } n \geq 1.$$

We now apply to this inequality and to the convex function $\phi(x) = (e^x)^p$ Lemma 3.4 of [12, Chapter II] to conclude that

$$\sum_{j=1}^{n} s_j(AB)^p \leq \sum_{j=1}^{n} s_j(A)^p s_j(B)^p \quad (n \geq 1),$$

so the result follows.

5. A Characterization of the inclusion Ran $(S(A, B)) \subset \mathscr{J}$

We now study conditions for the range of $S(A, B)$ to be contained in a proper two-sided ideal $\mathscr{J}$ of $\mathscr{L}(\mathscr{H})$. In this case, if $\mathscr{J}$ is a norm ideal with norm $\| \quad \|_{\mathscr{J}}$, then $\mathscr{S}_{\mathscr{J}}$ is bounded. For suppose $\{(X_n, AX_n B)\}$ is norm convergent in $\mathscr{L}(\mathscr{H}) \oplus \mathscr{J}$ to an element (X, Y); thus

$$\|X_n - X\| \to 0 \quad \text{and} \quad \|AX_n B - Y\|_{\mathscr{J}} \to 0.$$

Since $\|AX_n B - Y\| \leq \|AX_n B - Y\|_{\mathscr{J}}$ for each n, it follows that $Y = AXB$, and the Closed Graph Theorem implies that $\mathscr{S}_{\mathscr{J}}(A, B)$ is bounded. Our first result refines Proposition 4.1.

LEMMA 5.1. *If B is not compact and the range of $S(A, B)$ is contained in $\mathscr{J}$, then $A \in \mathscr{J}$.*

Proof. Since B is not compact, there exists an infinite dimensional subspace $\mathscr{N}$ such that $B|\mathscr{N}$ is bounded below. Let $V: B(\mathscr{N}) \to \mathscr{H}$ denote a partial isometry which maps $B(\mathscr{N})$ onto $\mathscr{H}$. Clearly VB is surjective, hence right invertible, and since $A(VB) \in \mathscr{J}$, it follows that $A \in \mathscr{J}$.

COROLLARY 5.2. *If A is not compact and the range of $S(A, B)$ is contained in $\mathscr{J}$, then $B \in \mathscr{J}$.*

Proof. Apply Lemma 5.1 to $S(B^*, A^*)$.

In the sequel, J denotes the ideal set of $\mathscr{J}$. We next recall the norm ideals C_Φ of [12]. Let c_0 denote the real space of all sequences of real numbers converging to 0 and let $\hat{c}$ denote the subspace of all sequences with a finite number of nonzero terms. Let Φ denote a *symmetric norming function* on $\hat{c}$ in the sense of [12, Chapter III, page 71]. Let c_Φ denote the *natural domain* of Φ, i.e.,

$$c_\Phi = \{a = \{a_n\} \in c_0 : \sup_n \Phi((a_1, a_2, \ldots, a_n, 0, 0, \ldots)) < \infty\};$$

for $a \in c_\Phi$, we set $\Phi(a) = \sup_n \Phi((a_1, \ldots, a_n, 0, 0, \ldots))$. Let C_Φ denote the set of all compact operators X in $\mathscr{L}(\mathscr{H})$ for which $s(X) \in c_\Phi$, and for each such operator, define $\|X\|_\Phi = \Phi(s(X))$. Then $(C_\Phi, \| \quad \|_\Phi)$ is a (symmetric) norm ideal [12, Chapter III, Theorem 4.1]; the Schatten p-ideals C_p correspond to the case $\Phi(a) = \|a\|_p$ $(1 \leq p \leq \infty)$. We shall now see how the s-numbers of A and B are related to the range of $S(A, B)$.

LEMMA 5.3. *Suppose there exist orthonormal sequences $\{e_n\}_{n=1}^\infty$ and $\{f_n\}_{n=1}^\infty$ in $\mathscr{H}$ such that $|A|e_n = s_n(A)e_n$ and $|B|f_n = s_n(B)f_n$ for each $n \geq 1$. If the range of $S(A, B)$ is contained in $\mathscr{J}$, then $s(A)s(B) \in J$; moreover, if $\mathscr{J} = C_\Phi$, then $\|\mathscr{S}_{C_\Phi}\| \geq \Phi(s(A)s(B))$.*

Proof. Let $A = U|A|$ and $B = V|B|$ denote the polar decompositions of A and B respectively. Let P denote the orthogonal projection onto span $\{f_1, f_2, \ldots\}$; and let W be a partial isometry such that $Wf_n = e_n$ (and $W^*e_n = f_n$) for each n.

Since the range of $S(A, B)$ is contained in $\mathcal{J}$, then

$$T = W^*U^*(AWV^*B)P \in \mathcal{J};$$

further, for $n \geq 1$,

$$
\begin{aligned}
Tf_n &= W^*U^*AW(V^*B)f_n \\
&= W^*(U^*A)W|B|f_n \\
&= W^*|A|Ws_n(B)f_n \\
&= W^*|A|s_n(B)e_n \\
&= W^*s_n(A)s_n(B)e_n \\
&= s_n(A)s_n(B)f_n.
\end{aligned}
$$

Also, since $T(1 - P) = 0$, it follows that $s_n(T) = s_n(A)s_n(B)$ $(n \geq 1)$, and so $s(A)s(B) = s(T) \in J$ (Lemma 1.1).

In the case $\mathcal{J} = C_\Phi$,

$$\Phi(s(A)s(B)) = \Phi(s(T)) = \|T\|_\Phi \leq \|W^*U^*\| \, \|AWV^*B\|_\Phi \|P\| \leq \|A(WV^*)B\|_\Phi.$$

Since $\|WV^*\| \leq 1$, it follows that $\|\mathscr{S}_{C_\Phi}\| \geq \|A(WV^*)B\|_\Phi \geq \Phi(s(A)s(B))$.

LEMMA 5.4. *Let A and B be in $\mathscr{L}(\mathscr{H})$. If $\mathrm{Ran}\,(S(A, B)) \subset \mathcal{J}$, then $s(A)s(B) \in J$. Moreover, if $\mathcal{J} = C_\Phi$, then $\|\mathscr{S}_{C_\Phi}\| \geq \Phi(s(A)s(B))$.*

Proof. If A and B are compact, the result follows from Lemma 5.3. Suppose that A is not compact; Corollary 5.2 implies that $B \in \mathcal{J}$. Lemma 4.2 implies that there exists a sequence $\{A_k\}_{k=1}^\infty \subset \mathscr{L}(\mathscr{H})$ such that (i) $\|A_k - A\| \to 0$, (ii) $s_n(A_k) = s_n(A)$ for all n, $k \geq 1$, (iii) for each $k \geq 1$, there exists an orthonormal sequence $\{e_n^{(k)}\}_{n=1}^\infty$ such that

$$|A_k|e_n^{(k)} = s_n(A)e_n^{(k)} \quad (n \geq 1).$$

Since $B \in \mathcal{J}$, the range of $S(A_k, B)$ is contained in $\mathcal{J}$, and Lemma 5.3 implies that $s(A)s(B) = s(A_k)s(B) \in J$.

For the case when $\mathcal{J} = C_\Phi$, we have

$$\|\mathscr{S}_{C_\Phi}(A_k, B) - \mathscr{S}_{C_\Phi}(A, B)\| \leq \|A_k - A\| \, \|B\|_\Phi \to 0;$$

since $\|\mathscr{S}_{C_\Phi}(A_k, B)\| \geq \Phi(s(A)s(B))$ for each k (Lemma 5.3), it follows that $\|\mathscr{S}_{C_\Phi}(A, B)\| \geq \Phi(s(A)s(B))$. The proof for the case when B is not compact is similar, using Lemma 5.1.

We now would like to establish a converse to Lemma 5.4.

LEMMA 5.5. *If $s(A)s(B) \in J$, then* Ran $(S(A, B)) \subset \mathscr{J}$. *Moreover, if $\mathscr{J} = C_\Phi$, then $\|\mathscr{S}_{C_\Phi}(A, B)\| \le 2\Phi(s(A)s(B))$.*

Proof. To prove that the range of $S(A, B)$ is contained in $\mathscr{J}$, it suffices to show that for each X in $\mathscr{L}(\mathscr{H})$, $s(AXB) \in J$ (Lemma 1.1).

Corollary 2.2 in Chapter 2 of [12] and the remarks of [12, page 62] imply that for $A, X, B \in \mathscr{L}(\mathscr{H})$,

$$s_{2n-1}(AXB) \le s_n(A)s_n(XB)$$

and

$$s_{2n}(AXB) \le s_{n+1}(A)s_n(XB) \le s_n(A)s_n(XB) \quad (n \ge 1).$$

Further, from [12, pages 27 and 61], $s_n(XB) \le \|X\|s_n(B)$ $(n \ge 1)$, so it follows that

(i) $s_{2n-1}(AXB) \le s_n(A)s_n(B)\|X\|$ $(n \ge 1)$,
(ii) $s_{2n}(AXB) \le s_n(A)s_n(B)\|X\|$ $(n \ge 1)$.

Lemma 1.2 in [5] implies that if $(a_1, a_2, a_3, \ldots)$ is in J, then so are the sequences

$$(a_1, 0, a_2, 0, a_3, 0, \ldots) \quad \text{and} \quad (0, a_1, 0, a_2, 0, a_3, 0, \ldots);$$

for $\alpha > 0$, it follows that $(\alpha a_1, \alpha a_1, \alpha a_2, \alpha a_2, \alpha a_3, \alpha a_3, \ldots)$ is in J. Since $s(A)s(B) \in \mathscr{J}$, the sequence

$$(\|X\|s_1(A)s_1(B), \|X\|s_1(A)s_1(B), \|X\|s_2(A)s_2(B), \|X\|s_2(A)s_2(B), \ldots)$$

belongs to J, so (i) and (ii) imply that $s(AXB)$ is in J, and Lemma 1.1 implies that AXB is in $\mathscr{J}$.

In the case when $\mathscr{J} = C_\Phi$ and $J = c_\Phi$, (i), (ii), and the properties of Φ [12, page 71] imply that for $k \ge 1$, there exist integers $m, p, r \ge 1$ such that

$\Phi(s_1(AXB), \ldots, s_k(AXB), 0, 0, \ldots)$

$$\le \Phi(\|X\|s_1(A)s_1(B), \|X\|s_1(A)s_1(B), \ldots, \|X\|s_m(A)s_m(B), 0, 0, \ldots)$$

$$\le \|X\|(\Phi(s_1(A)s_1(B), 0, s_2(A)s_2(B), 0, \ldots, s_p(A)s_p(B), 0, 0, \ldots)$$

$$+ \Phi(0, s_1(A)s_1(B), 0, s_2(A)s_2(B), 0, \ldots, s_r(A)s_r(B), 0, 0, \ldots))$$

$$= \|X\|(\Phi(s_1(A)s_1(B), s_2(A)s_2(B), \ldots, s_p(A)s_p(B), 0, 0, \ldots)$$

$$+ \Phi(s_1(A)s_1(B), s_2(A)s_2(B), \ldots, s_r(A)s_r(B), 0, 0, \ldots))$$

$$\le 2\Phi(s(A)s(B))\|X\|.$$

Thus $\|\mathscr{S}_{C_\Phi}(A, B)\| \le 2\Phi(s(A)s(B))$ and the proof is complete.

We now present our characterization of the range inclusion

$$\text{Ran}\ (S(A, B)) \subset \mathscr{J}.$$

THEOREM 5.6.   *Let A and B be in $\mathscr{L}(\mathscr{H})$. The range of S(A, B) is contained in the proper two-sided ideal $\mathscr{J}$ if and only if s(A)s(B) belongs to the ideal set of $\mathscr{J}$. In this case, if $\mathscr{J} = C_\Phi$, then*

$$\Phi(s(A)s(B)) \leq \|\mathscr{S}_{C_\Phi}\| \leq 2\Phi(s(A)s(B)).$$

Proof.   The result follows from Lemma 5.4 and Lemma 5.5.

For the Schatten p-ideals C_p $(1 \leq p \leq \infty)$ we are able to calculate the norm of $\mathscr{S}_p$ $(=\mathscr{S}_{C_p}(A, B))$ precisely.

THEOREM 5.7.   *Suppose $1 \leq p \leq \infty$. The range of S(A, B) is contained in C_p if and only if s(A)s(B) is in l_p; in this case,*

$$\|\mathscr{S}_p(A, B)\| = \|s(A)s(B)\|_p.$$

Proof.   By using Theorem 5.6, it suffices to show that

$$\|\mathscr{S}_p(A, B)\| \leq \|s(A)s(B)\|_p.$$

Suppose first that $p < \infty$. Theorem 5.6 implies that $s(A)s(B) \in l_p$. For $X \in \mathscr{L}(\mathscr{H})$, Proposition 4.6 implies that

$$\sum_{n=1}^\infty s_n(AXB)^p \leq \sum_{n=1}^\infty s_n(A)^p s_n(XB)^p \leq \sum_{n=1}^\infty s_n(A)(\|X\|^p s_n(B)^p)$$

$$\leq \|X\|^p \sum_{n=1}^\infty s_n(A)^p s_n(B)^p.$$

Thus $\|AXB\|_p \leq \|s(A)s(B)\|_p \|X\|$ and the result follows in this case. For $p = \infty$ and $X \in \mathscr{L}(\mathscr{H})$, we have

$$\|AXB\|_\infty = \|AXB\| \leq \|A\|\|B\|\|X\| = s_1(A)s_1(B)\|X\| = \|s(A)s(B)\|_\infty \|X\|,$$

and the result follows.

Remark.   The proof of Theorem 5.7 relies on Proposition 4.6, which in turn relies on convexity properties of the function e^{px}. There are other ideals C_Φ for which the conclusion of Theorem 5.7 is valid, namely, those for which the function Φ has the desired functional properties. We have not as yet determined whether Theorem 5.7 can be extended to all of the C_Φ ideals. More generally, the calculation of the norm of $\mathscr{S}_I$ for an arbitrary norm ideal $\mathscr{J}$ is an open problem.

The following example shows how the inclusion Ran $(S(A, B)) \subset \mathscr{J}$ may occur in a nontrivial way.

Example 5.8.   We exhibit operators A and B such that $A, B \notin C_p$ for every p, $1 \leq p < \infty$, but the range of $S(A, B)$ is contained in C_1. Let $\{a_n\}$ and $\{b_n\}$ denote monotone decreasing sequences in c_0^+ such that $\{a_n\}, \{b_n\} \notin l_p$ for every $p \geq 1$, but $\{a_n b_n\} \in l_1$. (A proof of the existence of such sequences was shown to us by J. D. Nelson.) Let A and B denote operators (e.g., diagonal-

izable operators) such that $s_n(A) = a_n$ and $s_n(B) = b_n$ $(n \geq 1)$. Theorem 5.7 implies that A and B satisfy our requirements. Note that A and B are compact; Lemma 5.1 and Corollary 5.2 imply that this is necessarily the case in an example of this type.

In the last example, the fact that Ran $(S(A, B))$ was contained in every C_p ideal did not impose on A or B membership in any C_p class. In contrast, we have the following complements to Proposition 4.1.

COROLLARY 5.9.   If Ran $(S(A, B)) \subset \mathscr{F}$, then A or B is in $\mathscr{F}$.

Proof. Theorem 5.6 implies that $s(A)s(B)$ has only a finite number of nonzero terms, so the same must be true for $s(A)$ or $s(B)$. (A simple proof independent of Theorem 5.6 is also easy to construct.)

COROLLARY 5.10.   If Ran $(S(A, B)) \subset \mathscr{J}$ for every ideal properly containing $\mathscr{F}$, then A or B is in $\mathscr{F}$.

Proof.   Apply [4, Corollary 4.7] and Corollary 5.9.

We conjecture that the property, Ran $(S(A, B)) \subset \mathscr{J}$ if and only if A or B is in $\mathscr{J}$, holds only for the ideals (0), $\mathscr{F}$, and $\mathscr{K}(\mathscr{H})$. Example 5.8 shows that this conjecture is valid for the C_p ideals $(1 \leq p \leq \infty)$.

6. Properties of the induced operator $\mathscr{S}_{\mathscr{J}}$

The Fong–Sourour characterization of the compact elementary operators [11] includes the result of K. Vala [24] that $S(A, B)$ is compact if and only if A and B are compact. Thus the operator $\mathscr{S}_\infty(A, B)$ (which is defined if and only if A or B is compact) is a compact operator if and only if both A and B are compact. Our first goal is to obtain an analogous result for the operators $\mathscr{S}_p(A, B)$ for $1 \leq p < \infty$. In the sequel we assume $1 \leq p < \infty$ and $s(A)s(B) \in l_p$, so that $\mathscr{S}_p(A, B)$ is defined (Theorem 5.7). The following result was shown to us by P. Eenigenburg.

LEMMA 6.1.   Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbf{R}^+$ such that $a_n \downarrow 0$, $b_n \downarrow 0$, and $\sum_{n=1}^\infty a_n b_n < \infty$. If $\{n_k\}_{k=0}^\infty$ is a strictly increasing sequence of positive integers, then

$$\lim_{k \to \infty} \left(\sum_{n=1}^\infty a_n b_{n_k+n} \right) = 0.$$

Proof.   Let $f_k(n) = a_n b_{n_k+n}$; since $\{b_n\}$ is decreasing, then $f_k(n) \geq f_{k+1}(n)$ for all, k, $n \geq 1$. Since $\lim_{n \to \infty} b_n = 0$, then $f_k \to 0$ pointwise in Z^+. Since $\sum_{n=1}^\infty$

$a_n b_{n_k+n} \le \sum_{n=1}^{\infty} a_n b_n < \infty$, the Monotone Convergence Theorem (applied to Z^+ with counting measure) implies that

$$\lim_{k \to \infty} \left(\sum_{n=1}^{\infty} a_n b_{n_k+n} \right) = \sum_{n=1}^{\infty} \left(\lim_{k \to \infty} a_n b_{n_k+n} \right) = 0.$$

THEOREM 6.2. *Let* $1 \le p < \infty$. $\mathscr{S}_p$ *is compact if and only if A and B are compact.*

Proof. Since $\|X\| \le \|X\|_p$ for every X in C_p, it follows easily that if $\mathscr{S}_p(A, B)$ is compact, then $S(A, B)$ is compact, and thus A and B are compact [11], [24].

Conversely, assume that A and B are compact operators with $s(A)s(B) \in l_p$; then $s_n(A) \downarrow 0$, $s_n(B) \downarrow 0$, and $\sum_{n=1}^{\infty} s_n(A)^p s_n(B)^p < \infty$. Lemma 6.1 implies that

(*) $$\lim_{k \to \infty} \left(\sum_{n=1}^{\infty} s_n(A)^p s_{n+k}(B)^p \right) = 0.$$

For $k \ge 1$, since $s_n(A) \downarrow 0$, there exists $n_k > 0$ such that

(**) $$s_{n_k+1}(A)^p s_1(B)^p + \cdots + s_{n_k+k}(A)^p s_k(B)^p < 1/k^p.$$

Let $A = U|A|$ and $B = V|B|$ denote the polar decompositions of A and B respectively. Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal sequence in $\mathscr{H}$ such that $|B|e_n = s_n(B)e_n$ for each n. For $k \ge 1$, define a finite rank operator Q_k as follows: $Q_k e_i = s_i(B)e_i$ $(1 \le i \le k)$; $Q_k x = 0$ if $(x, e_i) = 0$ for $1 \le i \le k$. Let $B_k = VQ_k$. Let $\{f_n\}_{n=1}^{\infty}$ denote an orthonormal sequence in $\mathscr{H}$ such that $|A|f_n = s_n(A)f_n$ for each n. For $k \ge 1$ define a finite rank operator P_k as follows: $P_k f_i = s_i(A)f_i$ $(1 \le i \le n_k)$; $P_k x = 0$ if $(x, f_i) = 0$ for $1 \le i \le n_k$. Now let $A_k = UP_k$.

Since A_k and B_k are finite rank operators,

$$\mathscr{S}_p(A_k, B_k), \quad \mathscr{S}_p(A_k - A, B_k) \quad \text{and} \quad \mathscr{S}_p(A, B_k - B)$$

are defined; moreover, it is easily verified that the range of $\mathscr{S}_p(A_k, B_k)$ is at most $k \cdot n_k$-dimensional, so that $\mathscr{S}_p(A_k, B_k)$ is a finite rank operator and perforce compact. To show that $\mathscr{S}_p(A, B)$ is compact, it thus suffices to prove that $\lim_{k \to \infty} \|\mathscr{S}_p(A_k, B_k) - \mathscr{S}_p(A, B)\| = 0$. For $X \in \mathscr{L}(\mathscr{H})$,

$\|\mathscr{S}_p(A_k, B_k)(X) - \mathscr{S}_p(A, B)(X)\|$

$$\le \|\mathscr{S}_p(A_k - A, B_k)(X)\| + \|\mathscr{S}_p(A, B_k - B)(X)\|$$

$$\le \|s(A_k - A)s(B_k)\|_p \|X\| + \|s(A)s(B_k - B)\|_p \|X\| \quad \text{(by Theorem 5.7)}$$

$$= \|X\| \left(\sum_{n=1}^{k} s_{n_k+n}(A)^p s_n(B)^p \right)^{1/p} + \|X\| \left(\sum_{n=1}^{\infty} s_n(A)^p s_{n+k}(B)^p \right)^{1/p},$$

and the desired convergence now follows from these inequalities via (*) and (**).

In [10] it was shown that an elementary operator on $\mathscr{L}(\mathscr{H})$ is surjective if and only if it is right invertible and bounded below if and only if it is left invertible. Moreover, the left and right spectra of elementary operators were calculated by R. Harte [13]. Using the results of [10] and [13] we show that $\mathscr{S}_{\mathscr{J}}$ is neither bounded below nor surjective.

PROPOSITION 6.3. *Let $\mathscr{J}$ be a proper two-sided ideal of $\mathscr{L}(\mathscr{H})$. The range of an elementary operator on $\mathscr{L}(\mathscr{H})$ does not coincide with $\mathscr{J}$.*

Proof. If Ran $(R(A, B)) = \mathscr{J}$, then $\mathscr{F} \subset$ Ran (R), so Theorem 2.3 in [10] implies that Ran $(R(A, B)) = \mathscr{L}(\mathscr{H})$, a contradiction.

COROLLARY 6.4. $\mathscr{S}_{\mathscr{J}}(A, B)$ *is not surjective.*

PROPOSITION 6.5. $\mathscr{S}_{\mathscr{J}}(A, B)$ *is not bounded below.*

Proof. Proposition 4.1 implies that A or B is compact. If A is compact, A is not bounded below, so there exists a sequence $\{X_n\}$ of rank one partial isometries such that $\|AX_n\| \to 0$. Since rank $(AX_n) \le 1$,

$$\|AX_n B\|_{\mathscr{J}} \le \|AX_n\|_{\mathscr{J}} \|B\| = \|AX_n\| \|B\| \to 0,$$

so $\mathscr{S}_{\mathscr{J}}$ is not bounded below. Similarly, if B is compact, there exists a sequence $\{V_n\}$ of rank one partial isometries such that $\|V_n B\| \to 0$, and thus $\|AV_n B\|_{\mathscr{J}} \le \|A\| \|V_n B\|_{\mathscr{J}} = \|A\| \|V_n B\| \to 0$, so the result follows.

7. Conclusion

We wish to discuss briefly some open questions suggested by our results. Concerning the range inclusion problem Ran $(R(A, B)) \subset \mathscr{J}$, we may assume that $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ are each dependent modulo $\mathscr{K}(\mathscr{H})$ [2]. Let us consider the case when A and B have this property and consist of mutually commuting compact operators; we may further assume that $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_n\}$ are each independent modulo $\mathscr{J}$. Under these conditions, if Ran $(R) \subset \mathscr{J}$, does it follow that Ran $(S(A_i, B_i)) \subset \mathscr{J}$ for $1 \le i \le n$? If so, then Theorem 5.6 implies that $s(A_i)s(B_i) \in J$ (the ideal set of $\mathscr{J}$) for each i. Theorem 5.6 provides the converse implication: if each $s(A_i)s(B_i)$ belongs to J, then Ran $(R) \subset \mathscr{J}$. Using the methods of Section 5, we are able to provide an affirmative answer to the preceding question if, in addition to the above hypotheses, we assume that the coefficient operators are positive.

Despite this evidence, we are able to show that the answer to the above question is negative in general. Indeed, for $\mathscr{J} = C_1$ and $n = 2$, we will exhibit

mutually commuting compact normal operators A_1, A_2, B_1, B_2 such that $\{A_1, A_2\}$ is independent mod $\mathscr{J}$, $\{B_1, B_2\}$ is independent mod $\mathscr{J}$,

$$\text{Ran } (S(A_1, B_1) + S(A_2, B_2)) \subset \mathscr{J},$$

but

$$\text{Ran } (S(A_i, B_i)) \nsubseteq \mathscr{J} \quad \text{for } i = 1, 2.$$

Thus the range inclusion problem remains open for $n = 2$.

Let $\{e_n\}_{n=1}^\infty$ denote an orthonormal basis for $\mathscr{H}$. Let M and N denote the diagonalizable normal operators on $\mathscr{H}$ defined as follows:

$$Me_n = (1/n^{1/2})e_n \ (n \geq 1); \quad Ne_n = (1/n)e_n \ (n \geq 1).$$

Consider the following mutually commuting compact operators on $\mathscr{H}' = \mathscr{H} \oplus \mathscr{H}$:

$$A_1 = M \oplus N, \quad B_1 = N \oplus M, \quad A_2 = -M \oplus 0_{\mathscr{H}}, \quad B_2 = -N \oplus M.$$

Let R denote the elementary operator on $\mathscr{H}'$ defined by

$$R(X) = A_1 X B_1 + A_2 X B_2.$$

It is straightforward to verify that $\{A_1, A_2\}$ is independent modulo C_1 and that $\{B_1, B_2\}$ is independent modulo C_1. Let

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

denote the operator matrix of an operator X in $\mathscr{L}(\mathscr{H}')$. A matrix calculation shows that the matrix of $R(X)$ is of the form

$$\begin{pmatrix} 2MX_{11}N & 0 \\ NX_{21}N & NX_{22}N \end{pmatrix};$$

since $s(M)s(N) \in l_1$ and $s(N)s(N) \in l_1$, Theorem 5.7 implies that each component of the above matrix is trace class, and it follows that $R(X) \in C_1$; thus Ran $(R) \subset C_1$. On the other hand, the matrix of $A_2 X B_2$ is equal to

$$\begin{pmatrix} MX_{11}N & -MX_{12}M \\ 0 & 0 \end{pmatrix};$$

since $M^2 \notin C_1$, if we set $X_{11} = 0$ and $X_{12} = 1$, it follows that $A_2 X B_2 \notin C_1$. Thus

$$\text{Ran } (S(A_i, B_i)) \nsubseteq C_1 \quad \text{for } i = 1, 2.$$

(Note also that $s(A_2)s(B_2) \notin l_1$ despite the fact that $A_2 B_2 \in C_1$.)

The preceding example warrants further examination. Let

$$A_1' = 2M \oplus 0, \quad B_1' = N \oplus 0, \quad A_2' = 0 \oplus N \quad \text{and} \quad B_2' = N \oplus N.$$

Note that for each $X \in \mathscr{L}(\mathscr{H}')$,

$$A_1' X B_1' + A_2' X B_2' = A_1 X B_1 + A_2 X B_2 = R(X).$$

Thus $R(A, B) = S(A_1', B_1') + S(A_2', B_2')$, and it is clear from Theorem 5.6 that Ran $(S(A_i', B_i')) \subset C_1$ for $i = 1, 2$. We are thus led to the following question.

Question 7.1. Let $R(A, B)$ denote an elementary operator on $\mathscr{L}(\mathscr{H})$ and let $\mathscr{J}$ denote a proper two-sided ideal of $\mathscr{L}(\mathscr{H})$. If Ran $(R(A, B)) \subset \mathscr{J}$, do there exist an integer $p \geq 1$ and operators $A_1', \ldots, A_p', B_1', \ldots, B_p'$, such that

$$R(A, B) = \sum_{i=1}^{p} S(A_i', B_i')$$

and such that

$$\text{Ran } (S(A_i', B_i')) \subset \mathscr{J}$$

for $1 \leq i \leq p$?

In view of Theorem 5.6 and [2, Theorem 1.1], an affirmative answer to Question 7.1 would solve the range inclusion problem.

In [22], J. Stampfli determined the norm of the generalized derivation $T(A, B)$, and [6] contains estimates of the norm of $\delta_A | \mathscr{J} \in \mathscr{L}(\mathscr{J})$. What is the norm of $\mathscr{T}_{\mathscr{J}}(A, B)$? In the case of inner derivations, we can obtain an estimate as follows. If Ran $(\delta_A) \subset \mathscr{J}$, let $\delta_{\mathscr{J}} \colon \mathscr{L}(\mathscr{H}) \to \mathscr{J}$ denote the induced operator and let λ denote the scalar such that $A - \lambda \in \mathscr{J}$ (Corollary 2.5). It follows from [6, Proposition 4.7] that

$$2\|A - \lambda\| \geq \|\delta_{\mathscr{J}}(A)\| \geq \|\delta_A | \mathscr{J}\| \geq \text{diam } (W(A)),$$

where $W(A)$ denotes the numerical range of A. Concerning closure properties of Ran $(\mathscr{T}_{\mathscr{J}})$, can Theorem 3.5(ii) be extended to arbitrary norm ideals $\mathscr{J}$?

Concerning the operator $\mathscr{S}_{\mathscr{J}}$, is the identity $\|\mathscr{S}_{\Phi}(A, B)\| = \Phi(s(A)s(B))$ valid? In [27] the semi-Fredholm domain of $S(A, B)$ is determined; it would be interesting to determine under what conditions the operators $S(A, B) - \lambda$ and $\mathscr{S}_{\mathscr{J}}(A, B)$ have closed range.

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