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FOURIER ANALYSIS AND DETERMINING SETS FOR RADON MEASURES ON $\mathbf{R}^n$

BY

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1. Introduction

Let C be a linear space of Radon measures on $\mathbf{R}^n$. Then we say a bounded Borel set $E \subseteq \mathbf{R}^n$ is a determining set for C if and only if $\mu \in C$, $\mu(x + E) = 0$ for all $x \in \mathbf{R}^n$ implies $\mu \equiv 0$ (equivalently, $\mu, \gamma \in C$ and $\mu(x + E) = \gamma(x + E)$ for all $x \in \mathbf{R}^n$ implies $\mu \equiv \gamma$). The general problem is: given a class of measures C, find conditions under which a given set E is a determining set. In this paper we study this problem for various classes of measures under different growth/decay conditions of the measures at ∞. Let M be the class of all Radon measures, M_T the class of "tempered" Radon measures, M_0 the class of measures "vanishing at ∞" and M_F the class of finite complex measures. Then we have $M_F \subseteq M_0$, $M_T \subseteq M$.

We describe the following interesting features:

(i) No bounded Borel set E is a determining set for M [3].
(ii) No "symmetric" bounded Borel set is a determining set for M_T (Corollary 3.4).
(iii) No "spherically symmetric" bounded Borel set is a determining set for M_0 (Theorem 4.3)
(iv) Every bounded Borel set of positive Lebesgue measure is a determining set for M_F (see [10]).

The problem of finding determining sets when one allows rotations as well as translations is an old one and is known in the literature as Pompeiu's problem [3], [13]. However, in this paper we restrict our attention, for the most part, to determining sets under translations. Some of the results presented here are our own while others rephrase old results in the language of determining sets. The main tool used here is the Fourier transform on $\mathbf{R}^n$. Since the basic question is measure theoretic it would be interesting if we could find geometric proofs of the results obtained without appealing to Fourier analysis (as for example in the proof of Helgason's support theorem for the Radon transform [5]).

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2. Notation and Terminology

For any unexplained terminology see [9]. Let λ denote the Lebesgue measure on $\mathbf{R}^n$ and $L^1(\mathbf{R}^n)$ the set of complex valued Borel measurable functions on $\mathbf{R}^n$ which are absolutely summable with respect to Lebesgue measure. Then $L^1(\mathbf{R}^n)$ is a Banach space when equipped with the norm defined by $\|f\| = \int_{R^n} |f(x)|\, d\lambda(x)$. (As usual we identify functions which are equal almost everywhere with respect to Lebesgue measure.) If $f \in L^1(\mathbf{R}^n)$, $x \in \mathbf{R}^n$, let ${}^x\!f$ be defined by ${}^x\!f(y) = f(x + y)$. Let V_f be the closure in the norm topology of $L^1(\mathbf{R}^n)$ of the linear span of $\{{}^x\!f;\ x \in \mathbf{R}^n\}$. If E is a Borel set, let 1_E denote its indicator function, i.e., $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ if $x \notin E$. Let $S(\mathbf{R}^n)$ be the space of smooth, rapidly decreasing functions equipped with the Schwartz topology (see [9]). By a tempered distribution we mean a continuous linear functional on $S(\mathbf{R}^n)$. Let $C_c^\infty(\mathbf{R}^n)$ denote the set of infinitely differentiable functions of compact support, $C_c(\mathbf{R}^n)$ the set of continuous functions of compact support and $C_0(\mathbf{R}^n)$ the set of continuous functions vanishing at infinity. A continuous linear functional on $C_c^\infty(\mathbf{R}^n)$ (equipped with the inductive limit topology) is called a distribution, and a distribution of order zero is called a (complex) Radon measure (see [9] for relevant definitions). Alternately a Radon measure is a continuous linear functional on $C_c(\mathbf{R}^n)$ equipped with the inductive limit topology (see [9]) (thus, for example, any continuous function f defines a Radon measure—usually denoted by $f(x)\, d\lambda(x)$). Let M be the set of all complex Radon measures on $\mathbf{R}^n$. A Radon measure μ is said to be "tempered" if μ defines a tempered distribution. (Thus for example, any L^p-function f and, more generally, any complex Radon measure of "polynomial growth" are tempered Radon measures.) Let

$$M_T = \{\mu \in M;\ \mu \text{ a tempered measure}\}.$$

We say $\mu \in M$ "vanishes at ∞" if $\mu(x + K) \to 0$ as $\|x\| \to \infty$ for each compact set K.

Let

$$M_0 = \{\mu \in M;\ \mu \text{ vanishes at } \infty\} \quad \text{and}$$

$$M_F = \{\mu \in M;\ \mu \text{ a finite complex measure}\}.$$

Observe that $M_F \subseteq M_0$, $M_T \subseteq M$. Finally let $*$ denote covolution (of functions, measures, distributions, etc.), and if f is an L^1-function or more generally a tempered distribution, $\hat{f}$ denotes its Fourier transform (see [9]).

Let $\mathbf{C}$ denote the field of complex numbers. An entire function f on $\mathbf{C}^n$ is said to be of "exponential type" if there exist positive constants A and r such that

$$|f(z)| \leq Ae^{r\|z\|} \quad \text{for all } z \in \mathbf{C}^n.$$

We now record three theorems which will be used later.

THEOREM 2.1 (Paley-Wiener). *Let μ be a tempered distribution on $\mathbf{R}^n$. If μ is of compact support, then $\hat{\mu}$ is an entire function on $\mathbf{C}^n$ of exponential* *type. Conversely if $\hat{\mu}$ extends to an entire function on* $\mathbf{C}^n$ *of exponential type and $\hat{\mu}$ is of slow growth on* $\mathbf{R}^n$, *then μ is compactly supported.*

THEOREM 2.2 (Malgrange [7]). *If f_1 and f_2 are entire functions on* $\mathbf{C}^n$ *of exponential type and if f_1/f_2 is entire, then f_1/f_2 is of exponential type.*

Finally, let Supp μ denote the (closed) support of μ, and $C_X(\text{Supp }\mu)$, the convex hull of the support of μ.

THEOREM 2.3 (Lions and Titchmarsh [4]). *If μ_1 and μ_2 are tempered distributions on* $\mathbf{R}^n$ *of compact support, then*

$$C_X(\text{Supp }(\mu_1 * \mu_2)) = C_X(\text{Supp }\mu_1) + C_X(\text{Supp }\mu_2).$$

3. $L^1(\mathbf{R}^n)$ and Determining Sets for M_T

We begin with the observation that there are no determining sets for the class M of all Radon measures on $\mathbf{R}^n$. More precisely we have the following theorem of Brown-Schreiber-Taylor (see Theorem 4.3 in [3]):

THEOREM 3.1. *Let E be a bounded Borel set of positive Lebesgue measure in* $\mathbf{R}^2$. *Then there exists a non-trivial continuous function f on* $\mathbf{R}^2$ *such that* $\int_{E+x} f(y)d\lambda(y) = 0$ *for all* $x \in \mathbf{R}^2$. (As the authors point out, the results in [3] can be generalized to any $n \geqslant 2$.)

In view of this we look for determining sets for M_T, the class of tempered Radon measures on $\mathbf{R}^n$. We begin by proving the following proposition:

PROPOSITION 3.2. *Let E be a bounded Borel subset of* $\mathbf{R}^n$ *of positive Lebesgue measure. Then E is a determining set for M_T if and only if* $\hat{1}_E(x) \neq 0$ *for all x in* $\mathbf{R}^n$. *Equivalently, E is a determining set for M_T if and only if* $V_{1_E} = L^1(\mathbf{R}^n)$.

Proof. We first observe that if there exists $x_0 \in \mathbf{R}^n$ with $\hat{1}_E(x_0) = 0$, then $x_0 \neq 0$ because $\hat{1}_E(0) = \lambda(E)$. Now consider the complex measure $d\mu = e^{ix_0 \cdot x}\, d\lambda(x)$, where $\cdot$ denotes the usual inner product in R^n. Then $0 \neq \mu \in M_T$ and $\mu(E + x) = 0$ for all $x \in \mathbf{R}^n$. On the other hand, suppose $\hat{1}_E(x) \neq 0$ for all $x \in \mathbf{R}^n$. Suppose $\mu \in M_T$ and $\mu(x + E) = 0$ for all $x \in \mathbf{R}^n$. This implies $1_E * \tilde{\mu} = 0$, where $\tilde{\mu}(A) = \mu(-A)$. Since E is bounded and $\mu \in M_T$, $1_E * \tilde{\mu}$ defines a tempered distribution (see [9]) and it has a Fourier transform in the sense of tempered distributions. Passing to Fourier transforms we have $\hat{1}_E\hat{\tilde{\mu}} = 0$. (Note $\hat{1}_E$ is a C^∞-function since E is bounded.) Since $\hat{1}_E(x) \neq 0$ for $x \in \mathbf{R}^n$, we conclude $\hat{\tilde{\mu}} = 0$ (as a distribution) and hence $\tilde{\mu} = 0$; i.e., $\mu = 0$; i.e., E is a determining set for M_T.

Finally, the last part follows immediately from the Wiener-Tauberian theorem (see [9]).

Example of determining sets for M_T. Take $n = 1$ and let

$$E = [-1, 1] \cup [2 - \alpha, 2 + \alpha]$$

where α is irrational and $0 < \alpha < 1$. For $\mathbf{R}^n$, $n > 1$, we can take the n-fold product of E.

We thank M. G. Nadkarni and R. L. Karandikar for this example.

In view of the last part of Proposition 3.2, a natural question to ask at this stage is: Given a Borel set E such that $0 < \lambda(E) < \infty$, is there a criterion to decide whether $V_{1_E} = L^1(\mathbf{R}^n)$? In this connection we prove a negative result, i.e., a large class of sets do not have this property. More generally we prove the following negative result:

PROPOSITION 3.3. *Let $f \in L^1(\mathbf{R}^n)$ be real valued, bounded and satisfy the following conditions:*

(i) *There exists $\lambda_0 \in \mathbf{R}^n$ such that $f(x + \lambda_0) = f(-x + \lambda_0)$* a.e (x).

(ii) *There does not exist a continuous function g on $\mathbf{R}^n$ such that $f = g$ almost everywhere.*

Then $V_f \neq L^1(\mathbf{R}^n)$.

An immediate consequence of this theorem is the following:

COROLLARY 3.4. *Let E be a Borel set in $\mathbf{R}^n$ with $0 < \lambda(E) < \infty$. Suppose $1_E(x) = 1_E(-x)$* a.e (x). *Then $V_{1_E} \neq L^1(\mathbf{R}^n)$. Thus such an E which is moreover bounded cannot be a determining set for M_T.*

Proof of Proposition 3.3. By translating if necessary we may assume $\lambda_0 = 0$. In this case, since $f(x) = f(-x)$, $\hat{f}$ is a real valued function. To prove the theorem it is enough to show that $\hat{f}$ has a zero. Suppose not. Since $\hat{f}$ is a continuous real valued function on $\mathbf{R}^n$ we may assume $\hat{f}(t) > 0$ for $t \in \mathbf{R}^n$. Now, it is known that if $f \in L^1(\mathbf{R}^n)$ is bounded and $\hat{f}(t) \geq 0$ for all t, then $\hat{f} \in L^1(\mathbf{R}^n)$. (See Remark (i) following this proof.) However, by Fourier inversion, this implies that $f(x) = g(x)$ (a.e) where g is a continuous function. This contradicts condition (ii) and the proof of the theorem is complete. The corollary is immediate from the theorem. (Corollary 3.4 can be regarded as an analogue of Theorem 3.1; i.e., we have shown that a "symmetric set" E can never be a determining set for M_T.)

Remarks. (i) If $f \in L^1 \cap L^\infty$, then $f \in L^2$. Choose an approximate identity $\{u_n\}$ in L^1 such that $0 \leq \hat{u}_n \in L^1 \cap L^2$ (many such exist). Then $\hat{u}_n \to 1$ uniformly on compact sets. If $\hat{f} \geq 0$, we have $0 \leq \int \hat{u}_n \hat{f} = \int u_n f \leq \|f\|_\infty$. Hence $\hat{f} \in L^1$ and in fact $\int \hat{f} \leq \|f\|_\infty$.

(ii) The corresponding theorem is false in $L^2(R^n)$. In fact, if f is any non trivial function in $L^2(\mathbf{R}^n)$ which vanishes outside a compact set, then it can be easily shown that the linear span of the translates of f is dense in $L^2(\mathbf{R}^n)$.

(iii) As remarked in the introduction, the Pompeiu problem concerns itself with the following question: What conditions on the set E ensure that E is a determining set for M if one allows rotations as well as translations? More precisely we say that a bounded set E of positive measure has the Pompeiu property if $\mu \equiv 0$ whenever $\mu \in M$ and $\mu(\sigma(E)) = 0$ for all rigid motions σ of $\mathbf{R}^n$. So the above question can be reformulated as follows: when does E have the Pompeiu proerty? This question is surprisingly deep and leads to problems of spectral synthesis. It has been completely answered by Brown-Schreiber-Taylor in [3] and the condition involves the "complex zeros" of $\hat{1}_E$. However in the same paper it is pointed out that the following theorem is, on the other hand, quite easy: for each $\alpha \geqslant 0$, let

$$C_\alpha = \{x \in \mathbf{R}^n : \|x\| = \alpha\}.$$

Then the following are equivalent: (i) If f is a bounded continuous function on $\mathbf{R}^n$ such that $\int_{\sigma(E)} f(x)d\lambda(x) = 0$ for all rigid motions σ of $\mathbf{R}^n$, then $f \equiv 0$; (ii) $\hat{1}_E$ does not vanish identically on C_α for any $\alpha \geqslant 0$. We wish to point out that this theorem can be strengthened. We say a bounded set E of positive measure has the Pompeiu property for M_T if $\mu \equiv 0$ whenever $\mu \in M_T$ and $\mu(\sigma(E)) = 0$ for all rigid motions σ of $\mathbf{R}^n$. Using the methods of Proposition 3.2 and standard properties of Fourier transforms and orthogonal transformations, one can prove that if

$$C_\alpha = \{x \in \mathbf{R}^n : \|x\| = \alpha\} \quad \text{for } \alpha \geqslant 0$$

then the following are equivalent: (i) E has the Pompeiu property for M_T; (ii) $\hat{1}_E$ does not vanish identically on C_α for any $\alpha \geqslant 0$.

4. Determining Sets for M_0

We begin with the following remark. Let $\mu \in M_0$ and let f be the indicator function of a ball with centre at 0 such that $\int f d\lambda = 1$. Let f_ε be the corresponding approximate identity. Then

$$f_\varepsilon * f_\varepsilon * \mu \to \mu \quad \text{as } \varepsilon \to 0$$

(in the sense of distributions). Now $f_\varepsilon * f_\varepsilon * \mu \;\varepsilon\; C_0(\mathbf{R}^n)$. In view of this, in most arguments, it is enough to take $\mu = g(x)d\lambda(x)$ with $g \in C_0(\mathbf{R}^n)$. It follows from a deep theorem about mean periodic functions on $\mathbf{R}$ (see [2], [6]) that if $f \in C_0(\mathbf{R})$, E is a bounded Borel subset of $\mathbf{R}$ of positive Lebesgue measure and $\int_{E+y} f(x)d\lambda(x) = 0$ for all $y \in \mathbf{R}$, then $f \equiv 0$. In view of the remark in the first paragraph we can rephrase this result as follows:

THEOREM 4.1. *Let E be a bounded Borel subset of $\mathbf{R}$ of positive Lebesgue measure. Then E is a determining set for $M_0(\mathbf{R})$.*

Using this result it is proved in [2] that:

THEOREM 4.2. *Let E be a bounded Borel subset of $\mathbf{R}^n$ of positive Lebesgue measure of the form $E_1 \times E_2 \times \cdots \times E_n$ where each E_i is a bounded Borel subset of $\mathbf{R}$. Then E is a determining set for M_0.*

The question of what happens if E is not of the above type is left open in [2]. The following theorem shows that for sets which are not product sets the situation can be very unsatisfactory.

THEOREM 4.3. *Let E be a bounded Borel subset of $\mathbf{R}^n$ $(n > 1)$ of positive Lebesgue measure. Also assume that E is spherically symmetric about 0 (i.e., $x \in E$ implies $Tx \in E$ for all orthogonal transformations T). Then E is* **not** *a determining set for M_0.*

Proof. Since E is spherically symmetric about 0, it is symmetric about 0 (i.e., $x \in E$ implies $-x \in E$). Hence by Corollary 3.4, there exists $0 \neq x_0 \in \mathbf{R}^n$ such that $\hat{1}_E(x_0) = 0$. Let $\|x_0\| = R > 0$. Then, because E is spherically symmetric, $\hat{1}_E$ vanishes on $\{x; \|x\| = R\}$. Let W_R be the uniform probability measure on $\{x; \|x\| = R\}$. Then it is well known that $\hat{W}_R \in C_0(\mathbf{R}^n)$. Now $\hat{1}_E W_R$ is the zero measure. Hence $(\hat{1}_E W_R)\hat{} \equiv 0$; i.e., $1_E * \hat{W}_R \equiv 0$; i.e., $\int_{E+y} \hat{W}_R d\lambda = 0$ for all $y \in \mathbf{R}^n$.Thus E is not a determining set for M_0.

The following interesting remark is due to B. V. Rao. Theorem 4.2 can be interpreted as follows. Let E be as in the theorem. Let $\mu \in M$ and suppose there exists $c \in \mathbf{C}$ such that for every compact set K, $(\mu - c\lambda)(x + K) \to 0$ as $\|x\| \to \infty$ (i.e., μ is asymptotically like the Lebesgue measure). Further if $\mu(x + E) = \mu(E)$ for all $x \in \mathbf{R}^n$, then $\mu = c\lambda$; i.e., if μ is a Radon measure which is asymptotically like the Lebesgue measure and is moreover translation invariant with respect to the single set E, then it is actually (a constant multiple of) the Lebesgue measure.

Theorem 4.3 (and its proof) is motivated by an example, given to us by Prof. J. P. Kahane, of a non trivial mean periodic function on $\mathbf{R}^2$ vanishing at ∞.

5. Support Theorems for Finite Measures

It is easy to prove (see [10]) that if E is a bounded Borel set of positive measure in $\mathbf{R}^n$, then E is a determining set for M_F. It is therefore natural to ask the following question: If $\mu \in M_F$ and $\mu(x + E) = 0$ for all $x \in \mathbf{R}^n$ such that $|x| > R$, can one say something about the support of μ? We prove in this section that if μ is "very rapidly decreasing", then indeed one can conclude that μ is of compact support. The theorem we are going to prove is essentially a reinterpretation of some famous results of Malgrange in [7]. Before stating our theorem, we make a definition.

DEFINITION 5.1. *$\mu \in M_F$ is said to be "very rapidly decreasing" if*

$$\int_{\mathbf{R}^n} e^{r\|x\|} d\,|\mu|\,(x) < \infty \quad \text{for all } r > 0.$$

Let

$$N = \{\mu \in M_F;\ \mu \text{ ``very rapidly decreasing''}\}.$$

THEOREM 5.1. *Let $\mu \in N$ and let E be a bounded Borel set of positive measure such that $E \subseteq \{x : \|x\| \leq r\}$. If $\mu(x + E) = 0$ for all x with $\|x\| > R$, then* Supp $\mu \subseteq \{x : \|x\| \leq R + r\}$.

Proof. Since $\mu \in N$ it is easy to show that $\hat{\mu}$ is an entire function on $\mathbf{C}^n$, bounded on $\mathbf{R}^n$. Further, the condition $\mu(x + E) = 0$ for $\|x\| > R$ implies that $\tilde{\mu} * 1_E = g$ where g is a measure of compact support, Supp $g \subseteq B_R$ (where $\tilde{\mu}(A) = \mu(-A)$). Passing to Fourier transforms we have $\hat{\tilde{\mu}}\, \hat{1}_E = \hat{g}$. Moreover $\hat{1}_E$ and $\hat{g}$ are entire functions on $\mathbf{C}^n$ of exponential type, being Fourier transforms of compactly supported measures. As already observed $\hat{\tilde{\mu}}$ is an entire function and hence it follows from Theorem 2.2 that $\hat{\tilde{\mu}}$ is of exponential type (and is moreover bounded on $\mathbf{R}^n$). Thus by Theorem 2.1, $\tilde{\mu}$ is of compact support; i.e., μ is of compact support. It follows easily from Theorem 2.3 that Supp $\mu \subseteq B_{R+r}$ and this concludes the proof of our theorem.

The theorem above is motivated by Helgason's support theorem for Radon transforms (see [5]). In Helgason's theorem one needs to assume only "rapid decrease" whereas we have to assume "very rapid decrease". We observe below that this condition is really necessary.

PROPOSITION 5.2. *Let E be a bounded Borel subset of* $\mathbf{R}$ *of positive Lebesgue measure such that $\hat{1}_E(z_0) = 0$ for some $z_0 \in \mathbf{C}$, $z_0 \notin \mathbf{R}$. Then there exists $f \in S(\mathbf{R})$, f* **not** *of compact support, such that $f * 1_E$ is of compact support.*

Proof. We first remark that such an E does exist; for example, take E to be the disjoint union of two closed bounded intervals of $\mathbf{R}$, suitably chosen—see the example of a determining set for M_T in §3.

Choose $g \in C_c^\infty(\mathbf{R})$ such that $\hat{g}(z_0) \neq 0$—this is certainly possible. Now

$$\frac{\hat{g}(x)}{x - z_0} \in S(\mathbf{R})$$

(because $z_0 \notin \mathbf{R}$ and $\hat{g}(x) \in S(\mathbf{R})$) and hence there exists $f \in S(\mathbf{R})$ such that

$$\hat{f}(x) = \frac{\hat{g}(x)}{x - z_0}, \quad x \in \mathbf{R}.$$

However f is *not* of compact support because $\hat{g}(z)/(z - z_0)$ is *not* entire. Now

$$\hat{f}(x)\,\hat{1}_E(x) = \hat{g}(x)\frac{\hat{1}_E(x)}{x - z_0}, \quad x \in \mathbf{R}.$$

Also the right hand side extends to an entire function on $\mathbf{C}$ of exponential type (because both $\hat{g}$, $\hat{1}_E(z)/(z - z_0)$ are entire functions of exponential type—note z_0 is a zero of $\hat{1}_E(z)$). Thus

$$(\hat{f} * \hat{1}_E) = (f * 1_E)\hat{}$$

is an entire function of exponential type and hence since it is bounded on $\mathbf{R}^n$, by Theorem 2.1, $f * 1_E$ is of compact support. As already observed, $f \in S(\mathbf{R})$ but f is *not* of compact support and the proposition is proved.

We can generalize this proposition to $\mathbf{R}^n$ by taking the n-fold product of E.

However if the set E is "sufficiently nice" we can get a support theorem without any decay conditions on the measure μ, as the next proposition indicates. For simplicity we work with $\mathbf{R}^2$ but the same proof can be used in $\mathbf{R}^n$.

PROPOSITION 5.3. *Let* $E = \{(x_1, x_2) : x_1^2 + x_2^2 \leq r^2\}$. *If* $\mu \in M_F(\mathbf{R}^2)$ *and* $\mu(x + E) = 0$ *for* $x \in \mathbf{R}^2$ *with* $\|x\| > R$, *then* Supp $\mu \subseteq B_{R+r}$.

Proof (sketch). We are given that $\mu * 1_E = g$ where g is a measure of compact support and supported in B_R. Passing to Fourier transforms, we have $\hat{\mu}(x)\,\hat{1}_E(x) = \hat{g}(x)$ for $x \in \mathbf{R}^2$. Now it is known that the zeros of $\hat{1}_E$ on $\mathbf{R}^2$ occur precisely on a sequence of circles

$$\{(x_1, x_2) : x_1^2 + x_2^2 = r_i^2\}, \quad 0 < r_1 < r_2 < \ldots.$$

Since $\hat{\mu}$ is a continuous function on $\mathbf{R}^2$, $\hat{g}$ also vanishes on these circles. But $\hat{g}$ is an entire function on $\mathbf{C}^2$ and hence $\hat{g}$ vanishes on

$$\{(z_1, z_2) \in \mathbf{C}^2;\ z_1^2 + z_2^2 = r_i^2\}.$$

Since $z_1^2 + z_2^2 - r_i^2$ is an irreducible entire function vanishing on this set, it divides $\hat{g}$. Also $\hat{1}_E$ vanishes precisely on those sets and it can easily be shown by an argument similar to above that if $(z_1^2 + z_2^2 - r_i^2)^k$ divides $\hat{1}_E$ then it divides $\hat{g}$. From this it follows that $\hat{g}/\hat{1}_E$ is an entire function on $\mathbf{C}^2$ and hence, since

$$\hat{\mu}(x) = \frac{\hat{g}(x)}{\hat{1}_E(x)} \quad \text{for } x \in \mathbf{R}^2,$$

$\hat{\mu}$ extends to an entire function on $\mathbf{C}^2$. The rest of the proof is exactly as in Theorem 5.1.

Remarks. (i) For finite measures there is no need to confine oneself to bounded sets E. For a discussion of determining sets without assuming boundedness of E, see [1], [10], [11].

(ii) For a discussion of determining sets for M_F in the context of general locally compact abelian groups etc., see [8].

(iii) The main idea in the proof of Proposition 5.3 is motivated by the proofs in [3].

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