ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1984

AN HARMONIC ANALYSIS FOR OPERATORS: F. AND M. RIESZ THEOREMS

BY

U. B. TEWARI AND SHOBHA MADAN

Dedicated to the memory of Professor Karel De Leeuw

1. Introduction

De Leeuw introduced and studied an harmonic analysis for operators in [2] and [3]. With each bounded linear operator T on a homogeneous Banach space B on the circle group $\mathbf{T}$, he associated a formal Fourier series,

$$T \sim \sum_{-\infty}^{\infty} \pi_n(T) \tag{1.1}$$

In [2], he established formal properties of the series (1.1) and remarked that analogues of all the results of Sections 2–5 of [2] are valid for any compact abelian group. In Section 6 of [2], De Leeuw obtained an extension to operators of the first theorem of F. and M. Riesz. This classical theorem asserts that a measure whose Fourier Stieltjes transform vanishes on the negative integers must be absolutely continuous. In this paper we obtain an extension of the above theorem to operators on certain homogeneous Banach spaces on a compact abelian group with ordered dual.

In [3], De Leeuw restricted himself to the case where B is $L^2(\mathbf{T})$. He extended the notions of support and analyticity to operators and established an analogue for operators of the second theorem of F. and M. Riesz. This theorem asserts that a function having all its negative Fourier coefficients zero, cannot vanish on a subset of $\mathbf{T}$ having positive Lebesgue measure, unless it is zero almost everywhere. In this paper we extend the notions of support and analyticity to operators on B, where B is $C(G)$ or one of the $L^p(G)$, $1 \leq p < \infty$, and G is a compact abelian group with ordered dual. We use an example due to De Leeuw and Glicksberg [4] to show that a natural generalization of the second F. and M. Riesz theorem for operators on $L^2(G)$ fails, unless some condition is imposed on the ordering of Γ, the dual group of G. The condition we impose is motivated by an extension of the classical theorem of F. and M. Riesz to measures on a compact abelian group with ordered dual due to De Leeuw and Glicksberg in [4]. Our generalization of the second theorem of F. and M. Riesz for

Received December 9, 1981.

operators on B where B is $C(G)$ or one of the $L^p(G)$ appears to be new even for $G = \mathbf{T}$, except for the case $p = 2$ which was established by De Leeuw in [3].

2. Operators on a Homogeneous Space and Their Fourier Series

In this section, we list the main definitions and results which are needed in what follows. These definitions and results are the natural generalizations of the corresponding concepts and results from [2]. The proofs are similar to those in [2] and hence are omitted.

Let G be a compact abelian group and let Γ denote the dual group of G. The translation operator R_x is defined by

$$(R_x f)(y) = f(y - x), \quad y \in G.$$

A *homogeneous Banach space* B is a dense linear subspace of $L^1(G)$ such that B is a Banach space under a norm $\| \ \|_B$ and the following hold:

(i) $\|f\|_1 \leq \|f\|_B, f \in B$.
(ii) If $f \in B$, then $R_x f \in B$ for every $x \in G$ and $\|R_x f\|_B = \|f\|_B$.
(iii) The functions in B translate continuously; i.e., if $f \in B$, then

$$\lim_{x \to 0} \|R_x f - f\|_B = 0.$$

We also assume that B is closed under multiplication by characters of G, that is, if $f \in B$ and $\gamma \in \Gamma$, then the function $M_\gamma f$, defined by

$$(M_\gamma f)(x) = (x, \gamma) f(x), \quad x \in G,$$

is also in B. It can be easily seen by using Closed Graph Theorem that each M_γ is a continuous operator on B.

Let $\mathscr{L}$ denote the Banach algebra of bounded linear operators on B. By an operator on B we shall always mean an element of $\mathscr{L}$. An operator T on B is said to be *invariant* if $TR_x = R_x T$ for each x in G. It is well known (see for example, [13]) that if T is an invariant operator on B then there is a unique function λ on Γ such that $(Tf)\hat{\ }(\gamma) = \lambda(\gamma)\hat{f}(\gamma)$ for every $f \in B$ and $\gamma \in \Gamma$. Here $\hat{f}$ denotes the Fourier transform of f. The set of invariant operators on B is denoted by $\mathscr{L}_0$. It is easily seen that $\mathscr{L}_0$ is a closed subalgebra of $\mathscr{L}$. An operator T in $\mathscr{L}$ is said to be *almost invariant* if

$$\lim_{x \to 0} \|TR_x - R_x T\| = 0.$$

The set of almost invariant operators is denoted by $\mathscr{L}_\#$. This set of operators has also been investigated by Loebl and Schochet in [14]. It is easily seen that $\mathscr{L}_\#$ is a closed subalgebra of $\mathscr{L}$.

For $\gamma \in \Gamma$, we denote by $\mathscr{L}_\gamma$ the subset of $\mathscr{L}$ consisting of all the operators T satisfying $TR_x = (x, \gamma) R_x T$, $x \in G$.

An operator $T \in \mathscr{L}$ is said to be *simple* if $T \in \mathscr{L}_\gamma$ for some $\gamma \in \Gamma$. If 0 denotes the identity in Γ, then $\mathscr{L}_0$ is precisely the set of invariant operators, and this agrees with our earlier notation.

LEMMA 2.1. *Let $T \in \mathscr{L}_\gamma$. Then the following are equivalent.*

(i) $T \in \mathscr{L}_\gamma$.
(ii) *There is U in $\mathscr{L}_0$ such that $T = UM_\gamma$.*
(iii) *There is V in $\mathscr{L}_0$ such that $T = M_\gamma V$.*

Let $T \in \mathscr{L}$. Since the functions in B translate continuously, it can be easily seen that the map $\Psi : G \to \mathscr{L}$ defined by $\Psi(x) = R_{-x}TR_x$ is continuous in the strong operator topology of $\mathscr{L}$. If $T \in \mathscr{L}_\#$ then the map Ψ is also continuous in the norm topology of $\mathscr{L}$.

Following De Leeuw [2], we define the Fourier transform of an operator $T \in \mathscr{L}$ as an operator valued function on Γ, given by the Bochner integral

$$[\pi_\gamma(T)](f) = \int_G (-x, \gamma)[R_{-x}TR_x](f)\,dx, \quad f \in B.$$

By the remarks in the previous paragraph, the integral on the right is well defined. For $T \in \mathscr{L}_\#$, we have the representation

$$\pi_\gamma(T) = \int_G (-x, \gamma)R_{-x}TR_x\,dx.$$

Using the translation invariance of the Haar measure on G, it can be easily seen that the map π_γ: $\mathscr{L} \to \mathscr{L}$ is a projection of $\mathscr{L}$ onto $\mathscr{L}_\gamma$. With each $T \in \mathscr{L}$, we associate a formal series $\Sigma_{\gamma\in\Gamma}\pi_\gamma(T)$, called the Fourier series of T. It is easily seen that if $T \in \mathscr{L}_\gamma$ then $\pi_{\gamma'}(T) = T$ if $\gamma' = \gamma$ and $\pi_{\gamma'}(T) = 0$ if $\gamma' \neq \gamma$.

To study the summability properties of the Fourier series of an operator $T \in \mathscr{L}$, we first define the convolution of a measure $\mu \in M(G)$ and an operator $T \in \mathscr{L}$. We remark that this definition has also been considered by Forelli [7] in a more general set up. For $f \in B$, we define

$$(\mu * T)(f) = \int_G R_xTR_{-x}f\,d\mu(x).$$

It is easily seen that the operator $\mu * T$ defined above belongs to $\mathscr{L}$. Also $\mu * T \in \mathscr{L}_\#$ if $T \in \mathscr{L}_\#$.

PROPOSITION 2.2. *Let $\mu \in M(G)$ and $T \in \mathscr{L}_\#$. Then:*

(i) $\|\mu * T\| \leq \|\mu\|\,\|T\|$.
(ii) $\pi_\gamma(\mu * T) = \hat{\mu}(\gamma)\pi_\gamma(T)$.
(iii) *If $g \in L^1(G)$ is such that $M_g \in \mathscr{L}$, where*

$$(M_g f)(x) = g(x)f(x) \quad \text{a.e.},$$

then $M_{\mu*g} \in \mathscr{L}$ and $\mu * M_g = M_{\mu*g}$.

For the circle group $\mathbf{T}$, De Leeuw [2] proved the $C - 1$ summability of the Fourier series of an operator $T \in \mathcal{L}_{\#}$. For arbitrary G, we replace the Féjer kernel by a bounded approximate identity $\{e_\alpha\}_{\alpha \in I}$ of $L^1(G)$ consisting of trigonometric polynomials and define the notion of e_α-summability.

DEFINITION. The Fourier series $\Sigma_{\gamma \in \Gamma} \pi_\gamma(T)$ of an operator $T \in \mathcal{L}$ is said to be e_α-summable to T in the strong operator (norm) topology of $\mathcal{L}$, if given $\varepsilon > 0$ and $f \in B$, there is an $\alpha_0 \in I$ such that for all $\alpha \geqslant \alpha_0$,

$$\left\| \sum_{\gamma \in \Gamma} \hat{e}_\alpha(\gamma) \pi_\gamma(T)(f) - Tf \right\|_B < \varepsilon \quad \left(\left\| \sum_{\gamma \in \Gamma} \hat{e}_\alpha(\gamma) \pi_\gamma(T) - T \right\| < \varepsilon \right).$$

Observe that $\Sigma_{\gamma \in \Gamma} \pi_\gamma(T)$ is e_α-summable to T in the strong operator (norm) topology if and only if $e_\alpha * T$ converges to T in the strong operator (norm) topology.

The following proposition has been proved by several authors; see, for example, [7] and [14], in varying degree of generality.

PROPOSITION 2.3. *Let $\{e_\alpha\}_{\alpha \in I}$ be an approximate identity of $L^1(G)$ satisfying $e_\alpha \geqslant 0$ and $\|e_\alpha\|_1 = 1$ for all $\alpha \in I$. Then $e_\alpha * T$ converges to T in the strong operator topology and if $T \in L_{\#}$ then $e_\alpha * T$ converges to T in the norm topology.*

Proposition 2.3 shows that $\mathcal{L}_{\#}$ is an essential Banach $L^1(G)$-module. This has also been observed by Forelli [7] and Loebl and Schochet [14]. By choosing an approximate identity of $L^1(G)$ as above, consisting of trigonometric polynomials, it is easily seen that $\mathcal{L}_{\#}$ is the norm closed subalgebra of $\mathcal{L}$ generated by the set $\mathcal{L}_0$ of invariant operators and the multiplication operators $\{M_\gamma : \gamma \in \Gamma\}$. Furthermore, it follows that if $T \in \mathcal{L}_{\#}$ then T belongs to the closed linear subspace of $\mathcal{L}$ generated by $\{\pi_\gamma(T) : \gamma \in \Gamma\}$. Finally we remark that all the formal properties of the Fourier series of an operator $T \in \mathcal{L}$, proved in [2] have their valid analogues for arbitrary G. The proofs are essentially the same and the reader should consult [2] for any such property which might be used in the following sections but has not been mentioned in this section.

3. The First F. and M. Riesz Theorem

As a generalization of the first F. and M. Riesz Theorem, De Leeuw [2] proved the following theorem for almost invariant operators.

THEOREM D_1. *Let T be an almost invariant operator on $C(\mathbf{T})$ such that*

$$T(C(\mathbf{T})_+) \subseteq C(\mathbf{T})_-.$$

Then T must be a compact operator. Here, $C(\mathbf{T})$ is the space of continuous *complex valued functions on the circle group and* $C(\mathbf{T})_+$ *and* $C(\mathbf{T})_-$ *are defined by*

$$C(\mathbf{T})_+ = \{f \in C(\mathbf{T}) : \hat{f}(n) = 0 \quad \text{if } n < 0\},$$

$$C(\mathbf{T})_- = \{f \in C(\mathbf{T}) : \hat{f}(n) = 0 \quad \text{if } n > 0\}.$$

De Leeuw also observes that the above theorem remains true if $C(\mathbf{T})$ is replaced by $L^1(\mathbf{T})$. We observe that the above theorem is not true, if $C(\mathbf{T})$ is replaced by an arbitrary homogeneous Banach space on $\mathbf{T}$. For example, consider $L^p(\mathbf{T})$, $1 < p < \infty$. It is well known (see, for example [5], page 94) that the operator T defined by $Tf(t) = \Sigma_{n \le 0} \hat{f}(n) e^{int}$ a.e. for $t \in \mathbf{T}$ and $f \in L^p(\mathbf{T})$, is an invariant operator on $L^p(\mathbf{T})$ such that $T(L^p(\mathbf{T})^+) \subseteq L^p(\mathbf{T})_-$. However, T is not a compact operator.

It can be easily seen that the theorem will remain true for operators on any homogeneous Banach space B on $\mathbf{T}$ if all invariant operators on B are given by convolution with measures on $\mathbf{T}$. The invariant operators on $C(\mathbf{T})$ or $L^1(\mathbf{T})$ are all given by convolution with measures on $\mathbf{T}$. For our generalization of De Leeuw's theorem for operators on a homogeneous Banach space B on a compact abelian group G with ordered dual, we shall assume that all the invariant operators on B are given by convolution with finite Borel measures on G.

For the proof of our theorem we shall depend on the generalizations of the classical theorems of F. and M. Riesz due to De Leeuw and Glicksberg [4]. We recall the terminology necessary for us.

The order on Γ is given by a continuous homomorphism $\Psi: \Gamma \to \mathbf{R}$, where $\mathbf{R}$ denotes the additive group of real numbers. This map induces a unique continuous homomorphism $\phi : \mathbf{R} \to G$ given by

$$(\phi(t), \gamma) = \exp(i\Psi(\gamma)t) \quad \text{for } t \in \mathbf{R} \text{ and } \gamma \in \Gamma.$$

Ψ and ϕ will have this meaning for the rest of this paper.

Let $\mu \in M(G)$.

(1) Ψ is said to be *ϕ-analytic* if $\hat{\mu}(\gamma) = 0$ for all γ such that $\Psi(\gamma) < 0$.

(2) μ is said to *translate continuously in the direction of* ϕ, if $\lim_{t\to 0}\|\mu_t - \mu\| = 0$, where μ_t is the measure given by $\mu_t(E) = \mu(E + \phi(t))$ for every Borel subset E of G.

We say that a measurable subset E of G is *null in the direction of* ϕ if for all $x \in E$, the set $\{t \in \mathbf{R} : x + \phi(t) \in E\}$ has Lebesgue measure zero. E will be called *thick in the direction of* ϕ if for each $x \in E$, the set $\{t \in \mathbf{R} : x + \phi(t) \in E\}$ has positive Lebesgue measure.

(3) μ *is said to be absolutely continuous in the direction of* ϕ if $|\mu|(E) = 0$ for every Borel set E which is null in the direction of ϕ.

(4) μ is said to be *non-vanishing in the direction of* ϕ if $|\mu|(E) > 0$ for each Borel subset E of G which is thick in the direction of ϕ and

for which $|\mu|(E + \phi(\mathbf{R})) > 0$. De Leeuw and Glicksberg [4] proved the following.

THEOREM DG-1. *Let $\mu \in M(G)$ be a ϕ-analytic measure on G. Then μ is absolutely continuous and nonvanishing in the direction of ϕ.*

We will need the following corollary of this theorem.

COROLLARY DG-1. *Let $\mu \in M(G)$ be such that for some $\gamma_0 \in \Gamma$, $\hat{\mu}(\gamma) = 0$ for all γ satisfying $\Psi(\gamma) \leqslant \Psi(\gamma_0)$ or $\hat{\mu}(\gamma) = 0$ for all γ satisfying $\Psi(\gamma) \geqslant \Psi(\gamma_0)$. Then μ is absolutely continuous in the direction of ϕ.*

For our formulation of the first F. and M. Riesz Theorem for almost invariant operators on a homogeneous Banach space B on G, we introduce the notion of ϕ-compactness.

A subset F of B is said to be *ϕ-compact* if it satisfies the following conditions:

(i) F is a norm bounded subset of B.
(ii) Given $\varepsilon > 0$, there exists a neighbourhood U of 0 in $\mathbf{R}$ such that $\|f_t - f\|_B < \varepsilon$ for all $f \in F$ and all $t \in U$. Here f_t is the function in B given by

$$f_t(x) = f(x + \phi(t)), \quad x \in G.$$

We observe that a subset F of B is compact if F satisfies condition (i) above, and

(ii′) Given $\varepsilon > 0$, there exists a neighbourhood V of 0 in G such that for all $f \in F$ and $x \in V$, $\|f_x - f\|_B < \varepsilon$ (see [6]).

It is immediate from the definition of ϕ-compactness that every compact subset of B is also ϕ-compact for any order introduced by a homomorphism Ψ of Γ into $\mathbf{R}$. However, there may be orderings on Γ for which not every ϕ-compact subset of B is compact. For example, let $G = \mathbf{T} \times \mathbf{T}$ and $\Psi : Z \times Z \to \mathbf{R}$ be defined by $\Psi(m, n) = m$. Then $\phi(t) = (e^{it}, 0)$ for every $t \in \mathbf{R}$.

$$\text{Let } F = \{g \in C(G): g(e^{it}, e^{iu}) = f(e^{iu}),\ \ f \in C(\mathbf{T}) \text{ and } \|f\|_\infty \leqslant 1\}.$$

It can be easily seen that F is a ϕ-compact but not a compact subset of $C(G)$. There is one situation, however, when every ϕ-compact subset of B is also compact. This happens when the mapping ϕ is an open map. In this connection it is important to note that ϕ is an open map if it is onto (see (5.29), [8]). We also remark that for the circle group $\mathbf{T}$ and any nontrivial homomorphism Ψ of Z into $\mathbf{R}$, the mapping ϕ is always onto. Thus for any nontrivial ordering introduced by a homomorphism Ψ on Z, ϕ-compactness is equivalent to compactness for any subset of a homogeneous Banach space on $\mathbf{T}$.

A linear operator T on a homogeneous Banach space B on G is said to be *ϕ-compact* if T maps norm bounded subsets of B into ϕ-compact subsets of B. It is obvious that a ϕ-compact operator is necessarily continuous.

For a homogeneous Banach space B on G, we define B_+ and B_- by

$$B_+ = \{f \in B : \hat{f}(\gamma) = 0 \text{ for all } \gamma \text{ such that } \Psi(\gamma) < 0\}$$

and

$$B_- = \{f \in B : \hat{f}(\gamma) = 0 \text{ for all } \gamma \text{ such that } \Psi(\gamma) > 0\}.$$

We are now ready to state our generalization of the first F. and M. Riesz Theorem for operators on a homogeneous Banach space B on G.

THEOREM 3.1. *Let B be a homogeneous Banach space on G such that $\mathscr{L}_0$ is isometrically isomorphic to $M(G)$. Let T be an almost invariant operator on B such that $T(B_+) \subseteq B_-$. Then T is ϕ-compact.*

The proof of Theorem 3.1 depends on the following three lemmas.

LEMMA 3.2. *Let B be a homogeneous Banach space on G such that $\mathscr{L}_0$ is isometrically isomorphic to $M(G)$. The invariant operator on B defined by a measure $\mu \in M(G)$ is ϕ-compact if and only if μ is absolutely continuous in the direction of ϕ.*

Proof. Let μ be absolutely continuous in the direction of ϕ. By Proposition 2.4 of [4], μ translates continuously in the direction of ϕ. Let F be a norm bounded subset of B bounded by K, say. Then $\mu * F$ is also norm bounded. Given $\varepsilon > 0$, let U be a neighbourhood of 0 in $\mathbf{R}$ such that

$$\|\mu_t - \mu\| < \varepsilon/K \quad \text{for all } t \in U.$$

Then

$$\begin{aligned}\|(\mu * f)_t - \mu * f\|_B &= \|\mu_t * f - \mu * f\|_B \\ &\leq \|\mu_t - \mu\| \, \|f\|_B \\ &< \varepsilon \quad \text{for all } t \in U \text{ and } f \in F.\end{aligned}$$

Therefore, $\mu * B$ is a ϕ-compact subset of B and the operator on B defined by μ is ϕ-compact.

Conversely, suppose μ defines a ϕ-compact operator on B. Let F be the unit ball of B. Then, for any $\varepsilon > 0$, we can find a neighbourhood U of 0 in $\mathbf{R}$ such that

$$\|(\mu * f)_t - \mu * f\|_B < \varepsilon \quad \text{for all } f \in F \text{ and } t \in U.$$

Then

$$\|\mu_t - \mu\| = \sup_{f \in F} \|\mu_t * f - \mu * f\|_B < \varepsilon \quad \text{for all } t \in U.$$

Therefore, μ translates continuously in the direction of ϕ. Again, Proposition 2.4 of [4] implies that μ is absolutely continuous in the direction of ϕ.

LEMMA 3.3. *The set of almost invariant ϕ-compact operators on a homogeneous Banach space B on G is a closed two sided ideal in $\mathscr{L}_{\#}$.*

Proof. Let T be an almost invariant ϕ-compact operator on B and let $S \in \mathscr{L}_{\#}$. Clearly TS is also a ϕ-compact operator. We now prove that ST is also a ϕ-compact operator. For this, let F be a norm bounded subset of B, bounded by K. Given $\varepsilon > 0$, let U be a neighbourhood of 0 in $\mathbf{R}$ such that

$$\|(Tf)_t - Tf\|_B < \frac{\varepsilon}{2\|S\|} \quad \text{for all } f \in F \text{ and } t \in U,$$

and

$$\|R_{\phi(t)}SR_{-\phi(t)} - S\| < \frac{\varepsilon}{2K\|T\|} \quad \text{for all } t \in U.$$

Then

$$\begin{aligned}\|(STf)_t - STf\|_B &\leqslant \|(STF)_t - S((Tf)_t)\|_B + \|S((Tf)_t) - STf\|_B \\ &\leqslant \|R_{\phi(t)}S - SR_{-\phi(t)}\| \, \|Tf\|_B + \|S\| \, \|(Tf)_t - Tf\|_B \\ &< \varepsilon \quad \text{for all } t \in U.\end{aligned}$$

This proves the ϕ-compactness of ST. The fact that the set of almost invariant ϕ-compact operators is a closed linear subspace of $\mathscr{L}_{\#}$ is easily seen.

The following lemma is an easy consequence of the definition of the Fourier transform of an operator.

LEMMA 3.4. *Let E and F be closed translation invariant subspaces of a homogeneous Banach space B on G and let $T \in \mathscr{L}$ be such that $T(E) \subseteq F$. Then $\pi_\gamma(T)(E) \subseteq F$ for each $\gamma \in \Gamma$.*

Proof of Theorem 3.1. Let $T \in \mathscr{L}_{\#}$ be such that $T(B_+) \subseteq B_-$. By Lemma 3.4 we get that $\pi_\gamma(T)(B_+) \subseteq B_-$ for every $\gamma \in \Gamma$. Since $\mathscr{L}_0$ is isometrically isomorphic to $M(G)$, Lemma 2.1 implies that there exists a measure $\mu_\gamma \in M(G)$, for each γ, such that $\pi_\gamma(T) = M_\gamma C_{\mu_\gamma}$ for each γ, where C_{μ_γ} denotes the operator of convolution by the measure μ_γ. Now for $\gamma' \in \Gamma$,

$$C_{\mu_\gamma}(\gamma') = \mu_\gamma * \gamma' = \hat{\mu}_\gamma(\gamma')\gamma'.$$

Therefore $M_\gamma C_{\mu_\gamma}(\gamma') = \hat{\mu}_\gamma(\gamma')(\gamma + \gamma')$.

For each $\gamma \in \Gamma$, let $\gamma_0 = 0$ if $\Psi(\gamma) \geqslant 0$ and $\gamma_0 = -\gamma$ if $\Psi(\gamma) < 0$. Then, if $\gamma' \in \Gamma$ is such that $\Psi(\gamma') \geqslant \Psi(\gamma_0) \geqslant 0$, we have

$$\pi_\gamma(T)(\gamma') = \hat{\mu}_\gamma(\gamma')(\gamma + \gamma') = 0,$$

because $\gamma + \gamma' \in B_+$. (If $\Psi(\gamma) \geqslant 0$, $\Psi(\gamma + \gamma') \geqslant 0$ and if $\Psi(\gamma) < 0$ then again $\Psi(\gamma + \gamma') = \Psi(\gamma') - \Psi(\gamma_0) \geqslant 0$.)

Therefore, $\hat{\mu}_\gamma(\gamma') = 0$ for all $\gamma' \in \Gamma$ such that $\Psi(\gamma') \geqslant \Psi(\gamma_0)$. By Corollary DG-1, μ_γ is absolutely continuous in the direction of ϕ. By Lemmas 3.2 and 3.3, the operator $M_\gamma C_{\mu_\gamma} = \pi_\gamma(T)$ is a ϕ-compact operator for each $\gamma \in \Gamma$. Since T lies in the closed linear span of the set $\{\pi_\gamma(T) : \gamma \in \Gamma\}$, it follows from Lemma 3.3 that T is ϕ-compact. This completes the proof of the theorem.

4. The Second F. and M. Riesz Theorem

An operator T on a homogeneous Banach space B on G is said to be ϕ-analytic if $\pi_\gamma(T) = 0$ for all γ such that $\Psi(\gamma) < 0$. We denote the set of all ϕ-analytic operators by $\mathscr{A}$. We state below some properties of ϕ-analytic operators. The proofs of these facts for the case $G = \mathbf{T}$ and $B = L^2(\mathbf{T})$ in [3] are easily generalized to our general situation and we omit the proofs.

1. Let f be an integrable function on G such that $M_f \in \mathscr{L}$. Then M_f is ϕ-analytic if and only if $\hat{f}(\gamma) = 0$ for all γ such that $\Psi(\gamma) < 0$.
2. $\mathscr{A}$ is a subalgebra of $\mathscr{L}$ closed in the strong operator topology of $\mathscr{L}$.
3. The subalgebra $\mathscr{A}$ is the strongly closed subalgebra of $\mathscr{A}$ generated by the operators in $\mathscr{L}_0$ and the set $\{M_\gamma \colon \Psi(\gamma) \geqslant 0\}$.

In the following proposition we give a characterization of ϕ-analytic operators. We remark that the proposition is a special case of Theorem 2.3 in [1]. De Leeuw [3] had proved it in the case $G = \mathbf{T}$ and $B = L^2(\mathbf{T})$. We give a simple proof independent of the arguments in [1].

Proposition 4.1. *Let B be a homogeneous Banach space on G and*

$$B_{\gamma'} = \{f \in B : \hat{f}(\gamma) = 0 \ for \ \Psi(\gamma) < \Psi(\gamma')\}$$

for each $\gamma' \in \Gamma$. An operator T on B is ϕ-analytic if and only if $T(B_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$.

Proof. Let $\gamma_0 \in \Gamma$ be such that $\Psi(\gamma_0) \geqslant 0$ and let $U \in \mathscr{L}_0$. If T is M_{γ_0} or U then it is clear that $T(B_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$. It now follows from property (3) above that if T is ϕ-analytic operator on B then $T(B_\gamma) \subseteq B_\gamma$ for every $\gamma \in \Gamma$.

Conversely, suppose T is an operator on B such that $T(B_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$. Since each B_γ is a closed translation invariant subspace of B, Lemma 3.4 implies that for every $\gamma' \in \Gamma$,

$$\pi_{\gamma'}(T)(B_\gamma) \subseteq B_\gamma \quad \text{for all } \gamma \in \Gamma.$$

Suppose $\Psi(\gamma') < 0$. We shall prove that $\pi_{\gamma'}(T) = 0$. Since trigonometric polynomials are dense in B, it is enough to show that $\pi_{\gamma'}(T)(\gamma) = 0$ for every $\gamma \in \Gamma$. Since $\pi_{\gamma'}(T) \in \mathscr{L}_{\gamma'}$, Lemma 2.1(iii) implies $\pi_{\gamma'}(T)(\gamma)$

$= C(\gamma + \gamma')$ for a scalar C depending on γ. Since $\pi_{\gamma'}(T)(\gamma) \in B_\gamma$ and $\Psi(\gamma + \gamma') < \Psi(\gamma)$, we conclude that $C = 0$ and hence $\pi_{\gamma'}(T)(\gamma) = 0$. This completes the proof of the proposition.

We now define the support and cosupport of an operator T on B where B is $C(G)$ or $L^p(G)$, $1 \leq p < \infty$.

First, we consider operators on $C(G)$. For any subset M of G, we denote by M^c, the complement of M in G and by $C(M)$ the subspace of $C(G)$ defined by

$$C(M) = \{f \in C(G) : \text{Support } f \subseteq M\}.$$

Let T be an operator on B. The *support* of T is defined to be the smallest closed subset K of G such that $Tf = 0$ for every $f \in C(K^c)$.

The *cosupport* of T is defined to be the smallest closed subset K of G such that $Tf \in C(K)$ for every $f \in C(G)$.

We now consider operators on $L^p(G)$, $1 \leq p < \infty$. For a measurable subset M of G, $L^p(M)$ denotes the subspace of $L^p(G)$ defined by

$$L^p(M) = \{f \in L^p(G) : f = 0 \text{ a.e. on } M^c\}.$$

A measurable subset M of G is said to be a supporting set for an operator T on $L^p(G)$ if $T(L^p(M^c)) = 0$. Let m be the Haar measure on G. We define the constant s_T by

$$s_T = \inf \{m(K) : K \text{ a supporting set for } T\}.$$

We can show (cf. [3]) that there is a supporting set K for T such that $m(K) = s_T$ and if M is any supporting set for T then $m(K \setminus M) = 0$. Such a supporting set K is called a *support* for T.

Cosupport of T is similarly defined. A measurable subset K of G is said to be a *cosupporting set* for T if $T(L^p(G)) \subseteq L^p(K)$. We define the constant cs_T by

$$cs_T = \inf\{m(K) : K \text{ a cosupporting set for } T\}.$$

Once again, we can show that there is a cosupporting set K for T such that $m(K) = cs_T$ and if M is any cosupporting set for T then $m(K \setminus M) = 0$. Such cosupporting set K is called a *cosupport* for T.

For the circle group $\mathbf{T}$ with the usual ordering on Z, De Leeuw [3] established the following generalization of the second F. and M. Riesz Theorem.

THEOREM D_2. *Let T be an analytic operator on $L^2(\mathbf{T})$. If $T \neq 0$, then $\mathbf{T}$ is both a support and a cosupport for T.*

In this form, this theorem can not be generalized to compact abelian groups with ordered duals, unless some condition is imposed on the ordering. We give below an example of a nonzero ϕ-analytic operator T on $L^2(G)$ such that G is not a support for T.

Example. Let $G = \mathbf{T} \times \{0, 1\}$, where $\{0, 1\}$ is the discrete group of two elements under addition modulo 2. Then $\Gamma = Z \times \{1, -1\}$. Consider the order on Γ given by the continuous homomorphism $\Psi : Z \times (1, -1\} \to \mathbf{R}$ defined by $\Psi(n, j) = n$. Then the induced map $\phi : \mathbf{R} \to \mathbf{T} \times \{0, 1\}$ is given by $\phi(t) = (e^{it}, 0)$. In this case $\phi(\mathbf{R}) = \mathbf{T} \times \{0\} = \mathbf{T}_0$. Let $f \in C(\mathbf{T})$ be such that $f \neq 0$ and $\hat{f}(n) = 0$ for $n < 0$. Define g: $\mathbf{T} \times \{0, 1\} \to \mathbf{C}$ by

$$g(t, j) \begin{cases} = f(t) & \text{if } j = 0, \\ = 0 & \text{if } j = 1. \end{cases}$$

Then $g \in C(G)$, $g \neq 0$ and support $g \subseteq \mathbf{T}_0$. Also $\hat{g}(m, k) = \hat{f}(m)$.

Hence $\hat{g}(m, k) = 0$ if $(m, k) < 0$. Therefore M_g is a ϕ-analytic operator on $L^2(G)$ but support M_g = support $g \subseteq \mathbf{T}_0$ which implies that G is not a support for M_g.

The condition which we impose on Γ for our generalization of the Theorem D_2 is motivated by the following theorem of De Leeuw and Glicksberg [4].

THEOREM DG-2. *Suppose $\phi(\mathbf{R})$ is dense in G. Let μ be a ϕ-analytic measure on G such that either*

(i) *μ vanishes identically on an open subset of G, or*
(ii) *μ is absolutely continuous and vanishes identically on a Borel set of positive Haar measure.*

Then $\mu = 0$.

We shall need the following corollary of this theorem.

COROLLARY DG-2. *Suppose $\phi(\mathbf{R})$ is dense in G. Let μ be a measure on G such that $\hat{\mu}(\gamma) = 0$ whenever $\Psi(\gamma) \leq \Psi(\gamma_0)$ or $\hat{\mu}(\gamma) = 0$ whenever $\Psi(\gamma) \geq \Psi(\gamma_0)$. If either*

(i) *μ vanishes identically on an open subset of G, or*
(ii) *μ is absolutely continuous and vanishes identically on a Borel set of positive Haar measure*

then $\mu = 0$.

We are now ready to state and prove our generalization of the second F. and M. Riesz Theorem. We remark that our theorem has wider scope than that of De Leeuw as it applies to operators on $L^p(G)$, $1 \leq p < \infty$ or $C(G)$ even for $G = \mathbf{T}$.

THEOREM 4.2. *Suppose $\phi(\mathbf{R})$ is dense in G. Let T be a nonzero ϕ-analytic operator on $L^p(G)$ or $C(G)$, $1 \leq p < \infty$. Then G is both a support and a cosupport for T.*

The proof of this theorem is similar to that of Theorem 5.1 in [3].

We start by proving the following:

LEMMA 4.3. (1) *If K is a measurable subset of G of positive Haar measure and $\gamma \in \Gamma$, then for* $1 \leq p < \infty$,

(i) $L^p_\gamma(G) \cap L^p(K^c) = \{0\}$, *and*
(ii) $L^p_\gamma(G) + L^p(K)$ *is dense in* $L^p(G)$.

(2) *If K is a closed subset of G with nonvoid interior, then*

(i) $C_\gamma(G) \cap C(K^c) = \{0\}$, *and*
(ii) $C_\gamma(G) + C(K)$ *is dense in* $C(G)$.

Proof. (1) Part (i) follows from Corollary DG-2. We prove (ii) by contradiction. Suppose $L^p_\gamma(G) + L^p(K)$ is not dense in $L^p(G)$. Then there is an $f \in L^q(G)$ (q is the conjugate index of p) such that $f \neq 0$ a.e. and

$$\int_G g(x)\overline{f(x)}\,dx = 0 \quad \text{for } g \in L^p(G)$$

or $g \in L^p(K)$. This implies that $\hat{f}(\gamma') = 0$ for $\Psi(\gamma') \geq \Psi(\gamma)$ and $f = 0$ a.e. on K. It follows from Corollary DG-2 that $f = 0$ a.e., which is a contradiction.

(2) Part (i) again follows from Corollary DG-2 and we prove (ii) by contradiction. Suppose $C_\gamma(G) + C(K)$ is not dense in $C(G)$. Then there exists a nonzero measure $\mu \in M(G)$ such that $\hat{\mu}(\gamma') = 0$ for $\Psi(\gamma') \geq \Psi(\gamma)$ and μ vanishes identically on the interior of K. It follows from Corollary DG-2 that $\mu = 0$, which is a contradiction.

This completes the proof of Lemma 4.3.

Proof of Theorem 4.2. Let us begin by considering the case when T is a ϕ-analytic operator on $L^p(G)$, $1 \leq p < \infty$.

We first show that G is a support for T. Suppose not, then there exists a measurable subset K of G of positive Haar measure such that $T(L^p(K)) = \{0\}$. By the ϕ-analyticity of T and Proposition 4.1,

$$T(L^p_\gamma(G) + L^p(K)) \subseteq L^p_\gamma(G) \quad \text{for all } \gamma \in \Gamma.$$

By Lemma 4.3, $T(L^p(G)) \subseteq L^p_\gamma(G)$ for every $\gamma \in \Gamma$. Since

$$\bigcap_{\gamma \in \Gamma} L^p_\gamma(G) = \{0\},$$

it follows that $T = 0$, which is a contradiction. Hence G is a support for T.

Next, suppose that G is not a cosupport for T. Then there exists a measurable subset K of G such that K^c has positive Haar measure and $T(L^p(G)) \subseteq L^p(K)$. By Proposition 4.1,

$$T(L^p_\gamma(G)) \subseteq L^p_\gamma(G) \cap L^p(K) \quad \text{for all } \gamma \in \Gamma.$$

Since K^c has positive Haar measure, Lemma 4.3 implies that

$$L^p_\gamma(G) \cap L^p(K) = \{0\}$$

and hence $T(L^p_\gamma(G)) = 0$. Since this is true for all $\gamma \in \Gamma$, it follows that $T = 0$, which is a contradiction.

Proceeding exactly as above, we obtain the proof for ϕ-analytic operators on $C(G)$. We omit the details.

This completes the proof of Theorem 4.2.

References

1. W. B. Arveson, *On groups of automorphisms of operator algebras*, J. Functional Analysis, vol. 15 (1974), pp. 217–243.
2. K. De Leeuw, *An harmonic analysis for operators I: Formal properties*, Illinois J. Math., vol. 19 (1975), pp. 593–606.
3. ———, *An harmonic analysis for operators II: Operators on Hilbert space and analytic operators*, Illinois J. Math., vol. 21 (1977), pp. 164–175.
4. K. De Leeuw and I. Glicksberg, *Quasi-invariance and analyticity of measures on compact groups*, Acta Math., vol. 109 (1963), pp. 179–205.
5. R. E. Edwards, *Fourier Series—A modern introduction*, vol. II, Rinehart and Winston, New York, 1967.
6. H. G. Feichtinger, *Multipliers from $L^1(G)$ to a homogeneous Banach space*, J. Math. Anal. Appl., vol. 61 (1977), pp. 341–356.
7. F. Forelli, *Analytic and quasi-invariant measures*, Acta Math., vol. 118 (1967), pp. 33–59.
8. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, vol. I, Springer-Verlag, New York, 1963.
9. ———, *Abstract harmonic analysis*, vol. II, Springer-Verlag, New York, 1970.
10. E. Hille and R. S. Phillips, *Functional analysis and Semigroups*, Amer. Math. Soc. Colloq. Publications, vol. XXXI, Amer. Math. Soc., Providence, R.I., 1957.
11. K. Hoffman, *Banach Space of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
12. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968.
13. R. Larsen, *An introduction to the theory of multipliers*, Springer, Berlin, 1971.
14. R. Loebl and C. Schochet, *Covariant representations of the Calkin algebra*, I, Duke Math. J., vol. 45 (1978), pp. 721–734.
15. W. Rudin, *Fourier analysis on groups*, Interscience tracts in pure and applied mathematics, No. 12, Interscience, New York, 1962.

Indian Institute of Technology
Kanpur, India.