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A RATIO ERGODIC THEOREM FOR GROUPS OF MEASURE-PRESERVING TRANSFORMATIONS

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Introduction

In this paper we use the method introduced in ergodic theory by A. P. Calderón [1] combined with a covering lemma due to Besicovich to obtain the pointwise convergence of averages formed with n-parameter groups of measure-preserving transformations.

Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. By an n-parameter group of measure-preserving transformations we mean a system of mappings $(\theta_t, t \in R^n)$ of X into itself having the following properties:

(i) $\theta_t(\theta_s x) = \theta_{t+s}x$; $\theta_0 x = x$ for every t and s in R^n and every x in X.

(ii) for every measurable subset E of X, $\theta_t(E)$ is measurable and its measure equals the measure of E, for any t in R^n.

(iii) For any function f measurable on X, the function $f(\theta_t x)$ is measurable on the product space $R^n \times X$, where the euclidean space R^n is endowed with Lebesgue measure.

Let p be a non-negative function in $L^1(\mu)$. For each function f integrable over X, we consider the ratios

$$R_\alpha(f, p)(x) = \frac{\int_{B_\alpha} f(\theta_t x)dt}{\int_{B_\alpha} p(\theta_t x)dt} \quad \text{if} \int_{B_\alpha} p(\theta_t x)dt > 0,$$

$R_\alpha(f, p)(x) = 0$ otherwise, where B_α is the ball in R^n of radius α and center at the origin.

In what follows we give sufficient conditions for the almost everywhere convergence of $R_\alpha(f, p)$, as $\alpha \to \infty$, in the set where the denominators eventually become positive and therefore it arises a continuous version of the Chacon and Ornstein theorem [2].

If f is integrable over X, we denote $\int_X f\, d\mu$ by $\int f(x)dx$.

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1. A Maximal Ergodic Inequality

For each $f \in L^1(\mu)$ we define the maximal operator S associated to $R_\alpha(f, p)$ by the formula

$$S(f, p)(x) = \sup_{\alpha>0} R_\alpha(|f|, p)(x).$$

In this section we prove that S satisfies an inequality of weak type (1.1). We will need the following lemmas.

LEMMA 1.1. *Let $\mathcal{T} = \{B_{r_i}(t_i)\}_{i\in I}$ be a family of balls in R^n with bounded radius, where, for each i, $B_{r_i}(t_i)$ is the ball of radius r_i with center at t_i. There exists a subfamily $\mathcal{T}_1 = \{B_{r_i}(t_i)\}_{i\in J}$ such that, if A denotes the set of centers of the balls in $\mathcal{T}$ and χ_i is the characteristic function of $B_{r_i}(t_i)$, then*

$$\chi_A \leqslant \sum_{i\in J} \chi_i \leqslant C,$$

where C is a constant depending only on the dimension n.

For the proof of this lemma we refer to de Guzmán [4].

LEMMA 1.2. *Let $q(t) \geqslant 0$ be integrable over each subset of R^n with finite measure. For each $g \in L^1(R^n)$ we define*

$$T_\alpha(g, q)(s) = \frac{\int_{B_\alpha} g(s+t)dt}{\int_{B_\alpha} q(s+t)dt}$$

if the denominator is positive, $T_\alpha(g, q)(s) = 0$ otherwise.

If we write

$$T^*(g, q)(s) = \sup_{\alpha>0} T_\alpha(|g|, q)(s)$$

then there exists a constant $C > 0$ such that

$$\int_{\{T^*(g,q)>\lambda\}} q(t)dt \leqslant \frac{C}{\lambda} \|g\|_{L^1(R^n)} \quad \text{for any } \lambda > 0.$$

Proof. For each positive integer k, we define

$$T_k^*(g, q)(s) = \sup_{0<\alpha\leqslant k} T_\alpha(|g|, q)(s),$$

so that $T_k^*(g, q)(s) \leqslant T_{k+1}^*(g, q)(s)$ and $\lim_{\alpha\to\infty} T_k^*(g, q)(s) = T^*(g, q)(s)$. For a given $\lambda > 0$ let us consider $E = \{s : T_k^*(g, q)(s) > \lambda\}$. If s belongs to E there exists $\alpha = \alpha(s) \leqslant k$ such that $T_\alpha(|g|, q)(s) > \lambda$; then

$$\int_{B_\alpha} q(s+t)dt = \int_{s+B_\alpha} q(t)dt < \frac{1}{\lambda}\int_{s+B_\alpha} |g(t)|dt.$$

By virtue of Lemma 1.1, if $\mathcal{T} = \{s + B_{\alpha(s)}\}_{s\in E}$ then there exists a subfamily $\mathcal{T}_1 = \{s_i + B_{\alpha(s_i)}\}_{i\in J}$ and a constant $C > 0$ such that

$$\chi_E \leqslant \sum_{i\in J} \chi_i \leqslant C,$$

where χ_i stands for the characteristic function of $s_i + B_{\alpha(s_i)}$.

Therefore

$$\begin{aligned}\int_E q(t)dt &\leqslant \int\Big(\sum_{i\in J}\chi_i(t)\Big)q(t)dt = \sum_{i\in J}\int_{s_i+B_{\alpha(s_i)}} q(t)dt \\ &\leqslant \sum_{i\in J}\frac{1}{\lambda}\int_{s_i+B_{\alpha(s_i)}}|g(t)|dt = \frac{1}{\lambda}\int\Big(\sum_{i\in J}\chi_i\Big)|g(t)|dt \\ &\leqslant \frac{C}{\lambda}\|g\|_{L^1(R^n)},\end{aligned}$$

and Lemma 1.2 follows by letting $k \to \infty$.

We can now state and prove the following theorem.

THEOREM 1.1. *Let p be a nonnegative function in $L^1(\mu)$. For each $f \in L^1(\mu)$ we define*

$$R_\alpha(f, p)(x) = \frac{\int_{B_\alpha} f(\theta_t x)dt}{\int_{B_\alpha} p(\theta_t x)dt},$$

if the denominator is positive, $R_\alpha(f, p)(x) = 0$ otherwise.

If

$$S(f, p)(x) = \sup_{\alpha>0} R_\alpha(|f|, p)(x),$$

then there exists a constant $C > 0$, depending only on the dimension n, such that for each $\lambda > 0$,

$$\int_{\{S(f,p)>\lambda\}} p(x)dx \leqslant \frac{C}{\lambda}\|f\|_{L^1(\mu)}$$

Proof. For each positive integer k, we write

$$F(t, x) = \begin{cases} f(\theta_t x) & \text{if } |t| \leqslant 2k \\ 0 & \text{if } |t| > 2k. \end{cases}$$

It follows from Fubini's theorem that $F(t, x)$ is an integrable function of t for almost all x.

Let $P(t, x)$ denote the function $p(\theta_t x)$ defined over the product space $R^n \times X$.

With the notation of Lemma 1.2, we have

$$T_\alpha(F, P)(s, x) = \frac{\int_{B_\alpha} F(s + t, x)dt}{\int_{B_\alpha} P(s + t, x)dt},$$

if the denominator is positive, $T_\alpha(F, P)(s, x) = 0$ otherwise.

Let us define

$$S_k(F, P)(s, x) = \begin{cases} \sup\limits_{0<\alpha\leq k} T_\alpha(|F|, P)(s, x) & \text{if } |s| \leq k \\ 0 & \text{if } |s| > k, \end{cases}$$

and

$$S(F, P)(s, x) = \sup_{\alpha>0} T_\alpha(|F|, P)(s, x),$$

So that

$$S_k(F, P)(s, x) \leq S_{k+1}(F, P)(s, x)$$

and

$$\lim_{k\to\infty} S_k(F, P)(s, x) = S(F, P)(s, x).$$

For a given $\lambda > 0$ let us consider

$$E = \{(s, x): S_k(F, P)(s, x) > \lambda\}$$

and its sections

$$E_s = \{x: (s, x) \in E\}; \; E^x = \{s: (s, x) \in E\}.$$

We observe that for $|s| \leq k$,

$$S_k(F, P)(s, x) = S_k(F, P)(0, \theta_s x),$$

and therefore $E_s = \theta_s^{-1}(E_0)$ for $|s| \leq k$ while $E_s = \emptyset$ if $|s| > k$. Then

$$\begin{aligned} \int_E P(s, x)ds\, dx &= \int_{|s|\leq k} ds \int_{E_s} P(s, x)dx \\ &= \int_{|s|\leq k} ds \int \theta_s^{-1}(E_0)\, p(\theta_s x)dx \\ &= \omega_n k^n \int_{E_0} p(x)dx, \end{aligned}$$

where ω_n is the measure of the unit ball in R^n.

On the other hand, by virtue of Lemma 1.2, we have

$$\begin{aligned}\int_E P(s, x)ds\, dx &= \int dx \int_{E^x} P(s, x)ds \\ &\leqslant \int dx \frac{C}{\lambda} \int_{|s| \leqslant 2k} |f(\theta_s x)| ds \\ &= \frac{C}{\lambda}(2k)^n\, \omega_n \|f\|_{L^1(\mu)}.\end{aligned}$$

Therefore

$$\int_{E_0} p(x)dx \leqslant \frac{2^n C}{\lambda} \|f\|_{L^1(\mu)},$$

and theorem 1.1 follows from the last inequality by letting $k \to \infty$. If we set

$$\mu_p(E) = \int_E p(x)dx,$$

for any measurable set E of X, then we can express the inequality of Theorem 1.1 by

$$\mu_p(\{S(f, p) > \lambda\}) \leqslant \frac{C}{\lambda} \|f\|_{L^1(\mu)}.$$

We will say that a function l measurable on X is invariant if for every t, $l(\theta_t x) = l(x)$ for almost all x. A measurable subset E of X will be called invariant if its characteristic function is invariant. The invariant subsets of X form a σ-field that we shall denote by $\mathcal{I}$. It is easily seen that a measurable function is invariant if and only if it is measurable with respect to $\mathcal{I}$.

In what follows we shall assume that the group $(\theta_t,\ t \in R^n)$ and the function p satisfy the following condition:

(A) For almost all x

$$\lim_{\alpha \to \infty} \frac{\int_{B_\alpha \Delta (s + B_\alpha)} p(\theta_t x)dt}{\int_{B_\alpha} p(\theta_t x)dt} = 0,$$

for every s in R^n, where Δ denotes the symmetric difference.

At the end of the next section we will prove that (A) is unnecessary for the almost everywhere convergence of $R_\alpha(f, p)$ if $\mu(X) < \infty$ or if $n = 1$.

2. Convergence and identification of the limit when p > 0 a.e.

THEOREM 2.1. *If* $(\theta_t,\ t \in R^n)$ *and* $p > 0$ a.e. *satisfy* (A) *then for any* f *in*

$L^1(\mu)$ the ratios

$$R_\alpha(f, p)(x) = \frac{\int_{B_\alpha} f(\theta_t x)dt}{\int_{B_\alpha} p(\theta_t x)dt}$$

converge almost everywhere in X as $\alpha \to \infty$.

Proof. Let us consider the set of all function h which can be represented in the form

$$h(x) = (p \cdot g)(x) - (p \cdot g)(\theta_s x),$$

where g is a bounded function and s is any point in R^n. For any function h of this form we have

$$\begin{aligned} \left| \int_{B_\alpha} h(\theta_t x)dt \right| &= \left| \int_{B_\alpha} \{(p \cdot g)(\theta_t x) - (p \cdot g)(\theta_{t+s} x)\}dt \right| \\ &= \left| \left\{ \int_{B_\alpha} - \int_{s+B_\alpha} \right\} (p \cdot g)(\theta_t x)dt \right| \\ &\leqslant \int_{B_\alpha \Delta (s+B_\alpha)} |(p \cdot g)(\theta_t x)|dt. \end{aligned}$$

Since $g(\theta_t x)$ is a bounded function of t for almost all x, we see by virtue of (A) that $R_\alpha(h, p)$ tends to zero for almost all x as $\alpha \to \infty$.

If $l(x)$ in $L^\infty(\mu)$ is invariant, for almost all x we have $l(\theta_t x) = l(x)$ for almost all t. Then for any function $q(x)$ of the form $q(x) = l(x)\ p(x)$, we have

$$R_\alpha(q, p)(x) = l(x) \quad \text{a.e.}$$

We conclude that the ratios $R_\alpha(f, p)$ converge almost everywhere if f is in the linear span V of the functions h and q. Our second step in the proof is to show that V is dense in $L^1(\mu)$. For this purpose, let us assume that a certain function $k_0(x)$ in $L^\infty(\mu)$ is orthogonal to all functions of V. Therefore

$$\begin{aligned} \int k_0(x)h(x)dx &= \int k_0(x)\{(p \cdot g)(x) - (p \cdot g)(\theta_s x)\}dx \\ &= \int g(x)p(x)\{k_0(x) - k_0(\theta_{-s} x)\}dx \\ &= 0 \end{aligned}$$

for any bounded function g and for any s in R^n. Since $p > 0$ a.e. we deduce that k_0 is invariant which implies that $k_0 \cdot p \in V$. Therefore $\int k_0^2(x)p(x)dx = 0$. Then $k_0 = 0$ a.e., which proves the density in $L^1(\mu)$ of the linear span

V. By virtue of the inequality

$$\mu_p(\{S(f, p) > \lambda\}) \leq \frac{C}{\lambda}\|f\|_{L^1(\mu)}$$

Theorem 2.1 follows from a standard argument.

Let f be of the form

$$f(x) = l(x)p(x) + h(x), \tag{1}$$

where l and h are as in the last theorem, and define

$$\chi(x) = \begin{cases} 1 & \text{if } l(x) \geq 0 \\ -1 & \text{if } l(x) < 0. \end{cases}$$

Then χ is invariant and we have $\int \chi(x)h(x)dx = 0$. Therefore

$$\int |l(x)|p(x)dx = \int f(x)\chi(x)dx \leq \int |f(x)|dx.$$

We deduce that for any given $f \in L^1(\mu)$ there can be at most one l (up to equivalence) for which (1) can hold for some h. Therefore, the mapping $f \to l \cdot p$ is well defined on V, and it is linear and bounded in the L^1 norm. We can thus conclude that this mapping has a unique extension to a bounded linear operator H of L^1 into itself such that $\int|Hf|dx \leq \int|f|dx$, for all $f \in L^1(\mu)$.

Following the method used by A. Garsia [3] for the identification of the limit in the Chacon-Ornstein theorem, we can now prove the following result.

THEOREM 2.2. *If for each f in $L^1(\mu)$,*

$$R(f, p)(x) = \lim_{\alpha \to \infty} R_\alpha(f, p)(x),$$

then $R(f, p)$ is invariant, and, for any $E \in \mathcal{I}$,

$$\int_E R(f, p)p\,dx = \int_E f\,dx.$$

Proof. If f is of the form (1), then

$$R(f, p)(x) = l(x) = \frac{Hf(x)}{p(x)} \quad \text{a.e.}$$

Therefore $R(f, p) = Hf/p$ for any $f \in V$.

Let now $f, f_\varepsilon \in L^1(\mu)$ with $\|f - f_\varepsilon\|_1 < \varepsilon$ and assume that $f_\varepsilon \in V$. Then

$$\left|R_\alpha(f, p) - \frac{Hf}{p}\right| \leq |R_\alpha(f - f_\varepsilon, p)| + \left|R_\alpha(f_\varepsilon, p) - \frac{Hf_\varepsilon}{p}\right| + \left|\frac{H(f_\varepsilon - f)}{p}\right|;$$

thus

$$\limsup_{\alpha\to\infty}\left|R_\alpha(f, p) - \frac{Hf}{p}\right| \leq S(f - f_\varepsilon, p) + \frac{|H(f - f_\varepsilon)|}{p}.$$

For a given $\lambda > 0$, we have

$$\mu_p(\{S(f - f_\varepsilon, p) > \lambda/2\}) \leq \frac{C}{\lambda}\|f - f_\varepsilon\|_{L^1(\mu)}$$

and

$$\mu_p\left(\left\{\frac{|H(f - f_\varepsilon)|}{p} > \lambda/2\right\}\right) \leq \frac{2}{\lambda}\|f - f_\varepsilon\|_{L^1(\mu)},$$

which implies that

$$\mu_p\left(\left\{\limsup_{a\to\infty}\left|R_\alpha(f, p) - \frac{Hf}{p}\right| > \lambda\right\}\right) \leq \frac{C + 2}{\lambda}\|f - f_\varepsilon\|_{L^1(\mu)}.$$

By letting $\varepsilon \to 0$ we deduce that $R(f, p) = Hf/p$ a.e.. Since $R(f_\varepsilon, p)$ is invariant for f_ε in V, it follows that $R(f, p)$ is invariant for all f in $L^1(\mu)$. Finally, we note that for any set $E \in \mathcal{I}$,

$$\int_E R(f_\varepsilon, p) \cdot p\, dx = \int_E f_\varepsilon\, dx,$$

and Theorem 2.2 follows by letting $\varepsilon \to 0$.

Remarks. (i) Convergence when $\mu(X) < \infty$. For any f in $L^1(\mu)$ we consider the averages

$$R_\alpha(f, 1) = \frac{1}{|B_\alpha|}\int_{B_\alpha} f(\theta_t x)dt,$$

where the vertical bars stand for Lebesgue measure. Since the function $\chi(x) = 1$ a.e. satisfies (A) we deduce from the preceding the almost everywhere convergence of $R_\alpha(f, 1)$. If $p > 0$ a.e. is in $L^1(\mu)$ we have

$$\sup_{\alpha>0}\frac{|B_\alpha|}{\int_{B_\alpha} p(\theta_t x)dt} = S(1, p)(x) < \infty \quad \text{a.e.;}$$

therefore

$$\lim_{\alpha\to\infty}\frac{1}{|B_\alpha|}\int_{B_\alpha} p(\theta_t x)dt > 0 \quad \text{a.e.,}$$

from which we deduce that $R_\alpha(f, p)$ converges almost everywhere for any $f \in L^1(\mu)$.

(ii) Convergence when $n = 1$. Let us consider the set of all functions $\bar{h}$ which can be represented in the form

$$\bar{h}(x) = g(x) - g(\theta_s x),$$

where g is a bounded function having support of finite measure.

It is not difficult to prove that, for any function $\bar{h}$ of this form, $R_\alpha(\bar{h}, p)$ tends to zero almost everywhere although p does not satisfy (A). It is also easily seen that Theorem 2.1 follows by replacing h by $\bar{h}$.

3. Convergence in the General Case

Let us consider $p \geqslant 0$ and for any u in R^n let us define $p_u(x) = p(\theta_u x)$. It is easily seen that if p satisfies (A) then p_u also does so. Thus, by virtue of Theorems 1.1 and 2.1 we conclude that $R_\alpha(f, p_u)(x)$ converges almost everywhere in $\{p_u > 0\}$. Since

$$\lim_{\alpha \to \infty} \frac{\int_{B_\alpha} p_u(\theta_t x)dt}{\int_{B_\alpha} p(\theta_t x)dt} = 1 \quad \text{a.e.},$$

by virtue of (A), the relation $R_\alpha(f, p_u) = R_\alpha(f, p) \cdot R_\alpha(p, p_u)$ shows that $R_\alpha(f, p)$ converges for almost all x in $\{p_u > 0\}$ to a finite limit $R(f, p)$. If we call E the set where $R_\alpha(f, p)$ does not converge then we have

$$\int_E dx \int_{B_\alpha} p(\theta_t x)dt = \int_{B_\alpha} dt \int_E p_t(x)dx = 0,$$

for every $\alpha > 0$, and from this it follows that $R(f, p)$ converges almost everywhere in the set where the denominators eventually become positive.

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