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SMALL INTO ISOMORPHISMS ON L_p SPACES

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Recently, Y. Benyamini proved that an isomorphism T of $C(K)$, K compact metric, into some $C(S)$, S compact Hausdorff is close to an isometry if $\|T\|\|T^{-1}\|$ is close to one [3]. In this paper we show that a similar result holds for isomorphisms of $L_p(\mu)$ into $L_p(\nu)$. The method of proof is completely different than that of [3]. Our argument depends heavily upon results of Dor [5] and Schechtman [10]. In fact one easy consequence of their results is that an isomorphism T of l_p into $L_p(\nu)$ is close to an isometry if $\|T\|\|T^{-1}\|$ is close to one (see Section 1). Hence the main point of this paper is to show how this can be extended to more complicated measure spaces.

There are a few other results in this direction. It was proved in [2] that if there is an isomorphism T of $L_p(\mu)$ *onto* $L_p(\nu)$ and $L_p(\mu)$ is separable then if $\|T\|\|T^{-1}\|$ is small enough $L_p(\mu)$ and $L_p(\nu)$ are isometric. The proof does not explicitly construct an isometry but rather observes that it is sufficient to show that the number of atoms in each measure space is the same. The form of isometries of $L_p(\mu)$ into $L_p(\nu)$ is well known, e.g., [1], [6], [7], [9].

This paper is organized as follows: In Section 1 we prove the main result for the special case of $T: L_p[0, 1] \to L_p(\nu)$. This case illustrates all of the major ideas needed to handle more general measures. In Section 2, we then describe the proof for the general case.

We will use standard notation and facts from Banach space theory as may be found in the books of Lindenstrauss and Tzafriri [8]. Throughout this paper p will be restricted to the values $[1, \infty) - \{2\}$. The case $p = 2$, of course, is special because there are many more isometries. This case can be handled easily by using the polar decomposition of an operator.

1. The separable case

We first state the results of Dor and Schechtman.

THEOREM A [Dor]. *Suppose $\{x_i : i \in N\}$ is a subset of $L_p(\nu)$ for some measure ν, such that for any set of scalars $\{a_i : i \in N\}$, with finitely many*

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non-zero,

$$(1 - \varepsilon)\left(\sum |a_i|^p\right)^{1/p} \leq \left\| \sum a_i x_i \right\| \leq (1 + \varepsilon)\left(\sum |a_i|^p\right)^{1/p}.$$

Then if ε is sufficiently small there exist disjoint measurable sets $\{A_i : i \in N\}$ such that $\|x_{i|A_i^c}\| < a_1(p, \varepsilon)$, where A_i^c denotes the complement of A_i. Moreover, $a_1(p, \varepsilon) \to 0$ as $\varepsilon \to 0$.

This result is stated in the paper of Dor only for the case of $L_p[0, 1]$. However, the proof of Proposition 2.2 of [5] yields a family of functions $\{\phi_i : i \in N\} \subset L_\infty(\nu)$ such that $\phi_i \geq 0$, $\Sigma \phi_i \leq 1$ a.e., and $\int |x_i|^p \phi_i d\nu \geq b(p, \varepsilon)$ for all $i \in N$. Because $b(p, \varepsilon) \to 1$ as $\varepsilon \to 0$, for ε sufficiently small there is a constant $a_1(p, \varepsilon)$ such that

$$\left[\int_{A_i} |x_i|^p\right]^{1/p} \geq 1 - a_1(p, \varepsilon) \quad \text{for all } i \in N$$

where $A_i = \left\{\phi_i > \frac{1}{2}\right\}$ and $a_1(p, \varepsilon) \to 0$ as $\varepsilon \to 0$. Clearly these A_i satisfy the conclusion of Theorem A.

THEOREM B (Schechtman). *Let $\{x_i : i \in N\} \subset L_p(\nu)$ satisfy the hypothesis of Theorem* A *and let $\{A_i : i \in N\}$ be measurable sets such that*

$$\int_{A_i} |x_i|^p \, d\nu \geq c \quad \text{for all } i \in N.$$

Then

$$\left\| \sum a_i x_{i|A_i^c} \right\| \leq a(\varepsilon, c)\left(\sum |a_i|^p\right)^{1/p}$$

for all sets of scalars $\{a_i : i \in N\}$ with finitely many nonzero, and $a(\varepsilon, c) \to 0$ as $\varepsilon \to 0$ and $c \to 1$.

As we noted earlier if $T : l_p \to L_p(\nu)$ and $\|T\|\|T^{-1}\|$ is sufficiently close to 1, then $x_i = \|Te_i\|^{-1} Te_i$, $i = 1, 2, \ldots$, where $\{e_i : i \in N\}$ is the usual unit vector basis of l_p, satisfies the hypothesis of Theorem A. Thus there are disjoint measurable subsets $\{A_i : i \in N\}$ such that

$$\int_{A_i} |x_i|^p \geq b(p, \varepsilon) \quad \text{for all } i \in N$$

where $(1 - \varepsilon)^{-1} = \|T\|\|T^{-1}\|$. Define an isometry $S : l_p \to [x_{i|A_i}]_{i \in N}$ by

$S(\Sigma\, a_i e_i) = \Sigma a_i \|x_{i|A_i}\|^{-1}\, x_{i|A_i}$. Then

$$\left\|(T - S)\left(\sum a_i e_i\right)\right\|$$
$$= \left\|\sum a_i(Te_i - \|Te_{i|A_i}\|^{-1}\, Te_{i|A_i})\right\|$$
$$\leq \left\|\sum a_i(1 - \|Te_{i|A_i}\|^{-1})\, Te_i\right\| + \left\|\sum a_i \|Te_{i|A_i}\|^{-1}\, Te_{i|A_i^c}\right\|$$
$$\leq \sup_i |1 - \|Te_{i|A_i}\|^{-1}| \|T\| \left(\sum |a_i|^p\right)^{1/p}$$
$$+ a(\varepsilon, b(p, \varepsilon)) \sup_i \|Te_{i|A_i}\|^{-1} \left(\sum |a_i|^p\right)^{1/p}.$$

Thus $\|T - S\| \to 0$ as $\varepsilon \to 0$.

Now we will prove the theorem stated in the abstract for the case of $L_p[0, 1]$. Let $T: L_p[0, 1] \to L_p(\nu)$ be an into isomorphism with

$$\|T\| \|T^{-1}\| < (1 - \varepsilon)^{-1}.$$

We want to choose a family (tree) of measurable sets

$$\{A_{ki} : 1 \leq i \leq 2^k,\ k = 0, 1, 2, \ldots\}$$

such that

(1) $A_{k+1,2i-1} \cup A_{k+1,2i} \subset A_{ki}$, $1 \leq i \leq 2^k$, $k = 0, 1, 2, \ldots,$
(2) $A_{ki} \cap A_{kj} = \emptyset$, $i \neq j$,

and a function $h \in L_p(\nu)$ such that

(3) $\int_{A_{ki}} |h|^p \, d\nu = 2^{-k}$, $1 \leq i \leq 2^k$, $k = 0, 1, 2, \ldots.$

Once this is done there is a natural isometry $S: L_p[0, 1] \to L_p(\nu)$ such that

$$S([(i - 1)2^{-k}, i2^{-k})) = h \cdot A_{ki}, \quad i \leq i \leq 2^k,\ k = 0, 1, 2, \ldots.$$

(Here we adopt the notational convention that if A is a measurable set we will write A rather than 1_A.) The difficulty is to choose the tree of sets and the function h so that $\|T - S\|$ is small.

First we will choose the tree of sets by modifying the proof of Theorem A (Proposition 2.2 of [5]). Let

$$X = \left(\sum_{k=0}^{\infty} \sum_{i=1}^{2^k} L_1(\nu)\right)_{l_1}^{*} = \left(\sum_{k=0}^{\infty} \sum_{i=1}^{2^k} L_\infty(\nu)\right)_{l_\infty},$$

with the w^* topology. (We may assume $L_1(\nu)^* = L_\infty(\nu)$; in fact, for this case, we may assume $L_1(\nu) = (L_1\,[0,1] \oplus l_1)_{l_1})$. Let

$$D = \{(\phi_{ki})_{k=0}^{\infty}{}_{i=1}^{2^k} : \phi_{ki} \in L_\infty(\nu),\ \phi_{ki} \geqslant 0,\ \phi_{0,1} \leqslant 1,$$
$$\text{and } \phi_{k+1,2i-1} + \phi_{k+1,2i} \leqslant \phi_{k,i} \text{ a.e., } 1 \leqslant i \leqslant 2^k,\ k = 0, 1, 2, \dots\}$$

It is easy to see that D is a bounded w^* closed (hence w^* compact) subset of X. Define a map $\Psi : D \to l_\infty$ by

$$\Psi((\phi_{ki}))_{lj} = \|TI_{lj}\|^{-p} \int |TI_{lj}|^p\, \phi_{lj}\, d\nu$$

where

$$I_{lj} = [(j-1)2^{-l}, j2^{-l}) \quad\text{and}\quad l_\infty = l_\infty(\{(l,j) : 1 \leqslant j \leqslant 2^l,\ l = 0, 1, 2, \dots\}).$$

Clearly Ψ is w^* continuous.

We will show that there is a constant $c(\varepsilon)$, which depends on p as well, such that for each $r \in N$ there is an element $d \in D$ such that

$$(\Psi(d))_{lj} \geqslant c(\varepsilon) \text{ for all } l \leqslant r,\ 1 \leqslant j \leqslant 2^l \quad\text{and}\quad \lim_{\varepsilon\to 0} c(\varepsilon) = 1.$$

Once this is accomplished it will follow from the w^* continuity of Ψ and the compactness of D that there is an element $(\phi_{ki}) \in D$ such that $\Psi((\phi_{ki}))_{lj} \geqslant c(\varepsilon)$ for all l, j. We then let $A_{ki} = \left\{\phi_{ki} > \frac{1}{2}\right\}$ and we have (1), (2) and

(4) $\|TI_{ki}\|^{-p} \int |TI_{ki}|^p \cdot A_{ki}\, d\nu \geqslant c_1(\varepsilon)$ where $c_1(\varepsilon) = 2\,c(\varepsilon) - 1$.

We will need the following lemma:

LEMMA 1. *Let $\{B_i\}_{i=1}^n$ and $\{C_i\}_{i=1}^n$ be families of disjoint measurable sets from measure spaces (X, B, μ) and (Y, C, ν), respectively, and let*

$$T : L_p(X, B, \mu) \to L_p(Y, C, \nu)$$

be an isomorphism (into) with $\|T\|\|T^{-1}\| < (1-\varepsilon)^{-1}$. If there exists a constant $c > 0$ such that $\|(TB_i) \cdot C_i\| > c\,\|TB_i\|$ for $i = 1, 2, \dots, n$, then

$$\|(T(UB_i)) \cdot UC_i\| > (c - a(\varepsilon, c^p))\,\|T^{-1}\|^{-1}\|T\|^{-1}\,\|T(UB_i)\|.$$

Proof.

$$\left\|\left(T\left(\bigcup_i B_i\right)\right) \cdot \bigcup_j C_j\right\| = \left\|\sum_i (TB_i)\left(\bigcup_j C_j\right)\right\|$$
$$= \left\|\sum_i (TB_i) \cdot C_i + \sum_i (TB_i) \cdot \bigcup_{j\neq i} C_j\right\|$$
$$\geqslant \left\|\sum_i (TB_i) \cdot C_i\right\| - \left\|\sum_i (TB_i) \cdot \bigcup_{j\neq i} C_j\right\|$$

$$\geq \left(\sum_i \|(TB_i) \cdot C_i\|^p\right)^{1/p} - \left\|\sum_i (TB_i) \cdot C_i^c\right\|$$

(the C_i's are disjoint)

$$\geq \left(\sum_i c^p \|T(B_i)\|^p\right)^{1/p}$$

$$- \left\|\sum_i \|TB_i\| \, (\|TB_i\|^{-1} (TB_i) \cdot C_i^c)\right\|$$

$$\geq c\left(\sum_i \|TB_i\|^p\right)^{1/p} - a(\varepsilon, c^p)\left(\sum_i \|TB_i\|^p\right)^{1/p}$$

by Theorem B where $x_i = \|TB_i\|^{-1} TB_i$ and $A_i = C_i$. Note that

$$\begin{aligned}\left(\sum \|TB_i\|^p\right)^{1/p} &\geq \|T^{-1}\|^{-1}\left(\sum \|B_i\|^p\right)^{1/p} \\ &= \|T^{-1}\|^{-1} \|\cup B_i\| \\ &\geq \|T^{-1}\|^{-1} \|T\|^{-1} \|T(\cup B_i)\|\end{aligned}$$

Thus

$$\left\|\left(T\left(\bigcup_i B_i\right)\right) \cdot \bigcup_j C_j\right\| \geq [c - a(\varepsilon, c^p)] \, \|T^{-1}\|^{-1} \|T\|^{-1} \|T(\cup B_i)\|$$

as claimed.

Fix $r \in N$ and consider the elements $x_i = \|TI_{ri}\|^{-1} TI_{ri}$, $i = 1, 2, \dots 2^r$. These elements satisfy the hypothesis of Theorem A and thus there are disjoint measurable sets C_i, $i = 1, 2, \dots 2^r$ such that

$$\left[\int_{C_i} |x_i|^p\right]^{1/p} \geq 1 - a_1(p, \varepsilon).$$

Hence

$$\|(TI_{ri}) \cdot C_i\| \geq (1 - a_1(p, \varepsilon)) \, \|TI_{ri}\|.$$

Define an element $(\psi_{lj}) \in D$ as follows:

$$\psi_{lj} = \begin{cases} 0 & \text{if } l > r, \\ \cup\{C_i : (j-1)2^{r-l} < i \leq j\, 2^{r-l} & \text{if } l \leq r. \end{cases}$$

By Lemma 1,

$$\left[\int |TI_{lj}|^p \psi_{lj}\right]^{1/p} \geq [(1 - a_1(p, \varepsilon)) - a(\varepsilon, (1 - a_1(p, \varepsilon))^p)] \, \|T^{-1}\|^{-1} \|T\|^{-1} \|TI_{lj}\|$$

for $l \leq r$.

Therefore

$$\|TI_{lj}\|^{-p}\int|TI_{lj}|^p\psi_{lj}\geq c(\varepsilon)$$

where

$$c(\varepsilon)=[(1-a_1(p,\varepsilon))-a(\varepsilon,(1-a_1(p,\varepsilon))^p)]^p\|T^{-1}\|^{-p}\|T\|^{-p}\quad\text{for } l\leq r.$$

Clearly $\lim_{\varepsilon\to 0}c(\varepsilon)=1$ and thus, as noted above, the w^* compactness of D shows that

$$\{A_{ki}:1\leq i\leq 2^k,\ k=0,1,2,\ldots\}$$

satisfies (1), (2), and (4).

Our next task is to find the function h. This will be accomplished in two steps. For each $k\in\mathbf{N}$ let

$$g_k=\sum_{i=1}^{2^k}(TI_{ki})A_{ki}$$

and let g be a weak limit point of (g_k) in the case $p>1$. If $p=1$ we let γ be a w^* limit point of (g_k) in $L_1(\nu)^{**}$ and let g be the part of γ absolutely continuous with respect to ν. Note that in either case if $h\in L_p(\nu)$

$$\|g-h\|\leq\overline{\lim}\,\|g_k-h\|.$$

The function g is almost what we want but we must change its modulus.

Claim. There is a constant $c_2(\varepsilon)$ such that

$$2^{-k/p}c_2(\varepsilon)^{-1}\leq\|gA_{ki}\|\leq 2^{-k/p}c_2(\varepsilon)$$

for all $1\leq i\leq 2^k$, $k\in\mathbf{N}$, and $\lim_{\varepsilon\to 0}c_2(\varepsilon)=1$.

Proof of the claim. First observe that if $k<l$ and

$$\mathscr{A}^l_{ki}=\{j:(i-1)2^{l-k}<j\leq i2^{l-k}\}$$

then

$$\begin{aligned}\|(g_k-g_l)\cdot A_{ki}\|^p&=\Big\|\sum_{j\in\mathscr{A}^l_{ki}}TI_{lj}\cdot A_{lj}-TI_{ki}\cdot A_{ki}\Big\|^p\\&\leq\Big\|\sum_{j\in\mathscr{A}^l_{ki}}TI_{lj}-A_{lj}-TI_{ki}\Big\|^p\\&=\Big\|\sum_{j\in\mathscr{A}^l_{ki}}TI_{lj}(A^c_{lj})\Big\|^p\leq a(\varepsilon,c_1(\varepsilon))^p\Big(\sum_{j\in\mathscr{A}^l_{ki}}\|TI_{lj}\|^p\Big)\end{aligned}$$

by Theorem B. Hence

$$\|(g_k - g_l) \cdot A_{ki}\|^p \leq a(\varepsilon, c_1(\varepsilon))^p \|T\|^p \Big(\sum_{j \in \mathscr{A}^l_{ki}} \|I_{lj}\|^p \Big)$$
$$= a(\varepsilon, c_1(\varepsilon))^p \|T\|^p 2^{-k}.$$

Passing to the limit on l (using the lower semi continuity of $\|\cdot\|$) we have

$$\|(g - g_k) \cdot A_{ki}\|^p \leq a(\varepsilon, c_1(\varepsilon))^p \|T\|^p 2^{-k},$$

and thus

$$\begin{aligned}
&\|T\| 2^{-k/p} + a(\varepsilon, c_1(\varepsilon)) \|T\| 2^{-k/p} \\
&\quad \geq \|g_k \cdot A_{ki}\| + \|(g - g_k) \cdot A_{ki}\| \\
&\quad \geq \|g \cdot A_{ki}\| \geq \|g_k \cdot A_{ki}\| - \|(g - g_k) \cdot A_{ki}\| \\
&\quad \geq c_1(\varepsilon)^{1/p} \|T^{-1}\|^{-1} 2^{-k/p} - a(\varepsilon, c_1(\varepsilon)) \|T\| 2^{-k/p}.
\end{aligned}$$

This establishes the claim with

$$c_2(\varepsilon) = \max \{(a(\varepsilon, c_1(\varepsilon)) + 1) \|T\|, (c_1(\varepsilon)^{1/p} \|T^{-1}\|^{-1} - a(\varepsilon, c_1(\varepsilon))\|T\|)^{-1}\}.$$

Next, let

$$h_k = \sum_{i=1}^{2^k} \|g \cdot A_{ki}\|^{-p} 2^{-k} A_{ki}.$$

Note that $h_k \in L_\infty(\nu)$ and that in fact the claim implies

(5) $\quad c_2(\varepsilon)^p \geq h_k \geq c_2(\varepsilon)^{-p}$.

Let h_o be a w^* limit point of (h_k) for k in the subsequence (if $p = 1$, subnet) for which g_k converges to g. Define $h = h_o^{1/p} g$. We will show that h satisfies (3).

$$\begin{aligned}
\int |h|^p \cdot A_{ki}\, d\nu &= \int |g|^p h_o \cdot A_{ki}\, d\nu \\
&= \lim_j \int |g|^p h_j \cdot A_{ki}\, d\nu \\
&= \lim_j \int |g|^p \sum_{r \in \mathscr{A}^j_{ki}} \|gA_{jr}\|^{-p} 2^{-j} A_{jr}\, d\nu \\
&= \lim_j 2^{-j} \sum_{r \in \mathscr{A}^j_{ki}} \|gA_{jr}\|^{-p} \int |g|^p A_{jr}\, d\nu \\
&= \lim_j 2^{-j} 2^{j-k} \\
&= 2^{-k}.
\end{aligned}$$

It follows that the operator $S: L_p[0, 1] \to L_p(\nu)$ such that $SI_{ki} = hA_{ki}$ is an isometry. It remains only to estimate $\|T - S\|$. For any sequence $(a_i)_{i=1}^{2^k}$ of scalars and any k,

$$\left\|T \sum_i a_i I_{ki} - S \sum_i a_i I_{ki}\right\|$$

$$= \left\|\sum_i a_i T I_{ki} - \sum_i a_i h \cdot A_{ki}\right\|$$

$$= \left\|\sum_i a_i T I_{ki} - \sum_i a_i g h_o^{1/p} \cdot A_{ki}\right\|$$

$$\leq \left\|\sum_i a_i T I_{ki} - \sum_i a_i g \cdot A_{ki}\right\| + \left\|\left(\sum_i a_i g A_{ki}\right)(1 - h_o^{1/p})\right\|$$

$$\leq \lim_l \left\|\sum_i a_i T I_{ki} - \sum_i a_i g_l A_{ki}\right\| + \left\|\sum_i a_i g A_{ki}\right\| \|1 - h_o^{1/p}\|_\infty$$

$$\leq \lim_l \left\|\sum_i a_i \sum_{r \in \mathscr{A}_{ki}^l} T I_{lr} - \sum_i a_i \sum_{r \in \mathscr{A}_{ki}^l} T I_{lr} A_{lr}\right\| + \left\|\sum_i a_i g A_{ki}\right\| (c_2(\varepsilon) - 1)$$

(by the definition of g_l and (5))

$$\leq \lim_l \left\|\sum_i a_i \sum_{r \in \mathscr{A}_{ki}^l} T I_{lr}(A_{lr}^c)\right\| + \left(\sum_i |a_i|^p \|g A_{ki}\|^p\right)^{1/p} (c_2(\varepsilon) - 1)$$

$$\leq \lim_l a(\varepsilon, c_1(\varepsilon)) \left(\sum_i \sum_{r \in \mathscr{A}_{ki}^l} |a_i|^p \|T I_{lr}\|^p\right)^{1/p} + \left(\sum_i |a_i|^p 2^{-k} c_2(\varepsilon)^p\right)^{1/p} (c_2(\varepsilon) - 1)$$

(by Theorem B and the claim)

$$\leq \lim_l a(\varepsilon, c_1(\varepsilon)) \|T\| \left(\sum_i \sum_{r \in \mathscr{A}_{ki}^l} |a_i|^p \|I_{lr}\|^p\right)^{1/p} + c_2(\varepsilon) \left\|\sum_i a_i I_{ki}\right\| (c_2(\varepsilon) - 1)$$

$$\leq [a(\varepsilon, c_1(\varepsilon)) \|T\| + c_2(\varepsilon)(c_2(\varepsilon) - 1)] \left\|\sum_i a_i I_{ki}\right\|$$

Therefore

$$\|T - S\| \leq \tau(\varepsilon)$$

where

$$\tau(\varepsilon) = a(\varepsilon, c_1(\varepsilon)) \|T\| + c_2(\varepsilon)(c_2(\varepsilon) - 1) \quad \text{and} \quad \lim_{\varepsilon \to 0} \tau(\varepsilon) = 0,$$

proving our result.

2. The general case

In this section we will prove our result for an isomorphism

$$T: L_p(\mu) \to L_p(\nu), \quad \|T\| \, \|T^{-1}\| < (1 - \varepsilon)^{-1}$$

with no restriction on μ. First let us recall that any abstract L_p space is isometric to $(\Sigma_{\kappa \in K} L_p([0, 1]^{m_\kappa}) \oplus l_p(K_1))_{lp}$ where K, K_1 and m_κ are cardinals (see [6, page 136]). To simplify our notation we will write $l_p(1) = L_p([0, 1]^0)$ and thus we may assume that

$$L_p(\mu) = \left(\sum_{k \in K} L_p[0, 1]^{m_\kappa} \right)_p$$

where some of the summands may be one dimensional. Our argument in the general case is very similar to that for $L_p[0, 1]$; essentially all we do is replace sequences by appropriate nets and proceed as before.

We begin by choosing our replacements for (I_{ki}). For each $\kappa \in K$, let

$$\{(n_s, k_s, i_s) : 1 \leqslant s \leqslant j\}$$

be a finite subset of $m_\kappa \times \mathbf{N} \cup \{0\} \times \mathbf{N}$ such that $1 \leqslant i_s \leqslant 2^{k_s}$ and $n_s \neq n_{s'}$, *if* $s \neq s'$; and define

$$C(\kappa, \{(n_s, k_s, i_s) : 1 \leqslant s \leqslant j\}) = \{x \in [0, 1]^{m_\kappa} : x(n_s) \in I_{k_s, i_s}\}$$

where, as before, $I_{k_s, i_s} = [(i_s - 1)2^{-k_s}, i_s 2^{-k_s})$. Let $C(\kappa, \phi) = [0, 1]^{m_\kappa}$. These cylinder sets will be used in the same way that the dyadic intervals were used before. Note that for κ fixed the sequence $\{(n_s, k_s) : 1 \leqslant s \leqslant j\}$ establishes a level of $[0, 1]^{m_\kappa}$ and that the levels are partially ordered by refinement, i.e.,

$$\{(n_s, k_s) : 1 \leqslant s \leqslant j\} \leqslant \{(n'_t, k'_t) : 1 \leqslant t \leqslant j'\}$$

if

$$\{n_s : 1 \leqslant s \leqslant j\} \subset \{n'_t : 1 \leqslant t \leqslant j'\}$$

and if $n_s = n'_t$, then $k_s \leqslant k'_t$.

Our first task is to find sets $A(\kappa, \{(n_s, k_s, i_s) : 1 \leqslant s \leqslant j\})$ for all $\kappa \in K$, and finite sets of triples (n_s, k_s, i_s) in $m_\kappa \times (\mathbf{N} \cup \{0\}) \times \mathbf{N}$, such that

$$\begin{aligned} (1) \qquad & A(\kappa, \{(n_s, k_s, i_s) : 1 \leqslant s \leqslant j\} \cup \{(n, k + 1, 2i - 1)\}) \\ & \cup A(\kappa, \{(n_s, k_s, i_s) : 1 \leqslant s \leqslant j\} \cup \{(n, k + 1, 2i)\}) \\ & \subset A(\kappa, \{(n_s, k_s, i_s) : 1 \leqslant s \leqslant j\} \cup \{(n, k, i)\}), \end{aligned}$$

$$(2) \quad A(\kappa, \{(n_s, k_s, i_s) : 1 \leqslant s \leqslant j\}) \cap A(\kappa', \{(n'_t, k'_t, i'_t) : 1 \leqslant t \leqslant j'\}) = \emptyset$$

(a) if $\kappa \neq \kappa'$ or

(b) if $\kappa = \kappa'$, $j = j'$, $(n_s, k_s) = (n'_s, k'_s)$, $s = 1, 2, \ldots j$, and $i_s \neq i'_s$ for some s, $1 \leq s \leq j$, and functions h_κ, $\kappa \in K$, such that

$$\int |h_\kappa|^p A(\kappa, \{(n_s, k_s, i_s) : 1 \leq s \leq j\})\, d\nu = 2^{-(k_1+k_2+\cdots+k_j)} \tag{3}$$

Let

$$X = \left(\sum_{\kappa\in K} \sum_{F\subset m_\kappa \times N\times N} L_1(\nu)\right)^*_{l_1} = \left(\sum_{\kappa\in K} \sum_{F\subset m_\kappa \times N\times N} L_\infty(\nu)\right)_\infty$$

where the second sum is over all finite (including the empty set) subsets

$$F = \{(n_s, k_s, i_s) : 1 \leq s \leq j\}$$

of $m_\kappa \times \mathbf{N} \times \mathbf{N}$ with $n_s \neq n_{s'}$, *if* $s \neq s'$, $1 \leq i \leq 2^{k_s}$, $1 \leq s \leq j$, in the w^* topology. Define a subset D of X as the set of all $(\zeta(\kappa, F))$ where $\kappa \in K$, $F \subset m_\kappa \times \mathbf{N} \times \mathbf{N}$ such that $\zeta(\kappa, F) \geq 0$, $\Sigma_{i=1}^\infty \zeta(\kappa_i, \emptyset) \leq 1$, for all (κ_i), a sequence of distinct elements in K,

$$\zeta(\kappa, F \cup \{(n, k+1, 2i-1)\}) + \zeta(\kappa, F \cup \{(n, k+1, 2i)\}) \leq \zeta(\kappa, F \cup \{(n, k, i)\})$$

for all $\kappa \in K$, $F \cup \{(n, k, i)\} \subset m_\kappa \times \mathbf{N} \cup \{0\} \times \mathbf{N}$, and F is as above. Note that in the index set for the definition of X we do not allow the tuple $(n, 0, 1)$ because this would be redundant, e.g.,

$$A(\kappa, \{(n, 0, 1)\}) = [0, 1]^{m_\kappa} = A(\kappa, \emptyset).$$

However, in the definition of D, we want to have the relation

$$\zeta(\kappa, \{(n, 1, 1)\}) + \zeta(\kappa, \{(n, 1, 2)\}) \leq \zeta(\kappa, \{(n, 0, 1)\}) = \zeta(\kappa, \phi) \quad \text{for all } n,$$

thus we allow the tuple $(n, 0, 1)$ with the understanding that

$$\zeta(\kappa, F \cup \{(n, 0, 1)\}) = \zeta(\kappa, F).$$

It is easy to see that D is a w^* closed, bounded and hence w^* compact subset of X. Define a map $\Psi : D \to l_\infty(\{(\kappa, F) : \kappa \in K, F \subset m_\kappa \times \mathbf{N} \times \mathbf{N}\})$ by

$$\Psi(d)(\kappa, F) = \|T(C(\kappa, F))\|^{-p} \int |T(C(\kappa, F))|^p \zeta(\kappa, F)\, d\nu.$$

Clearly Ψ is w^* continuous.

As in Section 1 we wish to show that there is an element $d \in D$ such that $\Psi(d)$ $(\kappa, F) \geq c(\varepsilon)$ for all (κ, F). Because of the w^* compactness of D and the w^* continuity of Ψ it is sufficient to show that for any finite subset $\{(\kappa_i, F_i)\}$ of the index set there is an element $d \in D$ such that $\Psi(d)(\kappa_i, F_i) \geq c(\varepsilon)$, for all i.

Observe that for any such finite set $\{(\kappa_i, F_i)\}$ there is a finite set $\{\kappa_1, \kappa_2, \dots \kappa_j\}$ of distinct elements in K and levels $\{(n_s^l, k_s^l) : 1 \leq s \leq j_l\}$, $1 \leq l \leq j$, such that the sets $C(\kappa_i, F_i)$ belong to the finite algebra generated by (in fact are unions of)

$$C(\kappa_l, \{(n_s, k_s^l, i_s^l) : 1 \leqslant s \leqslant j_l\}), \quad 1 \leqslant i_s^l \leqslant 2^{k_s^l}, 1 \leqslant s \leqslant j_l, 1 \leqslant l \leqslant j$$

and that these cylinder sets are disjoint. By applying Theorem A to the sequence

$$x(\kappa_l, F) = \|T(C(\kappa_l, F))\|^{-1} T(C(\kappa_l, F)),$$

$$F = \{(n_s^l, k_s^l, i_s^l) : 1 \leqslant s \leqslant j\}, 1 \leqslant i_s^l \leqslant 2^{k_s^l}, 1 \leqslant s \leqslant j_l, 1 \leqslant l \leqslant j\}$$

we get disjoint sets $\{B(\kappa_l, F)\}$ such that

$$\|x(\kappa_l, F) \cdot B(\kappa_l, F)\| \geqslant 1 - a_1(p, \varepsilon),$$

F as above, $1 \leqslant l \leqslant j$. Let $\zeta(\kappa_l, G) = \cup \{B(\kappa_l, F) : C(\kappa_l, F) \subset C(\kappa_l, G)\}$ and note that $(\zeta(\kappa_l, G)) \in D$. By lemma 1, if $C(\kappa_l, G)$ is a union of some of the sets $C(\kappa_l, F)$, F as above,

$$\|TC(\kappa_l, G)\|^{-p} \|(TC(\kappa_l, G)) \zeta(\kappa_l, G)\|^p \geqslant c(\varepsilon).$$

Therefore there is an element $(\zeta'(\kappa, G))$ in D such that

$$\|T(C(\kappa, G))\|^{-p} \|(TC(\kappa, G))\zeta'(\kappa, G))\|^p \geqslant c(\varepsilon)$$

for every $\kappa \in K$ and $G \subset m_\kappa \times \mathbf{N} \times \mathbf{N}$. Let $A(\kappa, G) = \left\{\zeta'(\kappa, G) > \frac{1}{2}\right\}$ and note that these sets satisfy (1), (2) and

$$\|T(C(\kappa, G))\|^{-p} \|(TC(\kappa, G)\zeta'(\kappa, G)\|^p \geqslant c(\varepsilon) \tag{4}$$

We turn now to the construction of the functions h_κ. The argument is very similar to that used in Section 1 to construct h. For each level

$$L = \{(n_s, k_s) : 1 \leqslant s \leqslant j\} \text{ of } [0, 1]^{m_\kappa}$$

let

$$g(\kappa, L) = \sum_G (TC(\kappa, G)) \cdot A(\kappa, G)$$

where the sum is over all G of the form $\{(n_s, k_s, i_s) : 1 \leqslant s \leqslant j\}$, $1 \leqslant i_s \leqslant 2^{k_s}$, i.e., G is on level L. Let $g(\kappa)$ be a weak limit of a weakly convergent subnet of $(g(\kappa, L))$ if $p > 1$ and the absolutely continuous part of a w^* limit if $p = 1$.

Claim.

$$2^{-k/p} c_2(\varepsilon) \geqslant \|g(\kappa) \cdot A(\kappa, G)\| \geqslant 2^{-k/p} c_2(\varepsilon)^{-1}$$

for G on level $L = \{(n_s, k_s) : 1 \leqslant s \leqslant j\}$ where $k = k_1 + k_2 + \cdots + k_j$.

Proof of Claim. Let L' be a level which refines L. Then

$$\|(g(\kappa, L) - g(\kappa, L')) \cdot A(\kappa, G)\| = \|(TC(\kappa, G)) \cdot A(\kappa, G) - \sum_H (TC(\kappa, H)) \cdot A(\kappa, H)\|$$

where the sum is over all H on level L' such that $C(\kappa, G) \supset C(\kappa, H)$. It follows that for such H, $A(\kappa, G) \supset A(\kappa, H)$ and thus

$$\|(TC(\kappa, G)) \cdot A(\kappa, G) - \sum_H (TC(\kappa, H)) A(\kappa, H)\|$$

$$= \left\| \sum_H (TC(\kappa, H)) \cdot A(\kappa, G) - \sum_H (TC(\kappa, H)) A(\kappa, H) \right\|$$

$$\leqslant \left\| \sum_H (TC(\kappa, H) \cdot A(\kappa, H)^c \right\| \leqslant a(\varepsilon, c_1(\varepsilon)) \left(\sum_H \|TC(\kappa, H)\|^p \right)^{1/p}$$

(by Theorem B)

$$\leqslant a(\varepsilon, c_1(\varepsilon))\|T\| \left(\sum_H \|C(\kappa, H)\|^p \right)^{1/p}$$

$$= a(\varepsilon, c_1(\varepsilon))\|T\| \|C(\kappa, G)\|$$

$$= a(\varepsilon, c_1(\varepsilon))\|T\| \, 2^{-k/p}.$$

Passing to the limit over the convergent subnet (using the lower semi-continuity of $\|\cdot\|$), we have

$$\|(g(\kappa, L) - g(\kappa)) \cdot A(\kappa, G)\| \leqslant a(\varepsilon, c_1(\varepsilon))\|T\| \, 2^{-k}$$

and thus the same computation as in Section 1 yields the claim.

Next we need to adjust the modulus of $g(\kappa)$. Let

$$h(\kappa, L) = \sum_G \|g(\kappa) \cdot A(\kappa, G)\|^{-p} \, 2^{-k} A(\kappa, G)$$

where, as before, G is on level L and $\|C(\kappa, G)\| = 2^{-k/p}$ for all G on level L. Let $h(\kappa)$ be a w^* limit point in $L_\infty(\nu)$ (The claim shows that $h(\kappa, L) \in L_\infty(\nu)$.) of a convergent subnet of the subnet of $(h(\kappa, L))$ having the same directed set $\mathscr{D}$ for which $(g(\kappa, L))_{L \in \mathscr{D}}$ converges to $g(\kappa)$.

Define $h_\kappa = g(\kappa) \, h(\kappa)^{1/p}$. First we will check (3). Let

$$G = \{(n_s, k_s, i_s) : 1 \leqslant s \leqslant j\}.$$

Then

$$\int |h_\kappa|^p \cdot A(\kappa, G) = \int |g(\kappa)|^p h(\kappa) \cdot A(\kappa, G)$$

$$= \lim \int |g(\kappa)|^p \sum_H \|g(\kappa) A(\kappa, H)\|^{-p} \, 2^{-l} A(\kappa, H)$$

where the limit is taken over the convergent subnet of $(h(\kappa, L))$ and the sum is over H in some level L' which refines the level of G with the measure of the set in level L', $\|C(\kappa, H)\|^p = 2^{-l}$, and $A(\kappa, H) \subset A(\kappa, G)$. We have

$$\begin{aligned}\int |g(\kappa)|^p \sum_H \|g(\kappa) \cdot A(\kappa, H)\|^{-p} 2^{-l} \cdot A(\kappa, H) &= \sum_H 2^{-l} \|g(\kappa) \cdot A(\kappa, H)\|^{-p} \int |g(\kappa)|^p \cdot A(\kappa, H) \\ &= \sum_H 2^{-l} = 2^{-k} \quad \text{where } k = k_1 + k_2 + \cdots + k_s\end{aligned}$$

proving (3).

Finally we estimate $\|T - S\|$ where S is the isometry which satisfies

$$S(C(\kappa, G)) = h_\kappa \cdot A(\kappa, G).$$

For any finite set $\{\kappa_l : 1 \leq l \leq j_l\}$ of levels of $m_{\kappa l}$,

$$L_l = \{(n_{ls}, A_{ls}) : 1 \leq s \leq j_l\},$$

and scalars $\{a(\kappa_l, G) : G \text{ on level } L_l, 1 \leq l \leq j\}$, we have

$$\begin{aligned}&\left\| T \sum_{l=1}^{j} \sum_G a(\kappa_l, G) C(\kappa_l, G) - S \sum_{l=1}^{j} \sum_G a(\kappa, G) C(\kappa_l, G) \right\| \\ &= \left\| \sum_{l=1}^{j} \sum_G a(\kappa_l, G) TC(\kappa_l, G) - \sum_{l=1}^{j} \sum_G a(\kappa_l, C)\, h_{\kappa_l} \cdot A(\kappa_l, G) \right\| \\ &= \left\| \sum_{l=1}^{j} \sum_G a(\kappa_l, G) TC(\kappa_l, G) - \sum_{l=1}^{j} \sum_G a(\kappa_l, G) g(\kappa_l) h(\kappa_l)^{1/p} \cdot A(\kappa_l, G) \right\| \\ &\leq \left\| \sum_{l=1}^{j} \sum_G a(\kappa_l, G) TC(\kappa_l, G) - \sum_{l=1}^{j} \sum_G a(\kappa_l, G) g(\kappa_l) \cdot A(\kappa_l, G) \right\| \\ &\quad + \left\| \sum_{l=1}^{j} \left(\sum_G a(\kappa_l, G) g(\kappa_l) \cdot A(\kappa_l, G) \right) ([0, 1]^{m_{\kappa l}} - h(\kappa_l)^{1/p}) \right\| \\ &\leq \overline{\lim} \left\| \sum_{l=1}^{j} \sum_G a(\kappa_l, G) TC(\kappa_l, G) - \sum_{l=1}^{j} \sum_G a(\kappa_l, G) g(\kappa_l, L_l) \cdot A(\kappa_l, G) \right\| \\ &\quad + \left\| \sum_{l=1}^{j} \sum_G a(\kappa_l, G) g(\kappa_l) \cdot A(\kappa_l, G) \right\| \max_l \left\| ([0, 1]^{m_{\kappa l}} - h(\kappa_l)^{1/p}) \cdot [0, 1]^{m_{\kappa l}} \right\|_\infty\end{aligned}$$

where the limit is over the product of the directed sets (which is directed in the obvious way) for the convergent subnet of $(g(\kappa_l, L))$, $1 \leq l \leq j$. Continuing our sequence of inequalities, we replace $g(\kappa_l, L_l)$ by its definition as a sum over H in level L_l and write $C(\kappa_l, G)$ as a union of sets from level L_l, $C(\kappa_l, H) \subset C(\kappa_l, G)$ for each l so that the inequalities continue with

$$\leqslant \overline{\lim} \left\| \sum_{l=1}^{j} \sum_{G} a(\kappa_l, G) \sum_{H} TC(\kappa_l, H) - \sum_{l=1}^{j} \sum_{G} a(\kappa_l, G) \sum_{H} TC(\kappa_l, H) \cdot A(\kappa_l, H) \right\|$$

$$+ \left\| \sum_{l=1}^{j} \sum_{G} a(\kappa_l, G) g(\kappa_l) \cdot A(\kappa_l, G) \right\| (c_2(\varepsilon) - 1) \quad \text{(by (5))}$$

$$\leqslant \overline{\lim} \left\| \sum_{l=1}^{j} \sum_{G} a(\kappa_l, G) \sum_{H} TC(\kappa_l, H) \cdot A(\kappa_l, H)^c \right\|$$

$$+ \left(\sum_{l=1}^{j} \sum_{G} |a(\kappa_l, G)|^p \|g(\kappa_l) \cdot A(\kappa_l, G)\|^p \right)^{1/p} (c_2(\varepsilon) - 1)$$

$$\leqslant \overline{\lim}\, a(\varepsilon, c_1(\varepsilon)) \left(\sum_{l=1}^{j} \sum_{G} \sum_{H} |a(\kappa_l, G)|^p \|TC(\kappa_l, H)\|^p \right)^{1/p}$$

$$+ \left(\sum_{l=1}^{j} \sum_{G} |a(\kappa_l, G)|^p \, \|C(\kappa_l, G)\|^p c_2(\varepsilon)^p \right)^{1/p} (c_2(\varepsilon) - 1)$$

(By Theorem B and the claim)

$$\leqslant \overline{\lim}\, a(\varepsilon, c_1(\varepsilon)) \|T\| \left(\sum_{l=1}^{j} \sum_{G} \sum_{H} |a(\kappa_l, G)|^p \|C(\kappa_l, H)\|^p \right)^{1/p}$$

$$+ c_2(\varepsilon) \left\| \sum_{l=1}^{j} \sum_{G} a(\kappa_l, G) C(\kappa_l, G) \right\| (c_2(\varepsilon) - 1)$$

$$\leqslant [a(\varepsilon, c_1(\varepsilon)) \|T\| + c_2(\varepsilon)(c_2(\varepsilon) - 1)] \left\| \sum_{l=1}^{j} \sum_{G} a(\kappa_l, G) C(\kappa_l, G) \right\|.$$

Therefore, with the previous notation, $\|T - S\| \leqslant \tau(\varepsilon)$ and $\lim_{\varepsilon \to 0} \tau(\varepsilon) = 0$.

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