ILLINOIS JOURNAL OF MATHEMATICS
Volume 27, Number 2, Summer 1983

ON TWO UNIVALENCE CRITERIA OF NEHARI

BY

BINYAMIN SCHWARZ

1. The Two Criteria

In [3] Nehari proved that the analytic function $f(z)$ is univalent in the unit disk $\Delta = \{z : |z| < 1\}$ if its Schwarzian derivative

$$\{f(z), z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

satisfies either

$$|\{f(z), z\}| \leq \frac{2}{(1 - |z|^2)^2}, \tag{1}$$

or

$$|\{f(z), z\}| \leq \pi^2/2. \tag{2}$$

In [4] he established the following more general criterion.

THEOREM 1. *The analytic function $f(z)$ is univalent in Δ if*

$$|\{f(z), z\}| \leq 2p(|z|),\ z \in \Delta, \tag{3}$$

where $p(x)$ is a function with the following properties: (a) *$p(x)$ is positive and continuous on $(-1, 1)$;* (b) *$p(-x) = p(x)$;* (c) *$(1 - x^2)^2 p(x)$ is nonincreasing if x varies from* 0 *to* 1; (d) *the differential equation*

$$y''(x) + p(x)y(x) = 0 \tag{4}$$

has a solution which does not vanish on $(-1, 1)$.

The functions $1/(1 - x^2)^2$ and $\pi^2/4$ have the properties (a)–(d) and thus yield the conditions (1) and (2). The function $p(x) = 2/(1 - x^2)$ yields the univalence condition

$$|\{f(z), z\}| \leq 4/(1 - |z|^2), \tag{5}$$

which was stated (without proof) by Pokornyi [6]. Using Theorem 1, Beesack

Received June 1, 1981.

© 1983 by the Board of Trustees of the University of Illinois
Manufactured in the United States of America

[1, pp. 217–218] proved and sharpened additional conditions stated in [6], and Friedland and Nehari [2] generalized conditions (1) and (5).

In a posthumously published paper [5] Nehari proved the following result.

THEOREM 2. *Let $f(z)$ be an analytic function in Δ and let $F(x)$ be a real-valued function on* $[0, 1)$ *with the following properties:*

(α) *F has three continuous derivatives on* $[0, 1)$ *and* $F'(x) > 0$;
(β) $F''(0) \geqslant 0$;
(γ) $\{F(x), x\} \geqslant 0$ *and* $(1 - x^2)^2\{F(x), x\}$ *is nonincreasing.*

If

$$|\{f(z), z\}| \leqslant \{F(|z|), |z|\},\ z \in \Delta, \tag{6}$$

then f is univalent in Δ.

(This is the more general form of Theorem I of [5] assuming only $F''(0) \geqslant 0$ instead of $F''(0) = 0$. For further relaxations of the conditions on $F(x)$ see the remarks on p. 346 and p. 350 of [5].)

Nehari found two classes of functions $F(x)$, each depending on a parameter μ, $0 \leqslant \mu \leqslant 1$, satisfying conditions (α)–(γ) and he thus obtained the following specific univalence criteria:

$$|\{f(z), z\}| \leqslant \frac{2(1 + \mu)(1 - \mu|z|^2)}{(1 - |z|^2)^2}, \quad 0 \leqslant \mu \leqslant 1, \tag{7}$$

and

$$|\{f(z), z\}| \leqslant \frac{2(1 - \mu^2)}{(1 - |z|^2)^2} + \frac{2\mu(2 + \mu)}{(1 + |z|^2)^2}, \quad 0 \leqslant \mu \leqslant 1. \tag{8}$$

Note that $\mu = 0$ in (7) and (8) yields (1), and $\mu = 1$ in (7) yields (5). The function $F(x) = \tan(\pi x/2)$ yields condition (2).

Nehari's proof of the recent result is independent of his proof of the earlier result. *It is the purpose of the present paper to show that the two theorems are equivalent.* This is done in the next section by means of two elementary lemmas. An application of this equivalence to matrix differential equations is given in the last section.

2. The Equivalence

All the functions in this section, with one obvious exception in the remark following Lemma 1, are real-valued.

LEMMA 1. (i) *Let the function $F(x)$, $0 \leqslant x < 1$, have the following properties:* (α) *F has three continuous derivatives on* $[0, 1)$ *and* $F'(x) > 0$; (β) $F''(0) \geqslant 0$. *Define*

$$p(x) = \frac{1}{2}\{F(x), x\}, \quad 0 \leqslant x < 1, \tag{9}$$

and

$$p(-x) = p(x), \quad 0 \leqslant x < 1. \tag{10}$$

Then the equation

$$y''(x) + p(x)y(x) = 0 \tag{4}$$

has a solution which does not vanish on $(-1, 1)$.

(ii) *Conversely, let* $p(x)$ *be a continuous, even function on* $(-1, 1)$, *such that* (4) *has there a nonvanishing solution. Then there exists an odd function* $F(x)$ *on* $(-1, 1)$ *with three continuous derivatives there,* $F''(0) = 0$, $F'(x) > 0$, *which satisfies*

$$p(x) = \frac{1}{2}\{F(x), x\}, \quad -1 < x < 1. \tag{9'}$$

This lemma clearly implies the equivalence of the two theorems when

$$p(x) = \frac{1}{2}\{F(x), x\} > 0.$$

However, they are also equivalent when $p(x) = 1/2\{F(x), x\} \geqslant 0$ because Theorem 1 remains valid for nonnegative $p(x)$. Indeed, if $p(x_0) = 0$, $0 \leqslant x_0 < 1$, then it follows by (c) that $p(x) = 0$ for $x_0 \leqslant x < 1$ and therefore $\{f(z), z\} \equiv 0$ in Δ; hence

$$f(z) = (az + b)/(cz + d)$$

and $f(z)$ is thus univalent.

Lemma 1 can be proved directly, but it is perhaps more natural to deduce it from the following lemma.

LEMMA 2. (i) *Let the function* $u(x)$ *have a continuous derivative on* $[0, 1)$ *and assume* $u(0) \leqslant 0$. *Define*

$$p(x) = -u'(x) - u^2(x), \quad 0 \leqslant x < 1, \tag{11}$$

and

$$p(-x) = p(x), \quad 0 \leqslant x < 1. \tag{10}$$

Then the differential equation (4) *has a solution which does not vanish on* $(-1, 1)$.

(ii) *Conversely, let* $p(x)$ *be a continuous, even function on* $(-1, 1)$, *such that* (4) *has there a nonvanishing solution. Then there exists an odd function* $u(x)$ *on* $(-1, 1)$ *with continuous derivative, which satisfies the Riccati equation*

$$p(x) = -u'(x) - u^2(x), \quad -1 < x < 1. \tag{11'}$$

First we show that Lemma 2 implies Lemma 1.

(i) Let $F(x)$ satisfy conditions (α) and (β) of Lemma 1. Set

$$u(x) = -\frac{F''(x)}{2F'(x)}, \quad 0 \leqslant x < 1. \tag{12}$$

Then $u(x)$ satisfies the assumptions of part (i) of Lemma 2. It follows that equation (4)—with $p(x)$ defined by (11) and (10)—has a solution which does not vanish on $(-1, 1)$. From (11) and (12), we have (9), and thus have proved part (i) of Lemma 1.

(ii) Let $u(x)$ be the function which exists by the assertion of part (ii) of Lemma 2. Set

$$F(x) = \int_0^x \exp\left(-2\int_0^t u(s)ds\right)dt, \quad -1 < x < 1. \tag{13}$$

Then $F(x)$ has all the properties stated in part (ii) of Lemma 1. We remark that conversely Lemma 1 also implies Lemma 2.

It remains to prove Lemma 2.

(i) Using $u(x)$ of part (i), we define

$$y_1(x) = \exp\left(\int_0^x u(t)dt\right), \quad 0 \leqslant x < 1. \tag{14}$$

Clearly $y_1(0) = 1$, $y_1'(0) = u(0) \leqslant 0$ and $y_1(x) > 0$ on $[0, 1)$. $y_1(x)$ is a solution of (4) on $[0, 1)$ with $p(x)$ defined by (11). Now let $y_2(x)$ be the even solution of (4) on $(-1, 1)$, defined by $y_2(0) = 1$, $y_2'(0) = 0$. We show that $y_2(x) \neq 0$ on $(-1, 1)$. If $y_1'(0) = u(0) = 0$, then $y_2(x) = y_1(x)$, and hence $y_2(x) > 0$ on $[0, 1)$ and, as $y_2(-x) = y_2(x)$, $y_2(x) > 0$ on $(-1, 1)$. If $y_1'(0) < 0$, define $y_3(x) = y_2(x) - y_1(x)$, $0 \leqslant x < 1$. Then $y_3(0) = 0$ and $y_3'(0) > 0$, and $y_3(x)$, a solution of (4) on $[0, 1)$, cannot vanish again in $[0, 1)$ as otherwise, by the Sturm separation theorem, also $y_1(x)$ would have to vanish in $[0, 1)$. Hence $y_3(x) \geqslant 0$ on $[0, 1)$ and thus also in this case $y_2(x) > 0$ on $[0, 1)$ and therefore also on $(-1, 1)$.

(ii) To prove part (ii) we remark that if (4) has a nonvanishing solution $y(x)$ on $(-1, 1)$, then the even solution $y_2(x)$, defined by $y_2(0) = 1$, $y_2'(0) = 0$ is positive on $(-1, 1)$. The function $u(x) = y_2'(x)/y_2(x)$ satisfies all the assertions of part (ii). This completes the proof of Lemma 2 and establishes the equivalence of the two univalence criteria of Nehari.

3. An Application

The univalence of the function $f(z)$ in a domain D is equivalent to the disconjugacy of the differential equation

$$w''(z) + q(z)w(z) = 0 \tag{15}$$

in D, where $q(z) = 1/2\{f(z), z\}$ [3]. So Theorems 1 and 2 may be stated as criteria for disconjugacy of (15). This version can be generalized to matrix differential equations of the form

$$w''(z) + Q(z)w(z) = 0. \tag{16}$$

Here $Q(z)$ is a $n \times n$ matrix, holomorphic in a simply connected domain D, $\infty \notin D$, and $w(z)$ is a solution vector. *The equation* (16) *is called disconjugate in* D, *if for any pair of points* (z_1, z_2) *in* D, $z_1 \neq z_2$, *the condition* $w(z_1) = w(z_2) = 0$ *is satisfied only for the trivial solution* $w(z) \equiv 0$. Let $\|Q\|_2$ denote the spectral norm of the $n \times n$ matrix Q. With this notation the following holds [7, Theorem 2.3].

THEOREM 3. *Let the matrix* $Q(z)$ *be holomorphic in* Δ *and assume that*

$$\|Q(z)\|_2 \leq p(|z|), \quad z \in \Delta, \tag{3'}$$

where $p(x)$ *is a function having properties* (a)–(d) *of Theorem* 1. *Then the equation* (16) *is disconjugate in* Δ.

This and Lemma 1 imply the following result.

THEOREM 4. *Let the matrix* $Q(z)$ *be holomorphic in* Δ *and assume that*

$$\|Q(z)\|_2 \leq \frac{1}{2}\{F(|z|), |z|\}, \quad z \in \Delta, \tag{6'}$$

where $F(x)$ *is a function having properties* (α)–(γ) *of Theorem* 2. *Then the equation* (16) *is disconjugate in* Δ.

We remark that the proof of Theorem 3 was a rather direct generalization of the proof of the scalar case (Theorem 1). The proof in [5] of Theorem 2 can, apparently, not be generalized so as to yield Theorem 4. So in order to generalize this result of Nehari to the multidimensional case, we had to show its equivalence to his previous result. As now all the scalar criteria (of [1]–[6]), mentioned in the present note, follow from Theorem 1, they can all be generalized to disconjugacy criteria for the matrix differential equation (16).

Acknowledgement. The author wishes to thank Professor D. London and the referee for their helpful comments on the material of this paper.

REFERENCES

1. P. R. BEESACK, *Nonoscillation and disconjugacy in the complex domain,* Trans. Amer. Math. Soc., vol. 81 (1956), pp. 211–242.
2. S. FRIEDLAND AND Z. NEHARI, *Univalence conditions and Sturm-Liouville eigenvalues,* Proc. Amer. Math. Soc., vol. 24 (1970), pp. 595–603.
3. Z. NEHARI, *The Schwarzian derivative and schlicht functions,* Bull. Amer. Math. Soc., vol. 55 (1949), pp. 545–551.

4. ———, *Some criteria of univalence,* Proc. Amer. Math. Soc., vol. 5 (1954), pp. 700–704.
5. ———, *Univalence criteria depending on the Schwarzian derivative,* Illinois J. Math., vol. 23 (1979), pp. 345–351.
6. V. V. Pokornyi, *On some sufficient conditions for univalence,* Dokl. Akad. Nauk SSSR, vol. 79 (1951), pp. 743–746 (Russian).
7. B. Schwarz, *Disconjugacy of complex second-order matrix differential systems,* J. Analyse Math., vol. 36 (1979), pp. 244–273.

Technion-Israel Institute of Technology
Haifa, Israel