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MANIFOLDS WHICH IMMERSE IN SMALL CODIMENSION

BY

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1. Introduction

The purpose of this note is to examine the possible cobordism classes of manifolds M^n which immerse in R^{n+k} for small values of k, specifically $1 \leqslant k \leqslant 3$.

The case $k = 1$ was first studied by Liulevicius [11], who analyzed the filtration of the unoriented cobordism ring N_* arising from the immersion codimension. This case was separated out for special attention by R. L. W. Brown [4] to give:

PROPOSITION 1. *If M^n immerses in R^{n+1} and is not a boundary, then M^n is cobordant to a point, RP^2 or RP^6.*

In a recent pair of papers, Kikuchi studied the case $k = 2$ and proved:

PROPOSITION 2 [8]. *If M^n is oriented, immerses in R^{n+2}, and is not an unoriented boundary, then M^n is unoriented cobordant to $\mathbf{C}P^{2^s-2}$ for some $s \geqslant 1$.*

PROPOSITION 3 [9]. *If M^n immerses in R^{n+2} and n is odd then M^n bounds.*

This note was largely inspired by these results, and one has the following improvements:

PROPOSITION 2′. *If M^n is oriented, immerses in R^{n+2}, and $n > 0$, then M^n is an unoriented boundary.*

PROPOSITION 3′. *If M^n immerses in R^{n+2} and does not bound, then M^n is cobordant to $RP^{2^{p+1}-2} \times RP^{2^{q+1}-2}$ for some $0 \leqslant p \leqslant q$.*

In later work, Kikuchi proved:

PROPOSITION 4 (Kikuchi [10]). *If M^n immerses in R^{n+3} and n is odd, then all Stiefel-Whitney numbers of M divisible by w_1 are zero.*

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The case of $k = 3$ is much harder, and one has the following partial results:

PROPOSITION 5. *If M^n immerses in R^{n+3} with n odd and is not a boundary, then there exist integers $0 < r \leqslant s$ so that M^n is cobordant to the Dold manifold*

$$P(2^r - 1, 2^{s+1} - 2) = S^{2^r-1} \times \mathbf{C}P^{2^{s+1}-2}/(-1) \times (\mathit{conjugation}).$$

PROPOSITION 6. *If M^n is oriented, immerses in R^{n+3}, and n is even and larger than 4, then M^n is an unoriented boundary.*

PROPOSITION 7. *If M^n is a nonbounding manifold of even dimension which immerses in R^{n+3}, then $n = (2^{p+1} - 2) + (2^{q+1} - 2) + (2^{r+1} - 2)$ with $0 \leqslant p \leqslant q \leqslant r$.*

PROPOSITION 8. *Let M^n be a nonbounding manifold immersed in R^{n+3}.*

(1) *For $n = 3 \cdot 2^{u+1} - 6$, $u > 0$, M^n is cobordant to*

$$RP^{2^{u+1}-2} \times \mathbf{C}P^{2^{u+1}-2}, \quad RP^{2^{u+2}-2} \times \mathbf{C}P^{2^{u-1}-2},$$

or their union.

(2) *For $n = 3 \cdot 2^{u+1} - 4$, $u > 0$, M^n is cobordant to $RP^{2^{u+2}-2} \times RP^{2^{u+1}-2}$.*

(3) *For $n = 3 \cdot 2^{u+1} - 2$, $u > 1$, M^n is cobordant to $RP^{2^{u+2}-2} \times RP^{2^{u+1}-2} \times RP^2$.*

For the remaining values of n, one has severe restrictions given for the class of M^n, but there is some ambiguity. To describe the result in vague terms, the classes possible for M^n immersed in R^{n+3} with n even form a subspace of the unoriented cobordism group N_n of dimension at most the number of s for which $0 \leqslant 2^{s+2} - 2 \leqslant n$.

In the final section, one considers the problem of finding the filtration of indecomposable n-dimensional manifolds posed by Liulevicius [11]. In terms of immersions, this result could be stated as follows:

PROPOSITION 9. *If M^n is an indecomposable n-dimensional manifold, $2^a \leqslant n < 2^{a+1}$, immersing in R^m, then*

$$m \geqslant \begin{cases} 2^{a+1} - 1 & \text{for } n \text{ even}, \\ 2^{a+1} & \text{for } n \text{ odd}. \end{cases}$$

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2. The case $k = 1$: Preliminaries

If one has a manifold M^n which immerses in R^{n+1}, then the normal Stiefel-Whitney class of M is $\bar{w} = 1 + \bar{w}_1$; i.e., $\bar{w}_i = 0$ for $i > 1$. The tangential Stiefel-Whitney class of M is then

$$w = \frac{1}{1 + \bar{w}_1} = 1 + \bar{w}_1 + \bar{w}_1^2 + \cdots + \bar{w}_1^n,$$

and if M^n is not a boundary one must have

$$\bar{w}_1^n[M^n] = w_n[M^n] \equiv \chi(M^n) \not\equiv 0 \mod 2$$

where χ is the Euler characteristic. Thus, n must be even and

$$w_n[M^n] = (v_{n/2})^2[M^n] \neq 0$$

where

$$v = Sq^{-1}w = 1 + \bar{w}_1 + \bar{w}_1^3 + \cdots + w_1^{2^p-1} + \cdots$$

is the *Wu* class. In particular, $n = 2(2^p - 1) = 2^{p+1} - 2$ for some $p \geqslant 0$, and there is at most one non-zero class possible for M^n, characterized by

$$w_{i_1} \cdots w_{i_r}[M^n] \neq 0 \quad \text{if } i_1 + \cdots + i_r = n,$$

or, equivalently,

$$\bar{w}_{j_i} \cdots \bar{w}_{j_s}[M_n] = \begin{cases} 1, & j_1 = \cdots = j_s = 1,\ s = n, \\ 0, & \text{otherwise } (j_1 + \cdots + j_s = n). \end{cases}$$

For the manifold RP^n, $n = 2^{p+1} - 2$, one has

$$w = (1 + \alpha)^{n+1} = (1 + \alpha)^{2^{p+1}-1} = \frac{1}{1 + \alpha}$$

where $\alpha \in H^1(RP^n; Z_2)$ is the nonzero class. Since $\alpha^n[RP^n] \neq 0$, this is precisely the desired class. Thus, one has shown:

PROPOSITION 1 (weak). *If M^n immerses in R^{n+1} and is not a boundary, then M^n is cobordant to $RP^{2^{p+1}-2}$ for some $p \geqslant 0$.*

Note. The remainder of the argument for $k = 1$, to eliminate the possibilities $p > 2$, is quite difficult and is a facet of the nonexistence of elements of Hopf invariant one. It is, of course, well known that a point, RP^2, and RP^6 immerse in codimension one, to give the cases which can actually occur. It should be noted that the above actually determined the classes of algebraic filtration one in the sense of Liulevicius [11].

The argument for Proposition 2 is, in fact, formally identical with the argument just given. If M^n is oriented and immerses in R^{n+2}, then $\bar{w} = 1 + \bar{w}_2$, with $\bar{w}_1 = w_1 = 0$ by orientability and $\bar{w}_i = 0$ for $i > 2$ because of the immersion. Since $Sq^1\bar{w}_2 = \bar{w}_3 + \bar{w}_2\bar{w}_1 = 0$, one just repeats the proof formally doubling the degree of every class. One also has $w(\mathbf{C}P^{2^{p+1}-2}) = 1/(1+\alpha)$, where $\alpha \in H^2(\mathbf{C}P^{2^{p+1}-2}; Z_2)$ is nonzero, to formally double the example. (*Note.* For details of the formal doubling, one should recall Milnor [12].)

To improve this result enough to get Proposition 2′ requires a bit more work. If M^n immerses in R^{n+k}, then the immersion $f : M^n \to R^{n+k}$ may be deformed in R^{n+k+r} for some large r to give an imbedding with normal bundle $\nu^{k+r} \cong \nu^k + r$ having r sections. (*Note.* Recall Hirsch [7].) If one applies the Pontrjagin-Thom construction to the embedding one obtains an element

$$[M,f] \in \lim_{r\to\infty} \pi_{n+k+r}\left(\sum{}^r MO_k\right)$$

in the cobordism group of immersions. Of course,

$$\lim_{r\to\infty} \pi_{n+k+r}\left(\sum{}^r MO_k\right) = \pi^s_{n+k}(MO_k) = \Omega^{fr}_{n+k}(MO_k)$$

is the stable homotopy or framed cobordism of the Thom space MO_k. (*Note.* This description is due to Wells [15].)

Being given a framed manifold P^{n+k} and map

$$P^{n+k} \xrightarrow{\phi} MO_k,$$

one can make ϕ transverse to $BO_k \subset MO_k$ to obtain a manifold M^n whose stable normal bundle is represented by a k-plane bundle (the normal bundle of M in P) which is immersible in R^{n+k} if $k \geqslant 1$.

Turning now to the situation of interest, let M^n be an oriented n-manifold with $n > 0$ immersed in R^{n+2} and let

$$\phi : P^{n+2} \to MSO_2 = MU_1 = \mathbf{C}P^\infty$$

be a map of a framed manifold into MSO_2 given as above so that ϕ is transverse to BSO_2 with inverse image M. (The constructions above pull back to SO_2 by orientability of ν). One then has

$$\overline{w}_2^{n/2}[M^n] \equiv \phi^*(U^{n/2+1})[P^{n+2}] \pmod 2$$

where $U \in H^2(MSO_2; Z)$ is the Thom class.

Now, P^{n+2} is a stably almost complex manifold with complex structure coming from the framing, having Chern class 1 and hence Todd class 1 $= Td$. Further $\phi : P \to \mathbf{C}P^\infty$ gives a complex bundle ξ over P with $c_1(\xi) = \phi^*(U)$, and one has

$$\begin{aligned}\phi^*(U^{n/2+1})[P^{n+2}] &= s_{n/2+1}(\xi)[P^{n+2}]\\ &= \left(\frac{1}{2}n+1\right)!\,\mathrm{ch}(\xi)[P^{n+2}]\\ &= \left(\frac{1}{2}n+1\right)!\,\mathrm{ch}(\xi)Td[P^{n+2}].\end{aligned}$$

By the Riemann-Roch theorem [1], $\mathrm{ch}(\xi)Td[P^{n+2}]$ is an integer, and so the integer $\phi^*(U^{n/2+1})[P^{n+2}]$ is divisible by $(1/2\ n + 1)!$ and, in particular, is even (n even, $n > 0$). Thus, $\bar{w}_2^{n/2}[M^n] = 0$, and one has proved:

PROPOSITION 2′. *If M^n is oriented, immerses in R^{n+2}, and $n > 0$, then M is an unoriented boundary.*

One can, of course, go much further and actually determine the possible classes for M^n in oriented cobordism (or complex cobordism).

LEMMA. *The image of the Hurewicz homomorphism*

$$h : \Omega_{2m}^{fr}(\mathbf{C}P^\infty) \to H_{2m}(\mathbf{C}P^\infty; Z)$$

is precisely $m!H_{2m}(\mathbf{C}P^\infty; Z)$.

Proof. The argument above gives image $(h) \subset m!H_{2m}(\mathbf{C}P^\infty; Z)$ where $2m = n + 2$. Taking the map

$$(S^2)^m = \underbrace{\mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1}_{m} \to \mathbf{C}P^\infty$$

which classifies $\xi_1 \otimes \cdots \otimes \xi_m$, the tensor product of the Hopf bundles, gives the opposite inclusion. ∎

PROPOSITION 2″. *Let M^n be an oriented manifold which immerses in R^{n+2}.*

(a) *For $n \not\equiv 0$ (4), M^n is an oriented boundary.*

(b) *For $n = 4k$, there is an integer j so that M^n is cobordant as oriented manifold to the submanifold*

$$N^{4k} \subset \mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1 \ ((2k + 1) - \text{times}),$$

dual to $(\xi_1)^j \otimes \xi_2 \otimes \cdots \otimes \xi_{2k+1}$, which does immerse in codimension 2.

Proof. The class of M is determined by Stiefel-Whitney numbers (all of which are zero for $n > 0$) and Pontrjagin numbers (for $n \equiv 0$ (4)). For $n = 4k$, the only nonzero normal Pontrjagin number is $\bar{\mathscr{P}}_1^k[M^{4k}]$, which is divisible by $(2k + 1)!$ by the above. For the manifold N^{4k} described,

$$c_1(\nu) = j\alpha_1 + \alpha_2 + \cdots + \alpha_{2k+1},$$

so

$$\begin{aligned} \bar{\mathscr{P}}_1^k[N^{4k}] &= [(j\alpha_1 + \alpha_2 + \cdots + \alpha_{2k+1})^2]^k[N^{4k}] \\ &= (j\alpha_1 + \alpha_2 + \cdots + \alpha_{2k+1})^{2k+1}[(\mathbf{C}P^1)^{2k+1}] \\ &= (2k + 1)!(j\alpha_1) \cdot \alpha_2 \cdots \alpha_{2k+1}[(\mathbf{C}P^1)^{2k+1}] \\ &= j \cdot (2k + 1)!, \end{aligned}$$

giving representatives for every possible class. ∎

Note. One should compare with the final remarks in Atiyah-Hirzebruch [2] giving the imbedding $N^{4k} \subset (S^2)^{2k+1} \subset R^{4k+3}$ in codimension 3 with a trivial summand in the normal bundle.

Note. In the complex case, the manifolds $N^{2k} \subset (\mathbf{C}P^1)^{k+1}$ dual to $(\xi_1)^j \otimes \xi_2 \otimes \cdots \otimes \xi_{k+1}$, with $\mathbf{C}P^1 = S^2$ having the framed structure give all possible classes.

3. The Case $k = 2$

If M^n is a manifold which immerses in R^{n+2}, then the normal Stiefel-Whitney class of M is $\overline{w} = 1 + \overline{w}_1 + \overline{w}_2$; i.e., $\overline{w}_i = 0$ for $i > 2$.

From the previous section, one has the manifolds $RP^{2^{p+1}-2} \times RP^{2^{q+1}-2}$ with $\overline{w} = (1 + \alpha)(1 + \beta)$, where $\alpha, \beta \in H^1(RP^{2^{i+1}-2}; Z_2)$ are the non-zero classes, $0 \leqslant p \leqslant q$. Provided $0 \leqslant p \leqslant q \leqslant 3$, these manifolds actually immerse in codimension 2. The goal of this section is to show that if M^n does not bound, then it is cobordant to one of these manifolds. *Note.* If $n = (2^{p+1} - 2) + (2^{q+1} - 2)$, then $n + 4 = 2^{p+1} + 2^{q+1}$ has at most two one's in its dyadic expansion. Thus p and q are determined uniquely by n.

Proposition 3 (Kikuchi [9]). *If M^n does not bound, then n is even.*

Proof. If n is odd, then

$$\overline{w}_1^{2s+1}\overline{w}_2^{2t}[M^n] = \overline{w}_1 \cdot \overline{w}_1^{2s}\overline{w}_2^{2t}[M^n] = Sq^1(\overline{w}_1^{2s}\overline{w}_2^{2t})[M^n] = 0$$

where $\overline{w}_1 = w_1 = v_1$ is the *Wu* class, and $Sq^1(x^2) = 0$. Furthermore,

$$\begin{aligned} \overline{w}_1^{2j+1}\overline{w}_2^{2k+1}[M^n] &= Sq^2(\overline{w}_1^{2j+1}\overline{w}_2^{2k})[M^n] \\ &= Sq^2(\overline{w}_1 \cdot (\overline{w}_1^{2j}\overline{w}_2^{2k}))[M^n] \\ &= \overline{w}_1(Sq^1(\overline{w}_1^j\overline{w}_2^k))^2[M^n] \\ &= Sq^1([Sq^1(\overline{w}_1^j\overline{w}_2^k)]^2)[M^n] \\ &= 0 \end{aligned}$$

where $\overline{w}_2 = v_2$ is the second *Wu* class. Thus, $\overline{w}_1^a\overline{w}_2^b[M^n] = 0$ for all a, b, with $a + 2b = n$, and M bounds. ■

Note. (1)

$$\overline{w}_i x\,[M^n] = \chi(Sq^i)(x)[M^n] \quad \text{for all } x \in H^{n-i}(M^n; Z_2),$$

just as $v_i x[M^n] = Sq^i x[M^n]$ for the *Wu* classes. This fact was first noted by Brown and Peterson [3] (see §8). Here $\chi(Sq) = Sq^{-1}$; i.e., χ is the canonical anti-automorphism of the Steenrod algebra.

(2) Kikuchi's first proof of this fact [9] is needlessly complicated. In [10], an easier proof, similar to the above, was given.

LEMMA. *If M^n is nonbounding, then $n = (2^{p+1} - 2) + (2^{q+1} - 2)$ for some p, q with $0 \leqslant p \leqslant q$.*

Proof. By the above, n is even. If $\overline{w}_1^a\overline{w}_2^b[M^n] \neq 0$, then a must be even since $a + 2b = n$. However,

$$\begin{aligned}\overline{w}_1^{2s}\overline{w}_2^{2t+1}[M^n] &= Sq^2(\overline{w}_1^{2s}\overline{w}_2^{2t})[M^n] \\ &= [Sq^1(\overline{w}_1^s\overline{w}_2^t)]^2[M^n] \\ &= (s + t)\overline{w}_1^{2s+2}\overline{w}_2^{2t}[M^n].\end{aligned}$$

Hence, if M^n has any nonzero Stiefel-Whitney number, it must have a nonzero number of the form

$$\overline{w}_1^{2u}\overline{w}_2^{2v}[M^n] = Sq^{n/2}(\overline{w}_1^u\overline{w}_2^v)[M^n] = v_{n/2} \cdot \overline{w}_1^u\overline{w}_2^v[M^n],$$

and so $v_{n/2} \neq 0$. By the splitting principle, one may formally write

$$\overline{w} = (1 + x)(1 + y) \quad \text{with } \dim x = \dim y = 1,$$

and

$$\begin{aligned}v &= Sq^{-1}\left(\frac{1}{(1 + x)(1 + y)}\right) \\ &= (1 + x + x^3 + \cdots + x^{2^r-1} + \cdots)(1 + y + y^3 + \cdots + y^{2^s-1} + \cdots)\end{aligned}$$

so that $v_i \neq 0$ only for i of the form $(2^r - 1) + (2^s - 1)$, $0 \leqslant r \leqslant s$. In particular, there must be integers p and q with $n/2 = (2^p - 1) + (2^q - 1)$, $0 \leqslant p \leqslant q$. ■

LEMMA. *If $n = (2^{p+1} - 2) + (2^{q+1} - 2)$ with $0 \leqslant p = q$, then M^n is cobordant to $\mathbf{C}P^{2^{q+1}-2} \sim (RP^{2^{q+1}-2})^2$.*

Proof. If $p = q = 0$, $n = 0$, and M is cobordant to a point. Thus, one may suppose $p = q > 0$. As above, one has

$$\begin{aligned}v &= (1 + x + x^3 + \cdots + x^{2^{q+1}-1})(1 + y + y^3 + \cdots + y^{2^{q+1}-1}) \\ &= 1 + \cdots + (xy)^{2^q-1} + (x^{2^{q+1}-1} + y^{2^{q+1}-1}) + \cdots\end{aligned}$$

so that

$$\begin{aligned}v_{n/2} = \overline{w}_2^{2^q-1} \quad \text{and} \quad 0 &= Sq^1(v_{n/2+1}) \\ &= Sq^1(x^{2^{q+1}-1} + y^{2^{q+1}-1}) = \overline{w}_1^{2^{q+1}},\end{aligned}$$

since $v_i = 0$ if $i > n/2$.

Now, suppose $\overline{w}_1^{2a}\overline{w}_2^{b}[M^n] \neq 0$ has the largest a value among zero Stiefel-Whitney numbers. Then $2a + 2b = n = 2^{q+2} - 4$, $b = 2^{q+1} - 2 - a$, and since $\overline{w}_1^{2a} \neq 0$, $2a < 2^{q+1}$ or $a \leq 2^q - 1$ so $b \geq 2^{q+1} - 2 - (2^q - 1) = 2^q - 1$. Thus

$$\begin{aligned} 0 \neq \overline{w}_1^{2a}\overline{w}_2^{b}[M^n] &= \overline{w}_1^{2a}\overline{w}_2^{b-(2^q-1)}\overline{w}_2^{2^q-1}[M^n] \\ &= (\overline{w}_1^{2a}\overline{w}_2^{b-(2^q-1)})^2[M^n] \\ &= \overline{w}_1^{4a}\overline{w}_2^{b'}[M^n]. \end{aligned}$$

Since a was chosen to be maximal, $4a \leq 2a$ and so $a = 0$. Thus, the only nonzero Stiefel-Whitney number of M^n is $\overline{w}_2^{2(2^q-1)}[M^n]$ and M^n is cobordant to $CP^{2^{q+1}-2}$. ■

LEMMA. *If* $n = (2^{p+1} - 2) + (2^{q+1} - 2)$ *with* $0 = p < q$, *then* M^n *is cobordant to* $RP^{2^{q+1}-2}$.

Proof. One has $n = 2^{q+1} - 2$ and

$$\begin{aligned} v &= (1 + x + x^3 + \cdots + x^{2^q-1})(1 + y + y^3 + \cdots + y^{2^q-1}) \\ &= 1 + \cdots + (x^{2^q-1} + y^{2^q-1}) + (x^{2^q-1}y + xy^{2^q-1}) + \cdots + (xy)^{2^q-1}. \end{aligned}$$

Thus $\overline{w}_2^{2^q-1} = v_n = 0$ gives a zero Stiefel-Whitney number, and

$$0 = v_{n/2+1} = x^{2^q-1}y + xy^{2^q-1}, = (x + y)(x^{2^q-1} + y^{2^q-1}) + x^{2^q} + y^{2^q},$$

i.e., $\overline{w}_1 v_{n/2} + \overline{w}_1^{2^q} = 0$ and also $\overline{w}_1 v_{n/2} + Sq^1 v_{n/2} = 0$. Then, for any x, one has

$$\begin{aligned} 0 &= (w_1 v_{n/2} + Sq^1 v_{n/2})x[M^n] \\ &= \{Sq^1(x \cdot v_{n/2}) + x \cdot Sq^1 v_{n/2}\}[M^n] \\ &= (Sq^1 x) \cdot v_{n/2}[M^n] \\ &= (Sq^1 x)^2[M^n]. \end{aligned}$$

If one has $0 \neq \overline{w}_1^{2a}\overline{w}_2^{b}[M^n]$, then $a > 0$ by the above. Then, with $a, t > 0$,

$$\begin{aligned} \overline{w}_1^{2a}\overline{w}_2^{2t}[M^n] &= (\overline{w}_1^{a}\overline{w}_2^{t})v_{n/2}[M^n] \\ &= \overline{w}_1^{a-1}\overline{w}_2^{t}(\overline{w}_1 v_{n/2})[M^n] \\ &= \overline{w}_1^{2^q+a-1}\overline{w}_2^{t}[M^n] \end{aligned}$$

and, by iteration, if there is a number $\overline{w}_1^{2a}\overline{w}_2^{b}[M^n] \neq 0$ with $b > 0$, then there is such a number with b odd. However,

$$\overline{w}_1^{2a}\overline{w}_2^{2t+1}[M^n] = Sq^2(\overline{w}_1^{2a}\overline{w}_2^{2t})[M^n] = [Sq^1(\overline{w}_1^{a}\overline{w}_2^{t})]^2[M^n] = 0.$$

Thus, the only nonzero number of M^n is $\bar{w}_1^{2(2^q-1)}[M^n]$ and M^n is cobordant to $RP^{2^{q+1}-2}$. ■

In order to study the remaining case, $n = (2^{p+1} - 2) + (2^{q+1} - 2)$ with $0 < p < q$, it is convenient to consider the diagram

$$\begin{array}{ccc} & M^n & \xrightarrow{i} & P^{n+2} \\ & \downarrow \nu^2 & & \downarrow \phi \\ RP^\infty \times RP^\infty \xrightarrow{u} & BO_2 & \xrightarrow{v} & MO_2 \end{array}$$

which gives

$$\begin{array}{ccccc} & & H^*(M; Z_2) & \xleftarrow{i^*} & H^*(P; Z_2) \\ & & \uparrow \nu^* & & \uparrow \phi \\ H^*(RP^\infty \times RP^\infty; Z_1) & \xleftarrow{u^*} & H^*(BO_2; Z_2) & \xleftarrow{v^*} & H^*(MO_2; Z_2) \\ \| & & & & \\ Z_2[x, y] & & & & \end{array}$$

Now, u^* is monic, and has image precisely the symmetric polynomials in x and y, by the usual splitting principle. Also, v^* is monic, and in strictly positive degrees hits the multiples of $w_2 = v^*(U)$, so that u^*v^* is monic and hits the symmetric functions in x and y which are divisible by xy.

It is also convenient to make use of the classes $s_{a,b} \in H^{a+b}(BO_2; Z_2)$ which are characterized by

$$u^*(s_{a,b}) = \begin{cases} x^a y^b + x^b y^a, & a \neq b, \\ x^a y^a, & a = b, \end{cases}$$

and if $a, b \geqslant 1$, $s_{a,b}$ actually comes from $H^*(MO_2; Z_2)$ with $s_{a,b} = s_{a-1,b-1}U$ (using s to denote the element in either space). Using the relation between M and the framed manifold P, one has

$$\begin{aligned} s_{a,b}[M^n] &= \nu^*(s_{a,b})[M^n] \\ &= \phi^*(s_{a,b}U)[P^{n+2}] \\ &= \phi^*(s_{a+1,b+1})[P^{n+2}] \\ &= s_{a+1,b+1}[P^{n+2}] \end{aligned}$$

if $a + b = n$, the homomorphisms ν^*, ϕ^* are omitted in writing characteristic numbers.

Now, consider M^n a nonbounding manifold immersed in R^{n+2} with

$$n = (2^{p+1} - 2) + (2^{q+1} - 2), \quad 0 < p < q.$$

The *Wu* class is given by

$$v = (1 + x + x^3 + \cdots + x^{2^{q+1}-1})(1 + y + y^3 + \cdots + y^{2^{q+1}-1})$$

and

$$w_2^{2^q-1} = x^{2^q-1}y^{2^q-1} = v_{2(2^q-1)} = 0,$$
$$s_{2^{q+1}-1} = x^{2^{q+1}-1} + y^{2^{q+1}-1} = v_{2^{q+1}-1} = 0.$$

For $0 \leqslant i \leqslant 2^{q+1} - 1$,

$$0 = Sq^i s_{2^{q+1}-1} = s_{2^{q+1}-1+i}$$

and hence $s_b = 0$ for $2^{q+1} - 1 \leqslant i \leqslant n$. Also $s_{a,a} = w_2^a = 0$ for $2^q - 1 \leqslant a$.

Now, for $a < b$, $a + b = n$, one has

$$\begin{aligned} s_{a,b} &= x^a y^b + x^b y^a \\ &= (x^a + y^a)(x^b + y^b) + (x^{a+b} + y^{a+b}) \\ &= s_a \cdot s_b + s_{a+b} \end{aligned}$$

and

$$s_{a,b} = (x^a y^a)(x^{b-a} + y^{b-a}) = \overline{w}_2^a s_{b-a}$$

giving $s_{a,b} = 0$ if $b > 2^{q+1} - 2$, or $a < 2^{p+1} - 2$ (since $s_b = s_{a+b} = 0$), or $a \geqslant 2^q - 1$ (since $\overline{w}_2^a = 0$). *Note.* For $a = b$, $n = 2a$, $s_{a,a} = w_2^a = 0$, and one need really only consider the case $a < b$.

Now, for $a + b = n \equiv 0$ (2), a and b are either both even or both odd. One has

$$s_{2r+1,2s+1}[M^n] = s_{2r+2,2s+2}[P^{n+2}] = Sq^{r+s+2}(s_{r+1,s+1})[P^{n+2}] = 0$$

since P is framed, and hence Steenrod operations into the top dimension are zero (the *Wu* class of P is 1). To be nonzero, one must have both a and b even.

Now consider $a < b$, $a + b = n$ with $s_{a,b}[M^n] \neq 0$ and such that $s_{a',b'}[M^n] = 0$ for $a < a' \leqslant b' < b$. Since a and b are even, let $a = 2r$, $b = 2s$. Suppose that $2s = b < 2^{q+1} - 2$ (equivalently $2r = a > 2^{p+1} - 2$). Recalling that $s_{u,v} = 0$ if $u < v$ and $u \geqslant 2^q - 1$, one must have $2r = a \leqslant 2^q - 2$.

Case 1. If $a = 2^q - 2$, $b = 2^q + 2^{p+1} - 2$, and

$$\begin{aligned} s_{a,b}[M^n] &= s_{2^q-1,2^{p+1}-1}[P^{n+2}] \\ &= \left\{ Sq^{2^{p+1}} s_{2^q-1,2^q-1} + \sum_{t=1}^{2^p} s_{2^q-1+t,2^q-1+(2^{p+1}-t)} \right\} [P^{n+2}] \\ &= \sum_{t=1}^{2^p} s_{2^q-2+t,2^q-2+2^{p+1}-t}[M^n], \end{aligned}$$

which is a sum of numbers $s_{a',b'}[\mathrm{M}^n]$ with $a < a' \leqslant b' < b$ and hence zero.

Case 2. If $a = 2r < 2^q - 2$, and, as assumed, $2s = b < 2^{q+1} - 2$, $2s + 2 - 2^q < 2^q$, then

$$\begin{aligned} s_{a,b}[M^n] &= s_{2r+1,2s+1}[P^{n+2}], \\ &= \{Sq^{2s+1-(2^q-1)}s_{2r+1,2^q-1} + \text{terms } s_{2r+1+u,2^q-1+v}\}[P^{n+2}] \\ &= \{\text{terms } s_{2r+u,2^q-2+v}\}[M^n] \end{aligned}$$

which is a sum of terms $s_{a',b'}[M^n]$ with $a < a' \leqslant b' < b$ and hence zero.

Note. Here, $2s + 2 - 2^q < 2^q$ makes the term $s_{2r+1,2s+1}$ occur with nonzero coefficient in the "squared" term. Also, each $u > 0$, so that $2r + u > a$ and, of course, $2^q - 2 + v \geqslant 2^q - 2 > a$, to guarantee $a' > a$.

Thus, $s_{a,b}[M^n] \neq 0$ implies $a = 2^{p+1} - 2$, $b = 2^{q+1} - 2$ if $a \leqslant b$. Now, consider

$$RP^{2^{p+1}-2} \times RP^{2^{q+1}-2},$$

with the nonzero classes

$$\alpha \in H^1(RP^{2^{p+1}-2}; Z_2), \beta \in (RP^{2^{q+1}-2}; Z_2)$$

giving

$$\overline{w} = (1 + \alpha)(1 + \beta),$$

and $s_{a,b}[RP^{2^{p+1}-2} \times RP^{2^{q+1}-2}] = \alpha^a\beta^b[RP^{2^{p+1}-2} \times RP^{2^{q+1}-2}]$ if $a \leqslant b$ (for $\alpha^b = 0$), which is nonzero only for $a = 2^{p+1} - 2$, $b = 2^{q+1} - 2$. Thus M^n is cobordant to

$$RP^{2^{p+1}-2} \times RP^{2^{q+1}-2}.$$

This gives the final case, and completes the proof of the following result.

PROPOSITION 3′. *If M^n immerses in R^{n+2} and is not a boundary, then M^n is cobordant to $RP^{2^{p+1}-2} \times RP^{2^{q+1}-2}$ for some $0 \leqslant p \leqslant q$.*

Note. (1) One has actually proved a bit more, in that the conclusion of Proposition 3′ will hold if you assume only that all Stiefel-Whitney numbers of M^n divisible by classes $\overline{w}_i$ with $i > 2$ are zero; i.e., if M^n has algebraic filtration 2 in the sense of Liulevicius [11]. Proposition 1 (weak) actually worked for manifolds of algebraic filtration 1. To prove this is just formalism based upon the methods of [14].

Given M^n with numbers involving $\overline{w}_i = 0$ for $i > k$ one has

$$\nu^* : H^*(BO; Z_2) \to Z_2 : x \to x(\nu)[M^n]$$

which can be realized as a homomorphism $\nu^* : H^*(BO_k; Z_2) \to Z_2$. One can find a Poincaré algebra M' in which the classes $\overline{w}_i$ are actually zero for $i > k$ and have the same characteristic number homomorphism ν^*.

Further, there is a homomorphism $\phi^* : \tilde{H}^*(MO_k; Z_2) \to Z_2$ defined by $\phi^*(xU) = \nu^*(x)$, which is induced as the characteristic number homomorphism for a Poincaré algebra P' of dimension $n + k$ with homomorphism $H^*(MO_k; Z_2) \to P'$ so that $w(P') = 1$. The calculations performed in $H^*(M; Z_2)$ and $H^*(P; Z_2)$ could instead have been done in M' and P' which depend only on the homomorphism ν^*.

(2) Propositions 1 (weak) and 3′ are equivalent to the assertions that

$$\tilde{H}^*(MO_1; Z_2) = \tilde{H}^*(RP^\infty; Z_2)$$

is generated as module over the Steenrod algebra $\mathscr{A}$ by the classes x^{2^s-1} and that $\tilde{H}^*(MO_2; Z_2)$ is generated over $\mathscr{A}$ by the classes

$$s_{2^s-1,2^t-1} = s_{2^s-2,2^t-2}U, \quad 1 \leqslant s \leqslant t.$$

4. The Case $k = 3$, n Odd

Proposition 4 (Kikuchi [10]). *If M^n immerses in R^{n+3} and n is odd, then all Stiefel-Whitney numbers of M divisible by $\bar{w}_1$ are zero.*

Note. Because Kikuchi's paper appears in a relatively inaccessible journal, a proof will be given.

Proof. Consider a Stiefel-Whitney number $\bar{w}_1^a\bar{w}_2^b\bar{w}_3^c[M^n]$ with $a + 2b + 3c = n \equiv 1\ (2)$. Then $a + c \equiv 1\ (2)$. Then

$$\bar{w}_1^{2p+1}\bar{w}_2^{2q}\bar{w}_3^{2r}[M^n] = Sq^1(\bar{w}_1^{2p}\bar{w}_2^{2p}\bar{w}_3^{2r})[M^n] = 0,$$

$$\bar{w}_1^{2p}\bar{w}_2^{2q}\bar{w}_3^{2r+1}[M^n] = \chi(Sq^3)(\bar{w}_1^{2p}\bar{w}_2^{2q}\bar{w}_3^{2r})[M^n] = 0,$$

since $Sq^1(x^2) = \chi(Sq^3)(y^2) = 0$, so that for b even, the number is zero. Now

$$\begin{aligned}\bar{w}_1^{2p+1}\bar{w}_2^{2q+1}\bar{w}_3^{2r}[M] &= Sq^2\{\bar{w}_1 \cdot \bar{w}_1^{2p}\bar{w}_2^{2q}\bar{w}_3^{2r}\}[M] \\ &= \bar{w}_1\{Sq^1(\bar{w}_1^p\bar{w}_2^q\bar{w}_3^r)\}^2[M] \\ &= Sq^1(x^2)[M] = 0\end{aligned}$$

and so the only possibly nonzero numbers are those with $a \equiv 0$, $b \equiv c \equiv 1\ (2)$. Let

$$\bar{w} = (1 + x)(1 + y)(1 + z)$$

via the splitting principle; then

$$v = (1 + x + x^3 + \cdots)(1 + y + y^3 + \cdots)(1 + z + z^3 + \cdots)$$

so

$$v_4 = x^3(y + z) + y^3(x + z) + z^3(x + y) = \bar{w}_1^2\bar{w}_2 + \bar{w}_1\bar{w}_3.$$

For $p > 0$, one has

$$\begin{aligned}\overline{w}_1^{2p}\overline{w}_2^{2q+1}\overline{w}_3^{2r+1}[M^n] &= \overline{w}_1\overline{w}_3(\overline{w}_1^{2p-1}\overline{w}_2^{2q+1}\overline{w}_3^{2r})[M^n] \\ &= \{\overline{w}_1^2\overline{w}_2 \cdot \overline{w}_1^{2p-1}\overline{w}_2^{2q+1}\overline{w}_3^{2r} + Sq^4(\overline{w}_1^{2p-1}\overline{w}_2^{2q+1}\overline{w}_3^{2r})\}[M^n],\end{aligned}$$

where the first term has $\overline{w}_1$ to an odd power, so is zero, and

$$\begin{aligned}&\overline{w}_1^{2p}\overline{w}_2^{2q+1}\overline{w}_3^{2r+1}[M^n] \\ &\quad = Sq^4(\overline{w}_1\overline{w}_2\{\overline{w}_1^{2p-2}\overline{w}_2^{2q}\overline{w}_3^{2r}\})[M^n] \\ &\quad = \overline{w}_1\overline{w}_2\{Sq^2(\overline{w}_1^{p-1}\overline{w}_2^{q}\overline{w}_3^{r})\}^2[M^n] + Sq^2(\overline{w}_1\overline{w}_2) \cdot \{Sq^1(\overline{w}_1^{p-1}\overline{w}_2^{q}\overline{w}_3^{r})\}^2[M^n].\end{aligned}$$

Expanding either term, $\{Sq^i(\overline{w}_1^{p-1}\overline{w}_2^{q}\overline{w}_3^{r})\}^2$ gives a sum of monomials in which $\overline{w}_1$ appears to an even power because it is a square and $\overline{w}_1\overline{w}_2$ and

$$Sq^2(\overline{w}_1\overline{w}_2) = \overline{w}_1^2(\overline{w}_3 + \overline{w}_2\overline{w}_1) + \overline{w}_1\overline{w}_2^2$$

give only one term with $\overline{w}_1$ to an even power; so

$$\begin{aligned}\overline{w}_1^{2p}\overline{w}_2^{2q+1}\overline{w}_3^{2r+1}[M^n] &= \overline{w}_1^2\overline{w}_3\{Sq^1(\overline{w}_1^{p-1}\overline{w}_2^{q}\overline{w}_3^{r})\}^2[M^n] \\ &= \chi(Sq^3)\{\overline{w}_1 Sq^1(\overline{w}_1^{p-1}\overline{w}_2^{q}\overline{w}_3^{r})\}^2[M^n] \\ &= 0.\end{aligned}$$

Thus the only possible Stiefel-Whitney numbers $\overline{w}_1^a\overline{w}_2^b\overline{w}_3^c[M^n]$ are those for which $a = 0$ and $b \equiv c \equiv 1$ (2). ∎

Hypothesis. Let M^n, n odd, be a nonbounding manifold which immerses in R^{n+3}. Then there are integers a, b with $\overline{w}_2^a\overline{w}_3^b[M^n] \neq 0$ so that if $\overline{w}_2^{a'}\overline{w}_3^{b'}[M^n] \neq 0$, then $b' \leqslant b$. By Kikuchi's result, a is odd and may be written as

$$a = 2^{r+1}v + (2^r - 1) \quad \text{with } v \geqslant 0.$$

Note. Using Kikuchi's result, all numbers divisible by $\overline{w}_1$ are zero, and applying [14] one may replace M by a Poincaré algebra having $\overline{w}_1 = \overline{w}_3^{b+1} = 0$.

LEMMA. *For all i with $0 \leqslant i \leqslant a$,*

$$\binom{a+b-i}{i} \equiv \begin{cases} 0 \bmod 2, & i \neq 2^s - 1, \\ 1 \bmod 2, & i = 2^s - 1. \end{cases}$$

Proof. One has

$$\begin{aligned}Sq(\overline{w}_2^{a-i}\overline{w}_3^b) &= (\overline{w}_2 + \overline{w}_3 + \overline{w}_2^2)^{a-i}\{\overline{w}_3(1 + \overline{w}_2 + \overline{w}_3)\}^b \\ &= \overline{w}_3^b(\overline{w}_2 + \overline{w}_3 + \overline{w}_2^2)^{a-i}(1 + \overline{w}_2 + \overline{w}_3)^b \\ &= \overline{w}_3^b\overline{w}_2^{a-i}(1 + \overline{w}_2)^{a+b-i}\end{aligned}$$

for $\overline{w}_3^{b+1} = 0$, so

$$Sq^{2i}(\overline{w}_2^{a-i}\overline{w}_3^b)[M^n] = \binom{a+b-i}{i}\overline{w}_2^a\overline{w}_3^b[M^n].$$

Now, modulo terms involving $\overline{w}_3$, $\overline{w} = 1 + \overline{w}_2$ and

$$v = Sq^{-1}\left(\frac{1}{1+\overline{w}_2}\right) = 1 + \overline{w}_2 + \overline{w}_2^3 + \cdots + \overline{w}_2^{2^s-1} + \cdots \quad \operatorname{mod} \overline{w}_3,$$

so

$$v \cdot \overline{w}_3^b = \overline{w}_3(1 + \overline{w}_2 + \overline{w}_2^3 + \cdots + \overline{w}_2^{2^s-1} + \cdots).$$

Thus,

$$Sq^{2i}(\overline{w}_2^{a-i}\overline{w}_3^b)[M^n] = v_{2i}\overline{w}_2^{a-i}\overline{w}_3^b[M^n] = \begin{cases} 0, & i \neq 2^s - 1, \\ \overline{w}_2^a\overline{w}_3^b[M^n], & i = 2^s - 1. \end{cases} \quad \blacksquare$$

COROLLARY. $b = 2^r x + (2^r - 1)$ *with* $x \geqslant 0$.

Proof. $2^r - 1 \leqslant a$, so

$$1 \equiv \binom{a + b - (2^r - 1)}{2^r - 1} = \binom{2^{r+1}v + b}{2^r - 1},$$

which implies that $2^0, 2^1, \ldots, 2^{r-1}$ all appear in the 2-adic expansion of b. Thus $b = 2^r x + (2^r - 1)$ for some $x \geqslant 0$. $\blacksquare$

Peering into the future to the cases desired, one has the following results.

COROLLARY. *If* $b = (2^r - 1)$, i.e., $x = 0$, *then* $a = 2^{s+1} - 2^{r+1} + (2^r - 1)$ *for some* $s \geqslant r$.

Proof. If $v = 0$, there is nothing to prove since one takes $s = r$. If $v > 0$, write $a = 2^s + 2^{s_1} + \cdots + 2^{s_q} + (2^r - 1)$ with $s > s_1 > \cdots > s_q \geqslant r + 1$. Then $2^s - 1 \leqslant a$, and

$$1 \equiv \binom{a + b - (2^s - 1)}{(2^s - 1)} = \binom{a + (2^r - 1) - (2^s - 1)}{2^s - 1}$$
$$= \binom{2^{s_1} + \cdots + 2^{s_q} + 2^r + (2^r - 1)}{2^s - 1}$$

so that $2^{s-1}, 2^{s-2}, \ldots, 2^{r+1}$ must all occur in the set $2^{s_1}, \ldots, 2^{s_q}$. Thus,

$$a = 2^s + 2^{s-1} + \cdots + 2^{r+1} + (2^r - 1) = 2^{s+1} - 2^{r+1} + (2^r - 1). \quad \blacksquare$$

COROLLARY. *If* $a = 2^{r+1}v + (2^r - 1)$ *with* $v > 0$, *then* $b = 2^r x + (2^r - 1)$ *with* x *even.*

Proof. Since $v > 0$, $a \geqslant 2^{r+1} - 1 = 2^r + (2^r - 1)$, and

$$1 \equiv \binom{a + b - (2^r + (2^r - 1))}{2^r + (2^r - 1)} = \binom{2^{r+1}v + 2^r x - 2^r + (2^r - 1)}{2^r + (2^r - 1)}$$

and in order that 2^r appear in the 2-adic expansion of $2^{r+1}v + 2^r x - 2^r$, one must have $2v + x - 1$ odd, so x is even. ■

Throughout the remainder of the arguments, considerable use will be made of the framed manifold $P = P^{n+3}$ in which M^n is imbedded and which satisfies

$$\overline{w}_1^i \overline{w}_2^j \overline{w}_3^k [M^n] = w_1^i w_2^k w_3^j \, U\,[P^{n+3}] = w_1^i w_2^j w_3^{k+1}[P]$$

where one considers $\tilde{H}^*(MO_3; Z_2) \subset H^*(BO_3; Z_2)$ as being the multiples of w_3. In P, one may also work modulo classes which give zero in all numbers, and hence consider w_1 and w_3^{b+2} as being zero.

LEMMA. *In the expression $b = 2^r x + (2^r - 1)$, x is even.*

Proof. Suppose, to the contrary, that x is odd. Then, by the previous corollary, one has $v = 0$ and $a = 2^r - 1$.

Since P is a framed manifold,

$$Sq^{2^r}(w_2^{2^{r+1}-1} w_3^{2^r x})[P] = 0$$

and

$$Sq(w_2^{2^{r+1}-1} w_3^{2^r x}) = w_3^{2^r x}(1 + w_2 + w_3)^{2^r x}(w_2 + w_3 + w_2^2)^{2^{r+1}-1}$$

where the lowest degree term is $w_3^{2^r x} w_2^{2^{r+1}-1}$ and one seeks the term of degree 2^r above that. Since $(1 + w_2 + w_3)^{2^r x} = 1 +$ terms of degree at least 2^{r+1}, this reduces to

$$w_3^{2^r x} w_2^{2^{r+1}-1}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^r}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^r - 1}$$

where

$$\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^r} = 1 + \left(\frac{w_3}{w_2}\right)^{2^r} + \text{terms of higher degree}$$

to give

$$\begin{aligned} & w_3^{2^r x + 2^r} w_2^{2^r - 1} + \text{terms in } w_3^{2^r x} w_2^{2^{r+1}-1}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^r - 1} \\ &= w_2^a w_3^{b+1} + \text{terms in } \sum_{i=0}^{2^r - 1} w_2^i \left(1 + \frac{w_3}{w_2}\right)^{2^r - 1 - i} \\ &= w_2^a w_3^{b+1} + \sum_{i=0}^{2^r - 1} \binom{2^r - 1 - i}{2^r - 2i} w_3^{2^r x + 2^r - 2i} w_2^{2^r - 1 + 3i}. \end{aligned}$$

Now,

$$\binom{2^r - 1 - i}{2^r - 2i},$$

the coefficient of z^{2^r-2i} in $(1 + z)^{2^r-1-i} = (1 + z^{2^r})/(1 + z)^{i+1}$, is

$$\binom{i + 2^r - 2i}{2^r - 2i} = \binom{2^r - i}{i}$$

and is nonzero mod 2 only for $i = 2^t$ with $0 \leqslant t \leqslant r - 1$. Noting that the term with $i = 2^0 = 1$ gives an even power of w_2, one obtains

$$w_2^a w_3^{b+1}[P] = \sum_{t=1}^{r-1} w_2^{2^r+2^{t+1}+2^t-1} w_3^{2^r x+2^r-2^{t+1}}[P].$$

Notice that this implies $r > 1$; i.e., $r = 1$ is impossible.

For $0 < t < r - 1$,

$$Sq^{2^{t+2}}(w_2^{2^r+2^t-1} w_3^{2^r x+2^r-2^{t+1}})[P] = 0,$$

and

$$\begin{aligned} Sq(&w_2^{2^r+2^t-1} w_3^{2^r x+2^r-2^{t+1}}) \\ &= w_2^{2^r+2^t-1} w_3^{2^r x+2^r-2^{t+1}}(1 + w_2 + w_3)^{2^{t+1}(\mathrm{odd})}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^r+2^t-1}, \end{aligned}$$

giving

$$\begin{aligned} &w_2^{2^r+2^{t+1}+2^t-1} w_3^{2^r x+2^r-2^{t+1}} + \text{terms in } w_2^{2^r+2^t-1} w_3^{2^r x+2^t-1} \\ &\qquad\qquad \left(1 + \frac{w_3}{w_2} + w_2\right)^{2^r}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^t-1}. \end{aligned}$$

Now

$$\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^r} = 1 + \text{terms of degree at least } 2^r \geqslant 2^{t+2},$$

while $(1 + w_3/w_2 + w_2)^{2^t-1}$ has highest term of degree $2^{t+1} - 2 < 2^{t+2}$. Thus one has

$$w_2^{2^r+2^{t+1}+2^t-1} w_3^{2^r x+2^r-1^{t+1}}[P] = \begin{cases} 0 & \text{if } t < r - 2, \\ w_2^{2^{r-2}-1} w_3^{b+1+2^{r-1}}[P] & \text{if } t = r - 2, \end{cases}$$

with the latter being zero since $w_3^{b+2} = 0$.

Thus, in the formula for $w_2^a w_3^{b+1}[P]$, only the term with $t = r - 1 > 0$ is nonzero, giving

$$w_2^a w_3^{b+1}[P] = w_2^{2^{r+1}+2^{r-1}-1} w_3^{2^r x}[P].$$

For $x > 1$,

$$Sq^{2^r}(w_2^{2^{r+1}+2^r+2^{r-1}-1}w_3^{2^r(x-1)})[P] = 0,$$

and

$$Sq(w_3^{2^{r+1}+2^r+2^{r-1}-1}w_3^{2^r(x-1)})$$
$$= w_3^{2^r(x-1)}w_2^{2^{r+1}+2^r+2^{r-1}-1}(1 + w_2 + w_3)^{2^r(x-1)}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+1}+2^r+2^{r-1}-1}$$

with

$$(1 + w_2 + w_3)^{2^r(x-1)}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+1}} = 1 + \text{terms of degree at least } 2^{r+1},$$

$$\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^r} = 1 + \left(\frac{w_3}{w_2}\right)^{2^r} + \text{higher terms}$$

and all terms in $(1 + w_3/w_2 + w_2)^{2^{r-1}-1}$ of degree less than 2^r. Thus, the only term is

$$w_2^{2^{r+1}+2^{r-1}-1}w_3^{2^r(x-1)+2^r}[P] = 0.$$

Notice that this implies $x = 1$.

Now consider the numbers

$$w_2^{2^{r+2}-2^q-2^{q-1}-1}w_3^{2^q}[P]$$

for $1 \leqslant q \leqslant r$, recalling $r > 1$. For $q = r$, this is $w_2^{2^{r+1}+2^{r-1}-1}w_3^{2^r}[P]$ which is $w_2^a w_3^{b+1}[P]$, since $x = 1$. For $q = 1$, this is zero since w_2 occurs to an even power.

One has (for $q \geqslant 2$)

$$0 = Sq^{2^{q-1}}(w_2^{2^{r+2}-2^q-1}w_3^{2^{q-1}})[P]$$

since P is framed, and

$$Sq(w_2^{2^{r+2}-2^q-1}w_3^{2^{q-1}})$$
$$= w_3^{2^{q-1}}w_2^{2^{r+2}-2^q-1}(1 + w_2 + w_3)^{2^{q-1}}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+1}+\cdots+2^{q+1}+2^{q-1}+\cdots+1}$$

with

$$(1 + w_2 + w_3)^{2^{q-1}} = 1 + \text{terms of degree at least } 2^q,$$

and

$$\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+1}+\cdots+2^{q+1}+2^{q-1}} = 1 + \left(\frac{w_3}{w_2}\right)^{2^{q-1}} + \text{higher terms},$$

to give

$$w_2^{2^{r+2}-2^q-2^{q-1}-1}w_3^{2^q} + \text{terms in } w_3^{2^{q-1}}w_3^{2^{r+2}-2^q-1}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{q-1}-1}.$$

Now the term of degree 2^{q-1} in

$$\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{q-1}-1} = \sum_{i=0}^{2^{q-1}-1} w_2^i\left(1 + \frac{w_3}{w_2}\right)^{2^{q-1}-1-i}$$

is

$$\sum_{i=0}^{2^{q-1}-1} w_2^i \binom{2^{q-1}-1-i}{2^{q-1}-2i}\left(\frac{w_3}{w_2}\right)^{2^{q-1}-2i}$$

and

$$\binom{2^{q-1}-1-i}{2^{q-1}-2i} \not\equiv 0 \mod 2$$

if and only if $i = 2^t$, $0 \leq t \leq q-2$. Thus

$$0 = w_2^{2^{r+2}-2^q-2^{q-1}-1}w_3^{2^q}[P] + w_2^{2^{r+2}-2^{q-1}-2^{q-2}-1}w_3^{2^{q-1}}[P] + \sum_{t=1}^{q-3} w_2^{2^{r+2}-2^q-2^{q-1}+2^{t+1}+2^t-1}w_3^{2^q-2^{t+1}}[P].$$

Note. The term with $t = 0$ for the sum has w_2 occurring to an even power and so is zero. The term $t = q-2$ is separately written. Further, for small values of q, the final summation does not occur.

In order to analyze the terms

$$w_2^{2^{r+2}-2^q-2^{q-1}+2^{t+1}+2^t-1}w_3^{2^q-2^{t+1}}[P] \quad \text{for } 1 \leq t \leq q-3,$$

consider

$$0 = Sq^{2^{t+2}}(w_2^{2^{r+2}-2^q-2^{q-1}+2^t-1}w_3^{2^q-2^{t+1}})[P].$$

One has

$$Sq(w_2^{2^{r+2}-2^q-2^{q-1}+2^t-1}w_3^{2^q-2^{t+1}}) = w_3^{2^q-2^{t+1}}w_2^{2^{r+2}-2^q-2^{q-1}+2^t-1}$$
$$(1 + w_2 + w_3)^{2^{t+1}(\text{odd})}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+2}-2^q-2^{q-1}+2^t-1}$$

and seeks the term of degree 2^{t+2} above the initial point. Now,

$$(1 + w_2 + w_3)^{2^{t+1}(\text{odd})} = 1 + w_2^{2^{t+1}} + \text{higher terms}$$

and

$$\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+2}-2^q-2^{q-1}} = 1 + \left(\frac{w_3}{w_2}\right)^{2^{q-1}} + \text{higher terms}$$

with $q - 1 \geqslant t + 2$, which contributes only for $q - 1 = t + 2$. Finally, the term of highest degree in $(1 + w_3/w_2 + w_2)^{2^t-1}$ is $w_2^{2^t-1}$, of degree less 2^{t+2}. Thus, the expansion gives

$$w_2^{2^{r+2}-2^{q+1}+2^{q-2}+2^{q-3}-1} w_3^{2^q}(1 + w_2 + w_3)^{2^q}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+2}-2^{q+1}+2^{q-2}+2^{q-3}-1},$$

Now, considering

$$w_2^{2^{r+2}-2^{q+1}+2^{q+3}-1} w_3^{2^q+2^{q-2}}[P],$$

one has

$$Sq^{2^{q-2}}(w_2^{2^{r+2}-2^{q+1}+2^{q+2}+2^{q-3}-1} w_3^{2^q})[P] = 0$$

and

$$Sq(w_2^{2^{r+2}-2^{q+1}+2^{q-2}+2^{q-3}-1} w_3^{2^q}) =$$

$$w_2^{2^{r+2}-2^{q+1}+2^{q-2}+2^{q-3}-1} w_3^{2^q}(1 + w_2 + w_3)^{2^q}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+2}-2^{q+1}+2^{q-2}+2^{q-3}-1},$$

where

$$(1 + w_2 + w_3)^{2^q}\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+2}-2^{q+1}} =$$

$$1 + \text{terms of degree at least } 2^{q+1},$$

$$\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{q-2}} = 1 + \left(\frac{w_3}{w_2}\right)^{2^{q-2}} + \text{higher terms},$$

and $(1 + w_3/w_2 + w_2)^{2^{q-3}-1}$ has largest term of degree $2(2^{q-3} - 1) < 2^{q-2}$. Thus,

$$w_2^{2^{r+2}-2^{q+1}+2^{q-3}-1} w_3^{2^q+2^{q-2}}[P] = 0.$$

Hence, the numbers

$$w_2^{2^{r+2}-2^q-2^{q-1}-1} w_3^{2^q}[P] \quad \text{for } 1 \leqslant q \leqslant r$$

are all equal, so

$$0 \neq \overline{w}_2^a \overline{w}_3^b[M^n] = w_2^a w_3^{b+1}[P] \quad \text{for } q = r$$

equals the number for $q = 1$ which is zero, since w_2 occurs to an even power. This contradicts the assumption that x is odd. ■

LEMMA. *In the expression $b = 2^r x + (2^r - 1)$, $x = 0$.*

Proof. Assume, to the contrary, that $x \neq 0$. By the previous lemma, one may then let $x = 2u$ with $u > 0$.

Consider the numbers

$$Sq^{2^j}(w_2^{a+2^j}w_3^{b+1-2^j})[P] \quad \text{with } 0 \leqslant j \leqslant r.$$

One has

$Sq(w_2^{a+2^j}w_3^{b+1-2^j}) =$

$$w_3^{2^{r+1}u+2^r-2^j}(1 + w_2 + w_3)^{2^{r+1}u+2^r-2^j} \sum_{i=0}^{a+2^j} \binom{a+2^j}{i} w_3^i(w_2 + w_2^2)^{a+2^j-i}$$

and

$$(1 + w_2 + w_3)^{2^{r+1}u+2^r-2^j} = 1 + \text{terms of degree at least } 2^{j+1},$$

while terms in the sum with $i > 2^j$ give powers of w_3 exceeding $b + 1$. Now

$$a + 2^j = \begin{cases} 1 + \cdots + 2^{j-1} + 2^r + 2^{r+1}v & \text{if } j < r, \\ 1 + \cdots + 2^r + 2^{r+1}v & \text{if } j = r, \end{cases}$$

so the coefficients

are $$\binom{a+2^j}{i}$$

odd for $i \leqslant 2^j - 1$; if $i = 2^j$ the coefficient is odd only when $j = r$. Thus, the Sq-expression reduces to

$$\sum_{i=0}^{2^j-1} w_3^{2^{r+1}u+2^r-2^j+i}w_2^{a+2^j-i}(1 + w_2)^{a+2^j-i} + \varepsilon w_3^{2^{r+1}u+2^r}(w_2 + w_2^2)^a$$

where $\varepsilon = 0$ for $j < r$ and $\varepsilon = 1$ for $j = r$. In this, the term of the correct degree is

$$\sum_{i=0}^{2^j-1} \binom{a + 2^j - i}{\frac{2^j - i}{2}} w_3^{2^{r+1}u+2^r-2^j+i}w_2^{a+2^j-i+(2^j-i)/2} + \varepsilon\, w_2^a w_3^{b+1}.$$

Here, the binomial coefficient is considered zero unless $2^j - i$ is even, and then it is the same as

$$\binom{2a + 2^{j+1} - 2i}{2^j - i}.$$

Note. For $j = 0$, the sum has the term $i = 0$ only, and $j \neq r$ so the entire expression is zero. Thus, only $j \geqslant 1$ actually occurs, and $2^j - i$ is even only for i even. Now letting $q = 2^j - i$, the binomial coefficient is

$$\binom{2a + 2q}{q} \quad \text{with } 1 \leqslant q \leqslant 2^j \leqslant 2^r$$

and is

$$\binom{2^r + \cdots + 2 + 2q}{q},$$

which is nonzero only when q is a power of 2; i.e., $i = 2^j - 2^p$ with $p \geqslant 1$ give the only nonzero terms. Thus,

$$\varepsilon w_2^a w_3^{b+1}[P] = \sum_{p=2}^{j} w_2^{a+3\cdot 2^{p-1}} w_3^{2^{r+1}u+2^r-2^p}[P],$$

noting that the term with $p = 1$ gives an even power of w_2, so is zero.

Adding the above for $j = r$ and $j = r - 1$, one has

$$w_2^a w_3^{b+1}[P] = w_2^{a+3\cdot 2^{r-1}} w_3^{2^{r+1}u}[P].$$

Note. For $r = 1 = j$, the right side is zero, and this is just the previous expression, with the term for $j = r - 1$ being identically zero. For $r = 2$, this is the equation for $j = r$, and the equation for $j = r - 1$ is identically zero.

Notice that when $r = 1$, this gives the contradiction $w_2^a w_3^{b+1}[P] = 0$, so $x = 0$. Thus, one must have $r > 1$.

Now, consider

$$0 = Sq^{2^r}(w_2^{a+2^{r+1}+2^{r-1}} w_3^{2^{r+1}u-2^r})[P]$$

with

$$Sq(w_2^{a+2^{r+1}+2^{r-1}} w_3^{2^{r+1}u-2^r})$$
$$= w_3^{2^{r+1}u-2^r} w_2^{a+2^{r+1}+2^{r-1}}(1 + w_2 + w_3)^{2^{r+1}u-2^r}\left(1 + \frac{w_3}{w_2} + w_2\right)^{a+2^{r+1}+2^{r-1}}.$$

where

$$(1 + w_2 + w_3)^{2^{r+1}u-2^r} = 1 + \text{terms of degree at least } 2^{r+1},$$

and

$$a + 2^{r+1} + 2^{r-1} = 2^{r+1}(v + 1) + (2^r - 1) + 2^{r-1} =$$
$$2^{r+1}(v + 1) + 2^r + (2^{r-1} - 1),$$

with

$$\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^{r+1}(v+1)} = 1 + \text{terms of degree at least } 2^{r+1},$$

$$\left(1 + \frac{w_3}{w_2} + w_2\right)^{2^r} = 1 + \left(\frac{w_3}{w_2}\right)^{2^r} + \text{higher terms},$$

and $(1 + w_3/w_2 + w_2)^{2^{r-1}-1}$ has all terms of degree less than 2^r. Thus

$$w_2^{a+2^r+2^{r-1}} w_3^{2^{r+1}u}[P] = 0.$$

Hence, one has $\overline{w}_2^a\overline{w}_3^b[M] = w_2^a w_3^{b+1}[P] = 0$, which is a contradiction. Thus $x = 0$. ∎

Applying the corollary which analyzed the case $x = 0$, one now has:

Conclusion. There are integers $0 < r \leqslant s$ with

$$a = 2^{s+1} - 2^r - 1 \quad\text{and}\quad b = 2^r - 1.$$

PROPOSITION 5. *If M^n immerses in R^{n+3} with n odd, and is not a boundary, then there exist integers $0 < r \leqslant s$ so that M^n is cobordant to the Dold manifold*

$$P(2^r - 1, 2^{s+1} - 2) = S^{2^r-1} \times \mathbf{C}P^{2^{s+1}-2}/(-1) \times (\textit{conjugation}).$$

Proof. Recalling the facts about Dold manifolds [6], one lets

$$P(m, n) = S^m \times \mathbf{C}P^n/(-1) \times (\text{conjugation}).$$

$H^*(P(m, n); Z_2)$ is $Z_2[c, d]$ modulo the relations $c^{m+1} = d^{n+1} = 0$, where $\dim c = 1$ and $\dim d = 2$. The Stiefel-Whitney class of $P(m, n)$ is

$$(1 + c)^m(1 + c + d)^{n+1}.$$

Taking $m = 2^r - 1$, $n = 2^{s+1} - 2$, with $0 < r \leqslant s$, one has

$$w \cdot (1 + c)(1 + c + d) = (1 + c)^{2^r}(1 + c + d)^{2^{s+1}} = 1$$

so

$$\overline{w} = (1 + c)(1 + c + d) = 1 + (d + c^2) + cd,$$

and the largest power of $\overline{w}_3$ which is nonzero is $\overline{w}_3^{2^r-1} = c^{2^r-1}d^{2^r-1}$. Multiplying by

$$\overline{w}_2^{2^{s+1}-2^r-1} = d^{2^{s+1}-2^r-1} + \text{terms divisible by } c,$$

one has

$$\overline{w}_2^{2^{s+1}-2^r-1}\overline{w}_3^{2^r-1}[P(2^r - 1, 2^{s+1} - 2)] \neq 0,$$

and this is the nonzero number having the largest power of $\overline{w}_3$.

Now, let M^n immerse in R^{n+3} with n odd and M not a boundary, and choose a, b so that $\overline{w}_2^a\overline{w}_3^b[M] \neq 0$, with b maximal. From the conclusion of the analysis, there are integers $0 < r \leqslant s$ with $a = 2^{s+1} - 2^r - 1$ and $b = 2^r - 1$, so that $n = 2a + 3b = 2^{s+2} + 2^r - 5$.

Consider the manifold $M' = M \cup P(2^r - 1, 2^{s+1} - 2)$. This is an odd-dimensional manifold with all numbers divisible by $\overline{w}_i (i > 3)$ zero, which was the only fact used in the analysis (note that P exists as a Poincaré algebra with $w(P) = 1$). If M' is not a boundary, there are integers a', b' with

$$\overline{w}_2^{a'}\overline{w}_3^{b'}[M'] \neq 0$$

and so that

$$\overline{w}_2^{c}\overline{w}_3^{d}[M'] \neq 0 \quad \text{only for } d \leqslant b'.$$

Since $\overline{w}_2^{c}\overline{w}_3^{d}[M'] = 0$ for $d \geqslant b$, one has $b' < b$. Now for some $0 < r' \leqslant s'$,

$$a' = 2^{s'+1} - 2^{r'} - 1 \quad \text{and} \quad b' = 2^{r'} - 1,$$

so $n = 2^{s'+2} + 2^{r'} - 5$. Since $n + 5 = 2^{s+2} + 2^r = 2^{s'+2} + 2^{r'}$ give two 2-adic expansions of $n + 5$, one must have $r = r'$, so $b = 2^r - 1 = 2^{r'} - 1 = b'$. Thus, M' must be a boundary, and M is cobordant to $P(2^r - 1, 2^{s+1} - 2)$. ■

Note. The Dold manifold $P(m, n)$ is cobordant to $S^m \times RP^n \times RP^n/(-1) \times$ twist. For $m = 1$ with $n = 2$ or 6 ($r = 1$ and $s = 1$ or 2) and for $m = 3$ with $n = 6$ ($r = 2$ and $s = 2$), these manifolds actually immerse in codimension 3, having dimensions 5, 13, and 15. To illustrate, $S^3 \times RP^6 \times RP^6/\sim$ immerses in $S^3 \times R^7 \times R^7/\sim$ which is the total space of the bundle $7\lambda + 7$ over RP^3. This total space imbeds in that of $8\lambda + 7 = 15$-dimensional trivial bundle, and $RP^3 \times R^{15}$ imbeds in R^{18} by imbedding RP^3 with trivial normal bundle. The composite immerses $S^3 \times RP^6 \times RP^6/\sim$ in R^{18}.

Note. The Dold manifold $P(m, n)$ is indecomposable in the unoriented cobordism ring if and only if

$$\binom{m + n - 1}{n}(n + 1)$$

is odd. In particular,

$$\binom{2^r - 1 + 2^{s+1} - 2 - 1}{2^{s+1} - 2} = \binom{2^{s+1} + 2^r - 4}{2^{s+1} - 2}$$

is always even if $r > 1$ and the manifolds $P(2^r - 1, 2^{s+1} - 2)$ with $1 < r \leqslant s$ are decomposable. They are not, however, decomposable in terms of classes of smaller filtration in the sense of Liulevicius [11], since any odd-dimensional manifold of lesser filtration is a boundary. This gives additional examples for the fact that this is not a nice filtration multiplicatively (i.e., a sum of monomials in generators has smaller filtration than any individual monomial).

Note. Kikuchi [10] actually contains more than Proposition 4. He shows that if M^n immerses in R^{n+3} with n odd and $\bar{w}_2^j\bar{w}_3^k[M^n] \neq 0$, then $j \equiv 1$ (mod 4) and $k = 1$ *or* $j \equiv k \equiv 3$ (mod 4). This is exactly the sort of phenomenon exploited to completely determine a and b, where b is the maximal such k. He also shows that for $n \equiv 1$ mod 8, M bounds.

5. The Case $k = 3$, n even

PROPOSITION 6. *If M^n is oriented, immerses in R^{n+3}, and n is even and larger than* 4, *then M^n is an unoriented boundary.*

Proof. If $\bar{w}_2^a\bar{w}_3^b[M^n] \neq 0$, $2a + 3b = n$, then b must be even. Now,

$$\begin{aligned}\bar{w}_2^{2p+1}\bar{w}_3^{2q}[M^n] &= \bar{w}_2(\bar{w}_2^{2p}\bar{w}_3^{2q}[M^n],\\ &= Sq^2(\bar{w}_2^{2p}\bar{w}_3^{2p})[P]\\ &= \{Sq^1(\bar{w}_2^p\bar{w}_3^q)\}^2[M^n],\\ &= Sq^1\{(\bar{w}_2^p \cdot \bar{w}_3^q) \cdot Sq^1(\bar{w}_2^p\bar{w}_3^q)\}[M^n],\\ &= 0\end{aligned}$$

since Sq^1 into the top dimension of M is zero. Also, if $q > 0$,

$$\begin{aligned}\bar{w}_2^{2p}\bar{w}_3^{2q}[M^n] &= \bar{w}_3 \cdot \bar{w}_3\{\bar{w}_2^{2p}\bar{w}_3^{2q-2}\}[M^n],\\ &= Sq^2Sq^1\{\bar{w}_3 \cdot (\bar{w}_2^p\bar{w}_3^{q-1})^2\}[M^n],\\ &= 0\end{aligned}$$

for $Sq^1\bar{w}^3 = \bar{w}_3\bar{w}_1 = 0$ and $Sq^1\{(\bar{w}_2^p\bar{w}_3^{q-1})^2\} = 0$. *Note.* $\chi(Sq^3) = Sq^2Sq_1$. Thus, the only possibly nonzero Stiefel-Whitney number of M^n is $\bar{w}_2^{n/2}[M^n]$. If M^n is not an unoriented boundary, then exactly as in the case $k = 2$, M^n must be cobordant to $CP^{2^{s+1}-2}$ for some $s \geqslant 0$, and in particular, $n = 2(2^{s+1} - 2) \equiv 0$ mod 4.

Letting $n = 4m$, one has $\bar{\mathscr{P}}_1$ reducing to $\bar{w}_2^2$, where $\bar{\mathscr{P}} = 1 + \bar{\mathscr{P}}_1$ is the Pontrjagin class of the normal bundle ν^3, and hence $\bar{w}_2^{n/2}[M^n] = \bar{w}_2^{2m}[M^n]$ is the mod 2 reduction of the integer $\bar{\mathscr{P}}_1^m[M^n]$.

The argument now proceeds following the lines of Atiyah-Hirzebruch [2], as modified for immersions by Sanderson and Schwarzenberger [13], Theorem 2(b). One takes $X = M$ which has dimension $4m$, and M immerses in R^{n+3}, hence also in R^{n+5} so that $X \subseteq 4m + 2k + 1$ with $k = 2$. Also $4m = 2(2^{2+1} - 2)$ so

$$2m + k = 2^{s+1} - 2 + 2 = 2^{s+1}$$

which is divisible by 4 if $n > 4$. Taking z to be the Chern character of the trivial quaternionic line bundle over X (i.e., $z = 2$), one has

$$2^{2m+2-1}\,\hat{A}(X, 0, z^{(1/2)}) \in Z.$$

With the given choice of z, $\hat{A}(X, 0, z^{(1/2)}) = 2\hat{A}(X)$, and so $2^{2m+2}\hat{A}(M)$ is integral.

Now $2^{4m}\hat{A}_m = A_m$ and $A_m = 2^{2m}\overline{\mathcal{P}}_1^m/(2m+1)!$ from [2, §5], so

$$2^{2m+2}\hat{A}(M) = \frac{2^{2m+2}A(M)}{2^{4m}} = \frac{A(M)}{2^{2m-2}} = \frac{2^{2m}\overline{\mathcal{P}}_1^m[M^n]}{2^{2m-2}(2m+1)!} = \frac{4\overline{\mathcal{P}}_1^m[M^n]}{(2m+1)!}$$

Since $n > 4$, $2m + 1 \geqslant 5$ so that $(2m+1)!$ is divisible by $5!$ and hence by 8. Hence $\overline{\mathcal{P}}_1^m[M^n]$ is even and M^n is an unoriented boundary. ■

COROLLARY. *If M^n immerses in R^{n+3} and all Stiefel-Whitney numbers divisible by w_1 are zero, then M bounds or M^n is cobordant to $P(2^r - 1, 2^{s+1} - 2)$ for some $0 \leqslant r \leqslant s$.*

Proof. For n odd, this is Proposition 5. For n even, the above argument started by proving M bounds or M is cobordant to

$$CP^{2^{s+1}-2} = P(2^0 - 1, 2^{s+1} - 2). \qquad ■$$

PROPOSITION 7. *If M^n is a nonbounding manifold of even dimension which immerses in R^{n+3}, then $n = (2^{p+1} - 2) + (2^{q+1} - 2) + (2^{r+1} - 2)$ with $0 \leqslant p \leqslant q \leqslant r$.*

Proof. For a nonzero characteristic number $\overline{w}_1^a\overline{w}_2^b\overline{w}_3^c[M^n] \neq 0$, one has $a + 2b + 3c = n$, so $a + c \equiv 0 \bmod 2$.

With $a \equiv c \equiv 0 \bmod 2$, one has

$$\overline{w}_1^{2p}\overline{w}_2^{2q}\overline{w}_3^{2r}[M^n] = v_{n/2} \cdot \overline{w}_1^p\overline{w}_2^q\overline{w}_3^r[M^n],$$

or

$$\begin{aligned}\overline{w}_1^{2p}\overline{w}_2^{2q+1}\overline{w}_3^{2r}[M^n] &= Sq^2(\overline{w}_1^{2p}\overline{w}_2^{2q}\overline{w}_3^{2r})[M^n] \\ &= v_{n/2} \cdot (Sq^1(\overline{w}_1^p\overline{w}_2^q\overline{w}_3^r))[M^n].\end{aligned}$$

With $a \equiv c \equiv 1 \bmod 2$, one has

$$\begin{aligned}\overline{w}_1^{2p+1}\overline{w}_2^{2q}\overline{w}_3^{2r+1}[M^n] &= Sq^2Sq^1(\overline{w}_1^{2p+1}\overline{w}_2^{2q}\overline{w}_3^{2r})[M^n] \\ &= Sq^2(\overline{w}_1^{2p+2}\overline{w}_2^{2q}\overline{w}_3^{2r})[M^n] \\ &= v_{n/2}\{Sq^1(\overline{w}_1^{p+1}\overline{w}_2^q\overline{w}_3^r)\}[M^n],\end{aligned}$$

or

$$\begin{aligned}&\overline{w}_1^{2p+1}\overline{w}_2^{2q+1}\overline{w}_3^{2r+1}[M^n] \\ &\quad = (\overline{w}_1\overline{w}_2)\{\overline{w}_3 \cdot (\overline{w}_1^{2p}\overline{w}_2^{2q}\overline{w}_3^{2r})\}[M^n] \\ &\quad = (\overline{w}_3 + Sq^1\overline{w}_2)\{\overline{w}_3(\overline{w}_1^{2p}\overline{w}_2^{2q}\overline{w}_3^{2r})\}[M^n]\end{aligned}$$

$$= \overline{w}_3^2 \cdot (\overline{w}_1^{2p}\overline{w}_2^{2q}\overline{w}_3^{2r})[M^n] + Sq^2Sq^1\{(Sq^1w_2) \cdot (\overline{w}_1^{2p}\overline{w}_2^{2q}\overline{w}_3^{2r})\}[M^n]$$
$$= v_{n/2} \cdot (\overline{w}_1^{p}\overline{w}_2^{q}\overline{w}_3^{r+1})[M^n]$$

since the other term is zero.

Since some characteristic number having one of these forms is nonzero, one must have $v_{n/2} \neq 0$. Using the splitting principle to write $w = (1 + x)(1 + y)(1 + z)$, one has

$$v = (1 + x + x^3 + \cdots + x^{2^s-1} + \cdots)(1 + y + y^3 + \cdots)(1 + z + z^3 + \cdots)$$

and so $v_i \neq 0$ only for $i = (2^p - 1) + (2^q - 1) + (2^r - 1)$, giving the desired form for $n/2$. ■

LEMMA. *Let M^n be a nonbounding even-dimensional manifold which immerses in R^{n+3}. There are integers a, b, c with $\overline{w}_1^a\overline{w}_2^b\overline{w}_3^c[M^n] \neq 0$, so that if*

$$\overline{w}_1^{a'}\overline{w}_2^{b'}\overline{w}_3^{c'}[M^n] \neq 0$$

then $a' \leqslant a$, and further if $a' = a$, then $c' \leqslant c$. Then a is even and $c = 0$, $b = 2^{s+1} - 2$ for some $s \geqslant 0$.

Proof. That a, b, c exist follows by taking the nonzero monomial which is largest in lexicographic order. Now let $Q^{n-a} \subset M^n$ be the submanifold of M dual to a copies of the line bundle $\det(\nu(M))$, i.e., dual to $\overline{w}_1^a$, so that

$$i^*(x)[Q^{n-a}] = x \cdot \overline{w}_1^a[M^n]$$

where i is the inclusion and

$$w(Q) = \frac{i^*\mathrm{w}(M)}{(1 + i^*\overline{w}_1)^a} \quad \text{or} \quad \overline{w}(Q) = i^*\{(1 + \overline{w}_1 + \overline{w}_2 + \overline{w}_3)(1 + \overline{w}_1)^a\}.$$

Now, since any number of M involving $\overline{w}_1^{a+1}$ is zero, any monomial

$$i^*(\overline{w}_1)^x i^*(\overline{w}_2)^y i^*(\overline{w}_3)^z[Q]$$

with $x > 0$ is zero. In characteristic numbers, $i^*(\overline{w}_1)$ is then zero, and $\overline{w}(Q)$ behaves as if it were $1 + i^*(\overline{w}_2) + i^*(\overline{w}_3)$. Q is then a manifold with $\overline{w}_j$ giving zero numbers for $j > 3$ or $j = 1$ and

$$\overline{w}_2^x\overline{w}_3^y[Q^{n-a}] = (i^*\overline{w}_2)(i^*\overline{w}_3)^y[Q] = \overline{w}_1^a\overline{w}_2^x\overline{w}_3^y[M^n].$$

If a is even, one then has Q^{n-a} cobordant to $CP^{2^{s+1}-2}$ for some s, so that the only nonzero number of Q is $w_2^{2^{s+1}-2}[Q^{n-a}]$. Thus, for a even, $c = 0$ and $b = 2^{s+1} - 2$ for some $s \geqslant 0$. If a is odd, one has Q^{n-a} cobordant to $P(2^u - 1, 2^{s+1} - 2)$ for some $0 < u \leqslant s$, and then $c = 2^u - 1$, $b = 2^{s+1} - 2^u - 1$.

Now consider the case $a = 2p + 1$, $b = 2^{s+1} - 2^u - 1$, $c = 2^u - 1$, with $p \geqslant 0$, $0 < u \leqslant s$.

Claim. $\bar{w}_1^{2p}\bar{w}_3^{2^u}\bar{w}_1^x\bar{w}_2^y\bar{w}_3^z[M^n] \neq 0$ implies $x = z = 0$.

To see this, $x = 0$ by the choice of a and c. Since n is even, z must then be even. For y odd, let $z = 2z'$, $y = 2y' + 1$, and then

$$\begin{aligned}
&\bar{w}_1^{2p}\bar{w}_3^{2^u+2z'}\bar{w}_2^{2y'+1}[M] \\
&\quad = Sq^2(\bar{w}_1^{2p}\bar{w}_3^{2^u+2z'}\bar{w}_2^{2y'})[M], \\
&\quad = \{Sq^1(\bar{w}_1^{p}\bar{w}_3^{2^{u-1}+z'}\bar{w}_2^{y'})\}^2[M], \\
&\quad = \{y'\bar{w}_1^{p}\bar{w}_3^{2^{u-1}+z'+1}\bar{w}_2^{y'-1} + (p + 2^{u-1} + z' + y')\bar{w}_1^{p+1}\bar{w}_3^{2^{u-1}+z'}\bar{w}_2^{y'}\}[M], \\
&\quad = \{y'\bar{w}_1^{2p}\bar{w}_3^{2^u+z+2}\bar{w}_2^{2y'-2} + (p + 2^{u-1} + z' + y')\bar{w}_1^{2p+2}\bar{w}_3^{2^u+z}\bar{w}_2^{2y'}\}[M] \\
&\quad = y'\bar{w}_1^{2p}\bar{w}_3^{2^u}\bar{w}_2^{q-1}\bar{w}_3^{z+2}[M]
\end{aligned}$$

reduces to the case y even. Now, for $z > 0$, z is even and 2^u is even so

$$\bar{w}_1^{2p}\bar{w}_3^{2^u+z}\bar{w}_2^{2q}[M] = \bar{w}_1^{2p+1}\bar{w}_2^{2q+1}\bar{w}_3^{2^u+z-1}[M]$$

as in the last calculation for Proposition 7, and this last is zero by the choice of a and c.

Claim. $v_{2^{u+1}+2^u-2} = \bar{w}_1\bar{w}_3^{2^u-1} + \bar{w}_1^{2^u}\bar{s}_{2^u-1,2^u-1}$.

To see this, $2^{u+1} + 2^u - 2 + 3 = 2^{u+1} + 2^u + 1$ and since $r > 0$,

$$2^{u+1} + 2^u - 2 = (2^{u+1} - 1) + (2^u - 1) + (2^0 - 1)$$

is the unique expression as the sum of three integers of the form $2^i - 1$. Then by the splitting principle

$$\begin{aligned}
v_{2^{u+1}+2^u-2} &= \sum x^{2^{u+1}-1}y^{2^u-1} \\
&= \left(\sum x^{2^u-1}y^{2^u-1}\right)\left(\sum x^{2^u}\right) + x^{2^u}y^{2^u-1}z^{2^u-1} \\
&= \bar{s}_{2^u-1,2^u-1}\cdot\bar{w}_1^{2^u} + \bar{w}_1\bar{w}_3^{2^u-1},
\end{aligned}$$

with the Σ notation indicating the sum of the distinct monomials of the given form by permuting x, y, and z.

Now,

$$\begin{aligned}
\bar{w}_1^a\bar{w}_2^b\bar{w}_3^c[M^n] &= \bar{w}_1^{2p+1}\bar{w}_2^{2^{s+1}-2^{u+1}+2^u-1}\bar{w}_3^{2^u-1}[M^n], \\
&= (v_{2^{u+1}+2^u-2} + \bar{w}_1^{2^u}\bar{s}_{2^u-1,2^u-1})\bar{w}_1^{2p}\bar{w}_2^{2^{s+1}-2^{u+1}+2^u-1}[M^n], \\
&= Sq^{2^{u+1}+2^u-2}(\bar{w}_1^{2p}\bar{w}_2^{2^{s+1}-2^{u+1}+2^u-1})[M^n]
\end{aligned}$$

since $\bar{w}_1^{2p+2^u}$ is zero in numbers. Applying any Sq^i to $\bar{w}_1^{2p}$ will multiply by a power of $\bar{w}_1^2$ which gives zero in numbers if $i > 0$ and $Sq^i(\bar{w}_2^{2^{s+1}-2^{u+1}}) =$

0 for $i \not\equiv 0\ (2^{u+1})$, so

$$\begin{aligned}
&\overline{w}_1^a\overline{w}_2^b\overline{w}_3^c[M^n] \\
&\quad = \overline{w}_1^{2p}Sq^{2^{u+1}+2^u-2}(\overline{w}_2^{2^{s+1}-2^{u+1}+2^u-1})[M^n] \\
&\quad = \overline{w}_1^{2p}\{Sq^{2^{u+1}}(\overline{w}_2^{2^{s+1}-2^{u+1}})Sq^{2^u-2}\overline{w}_2^{2^u-1} + \overline{w}_2^{2^{s+1}-2^{u+1}}Sq^{2^{u+1}+2^u-2}\overline{w}_2^{2^u-1}\}[M^n],
\end{aligned}$$

with the second summand being zero since $\dim \overline{w}_2^{2^u-1} = 2^{u+1} - 2 < 2^{u+1} + 2^u - 2$ and, of course, the first summand is only present for $s > u$, and, when $s > u$,

$$\begin{aligned}
&\overline{w}_1^{2p}Sq^{2^{u+1}}(\overline{w}_1^{2^{s+1}-2^{u+1}})Sq^{2^u-2}\overline{w}_2^{2^u-1}[M^n] \\
&\qquad\qquad = \overline{w}_1^{2p}(\overline{w}_3 + \overline{w}_2\overline{w}_1)^{2^{u+1}}\overline{w}_2^{2^{s+1}-2^{u+2}}Sq^{2^u-2}\overline{w}_2^{2^u-1}[M^n],
\end{aligned}$$

which is zero since $\overline{w}_1^{2p+2^{u+1}}$ and $\overline{w}_1^{2p}\overline{w}_3^{2^{u+1}}$ always give zero in characteristic numbers.

Thus, a must be even. ■

Observation. With this lemma one has a reduction of the problem to very specific terms. For an integer n of the form $(2^{p+1} - 2) + (2^{q+1} - 2) + (2^{r+1} - 2)$, $0 \leqslant p \leqslant q \leqslant r$, one considers the set Σ_n of integers s with $0 \leqslant 2^{s+2} - 4 \leqslant n$ and lets $a = n - (2^{s+2} - 4)$.

Let $S_n \subset \Sigma_n$ be the set of s for which there is a manifold M_s^n having the following properties:

(a) All Stiefel-Whitney numbers of M_s^n divisible by the classes $\overline{w}_i$, $i > 3$, and $\overline{w}_1^{a+1}$ are zero.

(b) $\overline{w}_1^a\overline{w}_2^{2^{s+1}-2}[M_s^n] \neq 0$.

Let $I_n \subset S_n$ be the subset of those s for which there is a manifold M_s^n which also immerses in R^{n+3}.

Claim. If M^n is a nonbounding manifold of even dimension which immerses in R^{n+3} (or has all numbers involving $\overline{w}_i$, $i > 3$, zero), then M^n is cobordant to a sum of manifolds M_s^n for $s \in I_n$ (respectively $s \in \Sigma_n$).

To see this, one may induct on the integer a for which $\overline{w}_1^{a+1}$ is zero in numbers of M^n but $\overline{w}_1^a$ is not, assuming that any M' with smaller a' is cobordant to a sum of $M_{s'}^n$ with $s' \in I_n$ (or Σ_n) with $n - (2^{s'+2} - 4) \leqslant a'$. Then by the lemma, there is an s with $a = n - (2^{s+2} - 4)$, so that $\overline{w}_1^a\overline{w}_2^{2^{s+1}-2}[M^n] \neq 0$, and $s \in I_n$ (or Σ_n), and $M' = M^n \cup M_s^n$ has $a' < a$. Letting M' be cobordant to $M_{s_1}^n \cup \cdots \cup M_{s_q}^n$, one has M cobordant to $M_s^n \cup M_{s_1}^n \cup \cdots \cup M_{s_q}^n$.

Thus any choice of manifolds M_s^n with $s \in I_n$ or Σ_n provides a base in the unoriented cobordism ring for the set of classes one is seeking to determine.

LEMMA. *If M^n is an even-dimensional manifold immersing in R^{n+3} and $\overline{w}_1^n[M^n] \neq 0$, then n has the form $2^{t+1} - 2$ or $2^{t+2} - 4$ for some integer $t \geqslant 0$.*

Proof. Let χ be the canonical anti-automorphism of the Steenrod algebra, so that $\chi(Sq) = Sq^{-1}$ with degree i term $\chi(Sq^i)$. Then for dim $x = 1$, one has

$$\chi(Sq^i)x^j = \binom{2i+j}{i+j} x^{i+j}.$$

Now,

$$\begin{aligned}\binom{n+i}{n}\overline{w}_1^n[M^n] &= \binom{2i+(n-i)}{i+(n-i)} \overline{w}_1^{i+(n-i)}[M^n] \\ &= \chi(Sq^i)\overline{w}_1^{n-i}[M^n] \\ &= \overline{w}_i\overline{w}_1^{n-1}[M^n]\end{aligned}$$

by [3], and thus

$$\binom{n+i}{n} = \binom{n+i}{i} \equiv 0 \bmod 2 \quad \text{for } i > 3, \text{ with } 0 < i < n.$$

Now,

$$\sum_{i\geq 0}\binom{n+i}{i} x^i = 1/(1+x)^{n+1}$$

and, for $2^t \geq n+1 > 2^{t-1}$, this is

$$(1+x)^{2^t-(n+1)}/(1+x)^{2^t} = (1+x)^{2^t-(n+1)} \quad \text{mod terms } x^{2^t},$$

so $2^t - (n+1) \leq 3$, since the top degree term is $x^{2^t(n+1)}$ with $2^t - (n+1) < 2^{t-1} \leq n$. Thus $2^t - 4 \leq n \leq 2^t - 1$, and n being even gives the result. ■

LEMMA. (a) *$RP^{2^{t+2}-4}$ has all numbers divisible by $\overline{w}_i$ zero for $i > 3$ and satisfies $\overline{w}_1^{2^{t+2}-4}[RP^{2^{t+2}-4}] \neq 0$.*

(b) *For $M^n = RP^{2^{p+1}-2} \times RP^{2^{q+1}-2} \times RP^{2^{r+1}-2}$, $0 \leq p \leq q \leq r$,*

$$\overline{w}_1^{2^{r+1}-2^{q+1}+2^{p+1}-2}\overline{w}_2^{2^{q+1}-2}[M^n] \neq 0$$

and is the largest monomial which is nonzero.

Note. $RP^{2^{t+2}-4}$ *and* $RP^{2^{t+1}-2}$ $(0 = p = q \leq r = t)$ have $\overline{w}_1^n[M^n] \neq 0$ and provide examples with $\overline{w}_1^a\overline{w}_2^{2^{s+1}-2}[M^n]$ as largest nonzero monomial in the cases $s = 0$ completely determining when $0 \in \Sigma_n$. For case (b), one has examples having largest nonzero monomial $\overline{w}_1^a\overline{w}_2^{2^{s+1}-2}[M^n]$ with $s = q$, i.e., $q \in \Sigma_n$.

Proof. From $w(RP^{2^{t+2}-4}) = (1+\alpha)^{2^{t+2}-3}$, $\overline{w}(RP^{2^{t+2}-4}) = (1+\alpha)^3 = 1 + \alpha + \alpha^2 + \alpha^3$, so $\overline{w}_i = 0$ for $i > 3$. Since $\overline{w}_1 = \alpha$, $\overline{w}_1^{2^{t+2}-4}[RP^{2^{t+2}-4}] \neq 0$.

For $M^n = RP^{2^{p+1}-2} \times RP^{2^{q+1}-2} \times RP^{2^{r+1}-2}, \overline{w} = (1+\alpha)(1+\beta)(1+\gamma)$, where α, β, γ belong to corresponding factors p, q, and r. Now $\overline{w}_1 = \alpha + \beta + \gamma$ so that $\overline{w}_1^{2^t} = \gamma^{2^t}$ if $r \geq t \geq q+1$, and is zero for $t > r$. In particular, $\overline{w}_1^{2^{r+1}-2^{q+1}} = \gamma^{2^{r+1}-2^{q+1}}$ and taking $Q \subset M$ dual to $(2^{r+1} - 2^{q+1})\det(\nu)$, or, equivalently, to $\gamma^{2^{r+1}-2^{q+1}}$, one obtains the manifold

$$Q = RP^{2^{p+1}-2} \times RP^{2^{q+1}-2} \times RP^{2^{q+1}-2}$$

which is cobordant to $Q' = RP^{2^{p+1}-2} \times CP^{2^{q+1}-2}$, which has

$$\overline{w} = (1+\alpha)(1+\theta) = 1 + \alpha + \theta + \alpha\theta, \dim \theta = 2.$$

Then,

$$\overline{w}_1^{2^{r+1}-2^{q+1}+x}\overline{w}_2^y[M] = \overline{w}_1^x\overline{w}_2^y[Q] = \overline{w}_1^x\overline{w}_2^y[Q'] = \alpha^x\theta^y[Q']$$

and is nonzero for $x = 2^{p+1} - 2$, $y = 2^{q+1} - 2$ and is zero otherwise. ■

LEMMA. *Let M^n, n even, be a manifold with all numbers divisible by $\overline{w}_i$, $i > 3$, equal to zero. Consider the characteristic number $\overline{w}_1^a\overline{w}^{b_2}[M^n]$ with a and b even and with a having a gap at 2^t in its dyadic expansion, i.e.,*

$$a = a' + 2^{t+1} + a''$$

with $a' \equiv 0\ (2^{t+2})$, $a'' < 2^t$. Equivalently,

$$\binom{a-2^t}{2^t} \equiv 1 \mod 2.$$

Then, $\overline{w}_1^a\overline{w}_2^b[M^n] = 0$ provided that

(1) $2^t > a'' + 2b + 3$,

or

(2) $2^t > a'' + 2b$ *and* $t > 1$.

Proof. Let P^{n+3} be the manifold associated to M^n and consider

$$0 = Sq^{2^t}(\overline{w}_1^{a-2^t}\overline{w}_2^b\overline{w}_3)[P].$$

One has

$$Sq^{2^t}(\overline{w}_1^{a'+2^t+a''}\overline{w}_2^b\overline{w}_3) = \overline{w}_1^a\overline{w}_2^b\overline{w}_3 + \overline{w}_1^{a'+2^t}Sq^{2^t}(\overline{w}_1^{a''}\overline{w}_2^b\overline{w}_3)$$

since $Sq^i(\overline{w}_1^{a'+2^t}) = 0$ for $i \not\equiv 0 \mod 2^t$. If $2^t > a'' + 2b + 3 = \dim(\overline{w}_1^{a''}\overline{w}_2^b\overline{w}_3)$ then

$$\overline{w}_1^a\overline{w}_2^b[M] = \overline{w}_1^a\overline{w}_2^b\overline{w}_3[P] = 0.$$

If $t > 1$, and $2^t > a'' + 2b$, then

$$\begin{aligned} Sq^{2^t}(\overline{w}_1^{a''}\overline{w}_2^b\overline{w}_3) &= Sq^{2^t}(\overline{w}_1^{a''}\overline{w}_2^b)\overline{w}_3 + Sq^{2^t-2}(\overline{w}_1^{a''}\overline{w}_2^b)Sq^2\overline{w}_3 \\ &= Sq^{2^t-2}(\overline{w}_1^{a''}\overline{w}_2^b)\overline{w}_2\overline{w}_3 \end{aligned}$$

for $a'' \equiv b' \equiv 0 \bmod 2$ gives $Sq^i(\overline{w}_2^{a''}\overline{w}_2^{b}) = 0$ if i is odd. Then,

$$\begin{aligned}\overline{w}_1^a\overline{w}_2^b[M^n] &= \overline{w}_1^a\overline{w}_2^b\overline{w}_3[P^{n+3}] \\ &= \overline{w}_1^{a'+2^t}\overline{w}_2Sq^{2^t-2}(\overline{w}_1^{a''}\overline{w}_2^b)\overline{w}_3[P^{n+3}] \\ &= \overline{w}_1^{a'+2^t}\overline{w}_2Sq^{2^t-2}(\overline{w}_1^{a''}\overline{w}_2^b)[M^n] \\ &= Sq^2\{\overline{w}_1^{a'+2^t}Sq^{2^t-2}(\overline{w}_1^{a''}\overline{w}_2^b)\}[M^n] \\ &= \overline{w}_1^{a'+2^t}Sq^2Sq^{2^t-2}(\overline{w}_1^{a''}\overline{w}_2^b)[M^n]\end{aligned}$$

for $Sq^i(\overline{w}_1^{a'+2^t}) = 0$, unless $i \equiv 0 \bmod 2^t$. Now $Sq^2Sq^{2^t-2} = Sq^1Sq^{2^t-2}Sq^1$ and since $a'' \equiv b \equiv 0 \bmod 2$, $Sq^1(\overline{w}_1^{a''}\overline{w}_2^b) = 0$. Thus $\overline{w}_1^a\overline{w}_2^b[M^n] = 0$. ■

Note. This gives another more unpleasant proof characterizing those n with $\overline{w}_1^n[M^n] \neq 0$. Taking $b = 0$, one always has $2^t > a''$ and so $n = a$ cannot have a gap at any 2^t with $t > 1$. One should also note that condition (1) implies $2^t > 3$ and so really implies condition (2).

If $n + 6 = 2^{t+3}$, $t \geq 0$, i.e. $n + 6$ has only one power of 2 in its dyadic expansion, then

(1) $I_n = \Sigma_n = \{t\}$ for $t = 0$ or 1,
(2) $\{t\} \subset \Sigma_n \subset \{s \mid 0 < s \leq t\}$ for $t > 1$.

Proof. $2^{s+2} - 4 \leq 2^{t+3} - 6$ implies $s \leq t$, and $n + 6 = 2^{t+2} + 2^{t+1} + 2^{t+1}$ is expressed in the form $2^{p+1} + 2^{q+1} + 2^{r+1}$, $0 \leq p \leq q \leq r$, only if $r = t + 1$, $p = q = t$; hence $s = q = t$ belongs to Σ_n. For $t \geq 1$, $n = 2^{t+3} - 6$ is not of the form $2^{s+1} - 2$ or $2^{s+2} - 4$ so $0 \notin \Sigma_n$. Finally, RP^2 and $RP^2 \times RP^2 \times RP^6$ immerse in codimension 3 to give $I_n = \Sigma_n = \{t\}$ for $n = 2$ and 10. ■

Note. The test case for improving this result is $s = 1$, $t = 2$; i.e., $\overline{w}_1^{22}\overline{w}_2^2[M^{26}]$ is unsettled in dimension $n = 26$.

Suppose $n + 6 = 2^{t+2} + 2^{u+1}$ with $t \geq u \geq 0$, i.e., $n + 6$ has precisely two powers of 2 in its dyadic expansion.

(1) If $u = 0$, then $\{0, t\} \subset \Sigma_n \subset \{s \mid 0 \leq s \leq t\}$, and

 (a) if $t = 0$ or 1, then $I_n = \Sigma_n = \{s \mid 0 \leq x \leq t\}$,
 (b) if $t = 2$, then $\Sigma_{12} = \{s \mid 0 \leq s \leq 2\}$.

(2) If $t = u > 0$, then $\Sigma_n = \{t, t - 1\}$, and if $t = 1$, then $I_n = \Sigma_n$.
(3) If $t > u > 0$, then

$$\{u - 1, t\} \subset \Sigma_n \subset \{s \mid u - 1 \leq s \leq t\},$$

and if $t = 2$, $u = 1$, then

$$\Sigma_{14} = \{s \mid 0 \leq s \leq 2\}.$$

Proof. $2^{s+2} - 4 \leqslant n$ gives $2^{s+2} + 2 \leqslant 2^{t+2} + 2^{u+1}$ so $s \leqslant t$. If $u = 0$, then $n = 2^{t+2} - 4$, so $0 \in \Sigma_n$, and $n = (2^{t+1} - 2) + (2^{t+1} - 2) + (2^{0+1} - 2)$ so that $t = q \in \Sigma_n$. Since a point, RP^4 and $RP^2 \times RP^2$ immerse in codimension 3, $I_0 = \Sigma_0$ and $I_4 = \Sigma_4$. For the case $t = 2$, $\Sigma_{12} \subset \{s \mid 0 \leqslant s \leqslant 2\}$ and Table 2.3 of Liulevicius [11] shows three classes of filtration less than or equal to 3, so Σ_{12} must contain 3 elements.

If $t \geqslant u > 0$, one has

$$n + 6 = 2^{t+2} + 2^{u+1} = 2^{t+1} + 2^{t+1} + 2^{u+1} = 2^{t+2} + 2^{(u-1)+1} + 2^{(u-1)+1}$$

providing expansions $n = (2^{p+1} - 2) + (2^{q+1} - 2) + (2^{r+1} - 2)$ in which $q = t$ and $q = u - 1$. Thus $\{t, u - 1\} \subset \Sigma_n$.

For $s < u - 1$, one has

$$a = 2^{t+2} + 2^{u+1} - 2 - 2^{s+2} = 2^{t+2} + (2^u + \cdots + 2) - 2^{s+2}$$
$$\text{with } s + 2 \geqslant u.$$

Thus a has a gap at 2^{t+1} with $t + 1 > 1$, and $b = 2^{s+1} - 2$ gives

$$2^{t+1} > a'' + 2b = 2^{u+1} - 6.$$

Thus $\Sigma_n \subset \{s \mid u - 1 \leqslant s \leqslant t\}$.

If $t = u > 0$, Σ_n is completely determined, and if $t = 1$, RP^6 and $RP^2 \times RP^2 \times RP^2$ immerse in codimension 3 giving $I_6 = \Sigma_6 = \{0, 1\}$. For the case $t = 2$, $u = 1$, $\Sigma_{14} \subset \{s \mid 0 \leqslant s \leqslant 2\}$ while Table 2.3 of Liulevicius [11] shows three classes of filtration less than or equal to 3, so that Σ_{14} must contain 3 elements. ■

Note. The least unsettled case is $t = 3$, $u = 0$ in dimension 28.

Suppose $n + 6 = 2^{p+1} + 2^{q+1} + 2^{r+1}$ with $0 \leqslant p < q < r$, i.e., $n + 6$ has precisely three powers of 2 in its dyadic expansion.

(1) If $r = q + 1$, $p = 0$ or 1, then $\Sigma_n = \{q\}$, and if $p = 0$, $q = 1$, $r = 2$, then $I_8 = \Sigma_8$.

(2) If $r = q + 1$, $p \geqslant 2$, then $\{q\} \subset \Sigma_n \subset \{s \mid p - 1 \leqslant s \leqslant q\}$.

(3) If $r > q + 1$, then $\{q\} \subset \Sigma_n \subset \{s \mid q \leqslant s \leqslant r - 1\}$

Proof. $2^{s+2} - 4 \leqslant n$ gives $2^{s+2} + 2 \leqslant 2^{p+1} + 2^{q+1} + 2^{r+1}$ so $s \leqslant r - 1$, and when $r = q + 1$, $s \leqslant q$. One always has $\{q\} \subset \Sigma_n$ by considering products of projective spaces, and since $RP^2 \times RP^6$ immerses in codimension 3, $\{1\} \subset I_8$.

Considering $b = 2^{s+1} - 2$, $a = 2^{r+1} + 2^{q+1} + 2^{p+1} - 2 - 2^{s+2}$, with $s < q$, $s + 2 \leqslant q + 1$, a will have a gap at 2^r provided $r > q + 1$ or $r = q + 1$ and $s + 2 > p$. Since $r \geqslant q + 1 > q > p \geqslant 0$, $r > 1$, and

$$2^r > a'' + 2b = 2^{q+1} + 2^{p+1} - 2 - 4 \quad \text{for } r > q + 1,$$

while for $p = 0, 1$,

$$2^{q+1} + 2^{p+1} - 2 - 4 \leqslant 2^{q+1} - 2 < 2^{q+1} \quad \text{if } r = q + 1.$$

For $r = q + 1$, $p \geq 2$, and $s \leq p - 1$, a has a gap at 2^q with $q > p \geq 2$ and $2^q > a'' + 2b = 2^{p+1} - 6$.

Note. The first cases in which Σ_n is not determined is $p = 2$, $q = 3$, $r = 4$ giving $n = 50$ for the case $r = q + 1$; and for $r > q + 1$ one has $p = 0$, $q = 1$, $r = 3$ giving $n = 16$. The next case is $p = 0$, $q = 1$, $r = 4$ for which $n = 3$. For $n = 16$, the question is whether $2 \in \Sigma_{16}$, i.e., the number $\overline{w}_1^4\overline{w}_2^6[M^{16}]$. Having made several errors in trying to decide this case, I believe $2 \in \Sigma_{16}$ but would not want to swear to it.

Combining the cases in which Σ_n is completely determined one has the following result.

PROPOSITION 8. *Let M^n be a nonbounding manifold immersed in R^{n+3}.*

(1) *If $n = 3 \cdot 2^{u+1} - 6$, $u > 0$, M^n is cobordant to*

$$RP^{2^{u+1}-2} \times CP^{2^{u+1}-2} \quad or \quad RP^{2^{u+2}-2} \times CP^{2^{u-1}-2}$$

or their union,

(2) *If $n = 3 \cdot 2^{u+1} - 4$, $u > 0$, M^n is cobordant to $RP^{2^{u+2}-2} \times RP^{2^{u+1}-2}$,*

(3) *If $n = 3 \cdot 2^{u+1} - 2$, $u > 1$, M^n is cobordant to $RP^{2^{u+2}-1} \times RP^{2^{u+1}-2} \times RP^2$.*

6. Immersion of Indecomposables

DEFINITION (Liulevicius [11]). A manifold M^n has algebraic filtration k if M^n has some nonzero Stiefel-Whitney number divisible by $\overline{w}_k$ and every Stiefel-Whitney number involving a class $\overline{w}_i$ with $i > k$ is zero.

Note. This is equivalent to the assertion that the normal map

$$M^n \xrightarrow{\nu} BO$$

is cobordant to a map $M' \to BO_k$ but not to a map into BO_{k-1}.

The major result of this section is:

PROPOSITION 9′. *Let M^n be an indecomposable n manifold with $n = 2^a + b$, $0 \leq b < 2^a$. Then the algebraic filtration of M^n is at least $n - 2b - 1$. Furthermore, if n is odd, then the algebraic filtration of M^n is at least $n - 2b$.*

Note. $n - 2b = 2^{a+1} - n$.

If n is even or $n \equiv 1 \pmod 4$ this result is best possible:

Observation. If $n = 2^a + b$, $0 \leq b < 2^a$, n even, the projective space RP^n has algebraic filtration $n - 2b - 1$. If $n = 4k + 1 = 2^a + b$, $0 \leq b < 2^a$, the Dold manifold $P(1, 2k)$ has algebraic filtration $n - 2b$.

This verifies the conjecture of Liulevicius [11, §2].

Note. For $n = 11 = 8 + 3$, $n - 2b = 5$, and the indecomposable v_{11} of [11, Table 2.3] has filtration 5. Additional remarks will give information on the best possible filtration.

To verify the observation, one has $w(RP^n) = (1 + \alpha)^{n+1}$, and since $(1 + \alpha)^{2^{a+1}} = 1$,

$$\overline{w}(RP^n) = (1 + \alpha)^{2^{a+1}-n-1} = (1 + \alpha)^{2^a-b-1} = (1 + \alpha)^{n-2b-1}$$

so that $\overline{w}_i = 0$ for $i > n - 2b - 1$ and $\overline{w}_{n-2b-1}\overline{w}_1^{2b+1}[RP^n] \neq 0$. Also,

$$w(P(1, 2k)) = (1 + c)(1 + c + d)^{2k+1} \quad \text{with } (1 + c)^2 = 1 \text{ and } (1 + c + d)^{2^a} = 1,$$

since $4k < 2^{a+1}$ gives $2 \leq 2k \leq 2^a$, and so $\overline{w}(P(1, 2k)) = (1 + c)(1 + c + d)^{2^a-(2k+1)}$. Thus $\overline{w}_i = 0$ for

$$i > 2[2^a - (2k + 1)] + 1 = 2^{a+1} - 4k - 1 = 2^a - b = n - 2b.$$

Further,

$$\overline{w}_{n-2b} = cd^{(2^a-b-1)/2} \quad \text{and} \quad w_2 = d + xc^2$$

so $\overline{w}_{n-2b}\overline{w}_2^b[P(1, 2k)] \neq 0$.

The proof of the proposition will proceed in a sequence of lemmas.

LEMMA. *Let M^n be an indecomposable n manifold with $n = 2^a + b$, $0 \leq b < 2^a$. Then the algebraic filtration of M^n is at least $n - 2b - 1$.*

Proof. By the splitting principle, $\overline{w}(M^n) = \prod_{i=1}^r(1 + x_i)$ with $\dim x_i = 1$. Let $s_j = \sum_{i=1}^r x_i^j$ be the usual primitive class, so that M^n is indecomposable if and only if $s_n[M^n] \neq 0$. Recalling that

$$\chi(Sq^i)x^j = \binom{2i + j}{i + j} x^{i+j},$$

one has

$$\chi(Sq^i)s_j = \binom{2i + j}{i + j} s_{i+j}.$$

Then

$$\begin{aligned}\overline{w}_{n-2b-1}s_{2b+1}[M^n] &= \chi(Sq^{n-2b-1})s_{2b+1}[M^n] \\ &= \binom{2^{a+1}-2b-2+2b-1}{n}s_n[M^n] \\ &= \binom{2^{a+1}-1}{n}s_n[M^n],\end{aligned}$$

which is nonzero. ■

The following makes use of certain standard characteristic classes defined via the splitting principle. If

$$\overline{w} = \prod_{i=1}^{r}(1+x_i),\ \dim x_i = 1$$

then

$$s_{j_1,j_2,\ldots,j_s} = \sum x_1^{j_1}x_2^{j_2}\cdots x_s^{j_s}$$

is the symmetric function of the x_i which is the sum of all distinct monomials in the x's. In particular, s_i is the primitive class used in the previous lemma, and $\overline{w}_i = s_{1,\ldots,1}$ with i ones.

LEMMA. *If M^n is an n-dimensional manifold, then*

$$s_{j,\underbrace{1,\ldots,1}_{n-j}}[M^n] = \binom{2n-j+1}{n+1}s_n[M^n].$$

In particular, if $n = 2^a + b$, $0 \leqslant b < 2^a$.

$$s_{j,\underbrace{1,\ldots,1}_{n-j}}[M^n] = 0 \quad \text{if } j < 2b+2.$$

Proof. One has

$$\begin{aligned}s_{j-1}s_{\underbrace{1,\ldots,1}_{n-j+1}} &= \left(\sum x^{j-1}\right)\left(\sum x_1,\ldots,x_{n-j+1}\right) \\ &= \sum x_1^{j-1}x_2\cdots x_{n-j+2} + \sum x_1^{j}x_2\cdots x_{n-j+1} \\ &= s_{j-1,\underbrace{1,\ldots,1}_{n-j+1}} + s_{j,\underbrace{1,\ldots,1}_{n-j}}\end{aligned}$$

if $j-1>1$, and so

$$\begin{aligned}s_{j,\underbrace{1,\ldots,1}_{n-j}}[M^n] + s_{j-1,\underbrace{1,\ldots,1}_{n-j+1}}[M^n] &= s_{j-1}\cdot s_{\underbrace{1,\ldots,1}_{n-j+1}}[M^n] \\ &= \chi(Sq^{n-j+1})s_{j-1}[M^n] \\ &= \binom{2n-j+1}{n}s_n[M^n].\end{aligned}$$

For the case $j = n$, one has

$$s_n[M^n] = \binom{2n - n + 1}{n + 1} s_n[M^n],$$

and inducting downward on j, $2 < j \leq n$, one has

$$s_{j-1,1,\dots,1}[M^n] = \left\{ \binom{2n - j + 1}{n} + \binom{2n - j + 1}{n + 1} \right\} s_n[M^n]$$

$$= \binom{2n - (j - 1) + 1}{n + 1} s_n[M^n],$$

giving the result for $j - 1 > 1$. For the case $j = 1$, $s_{1,\dots,1}[M^n] = \overline{w}_n[M^n]$ is always zero (since M^n immerses in R^{2n-1}, for example) and

$$\binom{2n}{n + 1}$$

is even except for $n = 2^s - 1$, and then $s_n[M^n] = 0$ since every manifold is decomposable. *Note.* For $j = 0$,

$$\binom{2n + 1}{n + 1} = \binom{2n + 1}{n}$$

is also even except for $n = 2^s - 1$, if one considers that case.

For $n = 2^a + b$, with $b = 2^a - 1$, $s_n[M^n] = 0$, and if $0 \leq b < 2^a - 1$,

$$\binom{2n - j + 1}{n + j} = \binom{2^{a+1} + 2b - j + 1}{2^a + b + 1}$$

which is odd only for $2b - j + 1 < 0$. To see this, note that if $2b - j + 1 \geq 0$, 2^a occurs in the dyadic expansion of $2^a + b + 1$, so must appear in $2b - j + 1$, so $b = 2^{a-1} + b'$ with $0 \leq b' < 2^{a-1} - 1$; then

$$\binom{2^{a+1} + 2b - j + 1}{2^a + b + 1} = \binom{2^{a+1} + 2^a + 2b' - j + 1}{2^a + 2^{a-1} + b' + 1}$$

$$\equiv \binom{2^a + 2b' - j + 1}{2^{a-1} + b' + 1}$$

which has the same form. Thus $s_{j,1,\dots,1}[M^n] = 0$ if $2b - j + 1 \geq 0$. ∎

LEMMA. *If* $n = 2^a + (2j - 1)$, $0 \leq 2j - 1 < 2^a$, *then*

$$s_{2j,2j,\underbrace{1,\dots,1}_{n-4j}}[M^n] = 0$$

if M^n *is decomposable.*

Proof. By additivity of the characteristic number, one may suppose $M^n = P^p \times Q^q$ with $n = p + q$, $p < q$ ($p \leq q$ by choice, but n is odd, so

$p < q$). One has the general formula

$$s_\omega[P \times Q] = \sum_{\omega' \cup \omega'' = \omega} s_{\omega'}[P] \cdot s_{\omega''}[Q]$$

for characteristic numbers of a product, and since

$$s_{\underbrace{1,\ldots,1}_{k}}[N^k] = 0,$$

one has

$$s_{\underbrace{2j,2j,1,\ldots,1}_{n-4j}}[M^n] = s_{\underbrace{2j,1,\ldots,1}_{p-2j}}[P^p] \cdot s_{\underbrace{2j,1,\ldots,1}_{q-2j}}[Q].$$

for this to be nonzero, one must have $p \geqslant 2j$ and since $n = 2^a + (2j - 1)$, $q < 2^a$. One then has $q > 2^{a-1}$ since $q > p$, and so may write $q = 2^{a-1} + b$ with $0 \leqslant b < 2^{a-1}$. In order that $s_{2j,1,\ldots,1}[Q] \neq 0$, one must have $2j \leqslant 2b + 2$ by the previous lemma and hence $q = 2^{a-1} + b \leqslant 2^{a-1} + (j - 1)$. But then $p = n - q \geqslant 2^{a-1} + j > q$, contradicting the choice $p \leqslant q$. ■

LEMMA. *If M^n is indecomposable, $n = 2^a + (2j - 1)$, $0 \leqslant 2j - 1 < 2^a$, then*

$$s_{\underbrace{2j,2j,1,\ldots,1}_{n-4j}}[M^n] \neq 0.$$

Proof. It is sufficient to exhibit some n manifold for which the given number is nonzero, for if the number is nonzero on N^n then N^n must be indecomposable, and then the number will have the same value on M^n and N^n since $M^n - N^n$ is decomposable.

Since M^n is indecomposable, one may suppose n is not of the form $2^s - 1$. Hence one may write

$$2j - 1 = 2^r(2x + 1) - 1$$

with $x \geqslant 0$ and $2^a > 2^r(2x + 1) = 2j$, and $r > 0$.

Consider the manifold $N^n = H(2^r, 2^a + 2^{r+1}x)$ contained in $RP^{2^r} \times RP^{2^a+2^{r+1}x}$ dual to $\lambda_1 \otimes \lambda_2$, i.e., to the class $\alpha + \beta$. Then

$$w(N) = \frac{(1 + \alpha)^{2^r+1}(1 + \beta)^{2^a+2^{r+1}x+1}}{(1 + \alpha + \beta)},$$

so

$$\overline{w} = (1 + \alpha + \beta)(1 + \alpha)^{2^r-1}(1 + \beta)^{2^a-2^{r+1}x-1}$$

and one wishes to compute the number $s_{2^{r+1}x+2^r,2^{r+1}x+2^r,1\cdots1}[N^n]$ with $2^a - 2^{r+1}x - 2^r - 1$ ones.

Since $\overline{w}$ is already in the form $\Pi_1^m(1 + x_i)$ one may compute the number

$s_{2j,2j,1,\ldots,1}$ readily as the top degree term in

$$\sum_{p<q} x_p^{2j} x_q^{2j} \{(1 + x_1) \cdots (1 + x_p)^\wedge \cdots (1 + x_q)^\wedge \cdots (1 + x_m)\}$$

where a flex denotes that the given factor is omitted. One then begins by choosing x_i from the list

$$\{\alpha + \beta, \underbrace{\alpha, \cdots, \alpha}_{2^r - 1}, \underbrace{\beta, \cdots, \beta}_{2^a - 2^{r+1}x - 1}\}$$

giving the possible pairs $\{\alpha, \alpha\}$, $\{\alpha, \alpha + \beta\}$, $\{\alpha, \beta\}$, $\{\alpha + \beta, \beta\}$, and $\{\beta, \beta\}$.

The pair $\{\alpha, \alpha\}$ always contributes zero. For $r = 1$, only one α occurs in the list, so no such pair can be chosen. For $r > 1$,

$$\alpha^{2^{r+2}x + 2^r} \alpha^{2^{r+1}x + 2^r} = \alpha^{2^{r+2}x + 2^{r+1}} = 0.$$

For $x > 0$, $\alpha^{2^{r+1}x + 2^r} = 0$, and so the pairs $\{\alpha, \alpha + \beta\}$, $\{\alpha, \beta\}$ contribute zero if $x > 0$. For $x = 0$, the contribution is

$$(2^r - 1)\alpha^{2^r}(\alpha + \beta)^{2^r}\{(1 + \alpha)^{2^r - 2}(1 + \beta)^{2^a - 1}\}$$
$$+ (2^r - 1)(2^a - 1)\alpha^{2^r}\beta^{2^r}\{(1 + \alpha + \beta)(1 + \alpha)^{2^r - 2}(1 + \beta)^{2^a - 2}\}$$

where the integer coefficients are the number of pairs $\{\alpha, \alpha + \beta\}$ and $\{\alpha, \beta\}$, respectively. Since $\alpha^{2^r + 1} = 0$, the leading factor α^{2^r} kills all remaining α's to give

$$\alpha^{2^r} \cdot \beta^{2^r}\{(1 + \beta)^{2^a - 1}\} + \alpha^{2^r}\beta^{2^r}\{(1 + \beta)(1 + \beta)^{2^a - 2}\},$$

which is zero.

Then, the remaining terms $\{\alpha + \beta, \beta\}$ and $\{\beta, \beta\}$ give

$$(2^a - 2^{r+1}x - 1)(\alpha + \beta)^{2^{r+1}x + 2^r}\beta^{2^{r+1}x + 2^r}\{(1 + \alpha)^{2^r - 1}(1 + \beta)^{2^a - 2^{r+1}x - 2}\}$$
$$+ \binom{2^a - 2^{r+1} - 1}{2}\beta^{2^{r+1}x + 2^r}\beta^{2^{r+1}x + 2^r}\{(1 + \alpha + \beta)(1 + \alpha)^{2^r - 1}(1 + \beta)^{2^a - 2^{r+1}x -}$$

in which the coefficients are the number of pairs. Since $r > 0$, both coefficients are odd. Also, $(\alpha + \beta)^{2^{r+1}x + 2^r} = (\alpha^{2^r} + \beta^{2^r})\,\beta^{2^{r+1}x}$ since $\alpha^{2^r + 1} = 0$. The value of this characteristic class on N^n is obtained by multiplying by $(\alpha + \beta)$ and evaluating on $RP^{2^r} \times RP^{2^a + 2^{r+1}x}$; i.e., $s_{2j,2j,1,\ldots,1}[N^n]$ is the coefficient of $\alpha^{2^r}\beta^{2^a + 2^{r+1}x}$ in

$$(\alpha^{2^r} + \beta^{2^r})\beta^{2^{r+2}x + 2^r}[(1 + \alpha)^{2^r - 1}(1 + \beta)^{2^a - 2^{r+1}x - 2}](\alpha + \beta)$$
$$+ \beta^{2^{r+2}x + 2^{r+1}}[(1 + \alpha + \beta)(1 + \alpha)^{2^r - 1}(1 + \beta)^{2^a - 2^{r+1}x - 3}](\alpha + \beta).$$

The term involving α^{2^r} is

$$\alpha^{2^r}\beta^{2^{r+2}x + 2^r}[(1 + \alpha)^{2^r - 1}(1 + \beta)^{2^a - 2^{r+1}x - 2}](\alpha + \beta)$$
$$= \alpha^{2^r}\beta^{2^{r+2}x + 2^r + 1}(1 + \beta)^{2^a - 2^{r+1}x - 2}$$

in which β occurs only to odd powers, so contributes zero. The remainder is

$$\beta^{2^{r+2}x+2^{r+1}}(\alpha+\beta)[(1+\alpha)^{2^r-1}(1+\beta)^{2^a-2^{r+1}x-2}+(1+\alpha+\beta)(1+\alpha)^{2^r-1}(1+\beta)$$
$$=\beta^{2^{r+2}x+2^{r+1}}(\alpha+\beta)(1+\alpha)^{2^r-1}(1+\beta)^{2^a-2^{r+1}x-3}[1+\beta+1+\alpha+\beta]$$
$$=\beta^{2^{r+2}x+2^{r+1}}(\alpha^2+\alpha\beta)(1+\alpha)^{2^r-1}(1+\beta)^{2^a-2^{r+1}x-3}$$

in which the coefficient of α^{2^r} is

$$\beta^{2^{r+1}x-2^{r+1}}(1+\beta)^{2^a-2^{r+1}x-3}+\beta^{2^{r+2}x+2^{r+1}+1}(1+\beta)^{2^a-2^{r+1}x-3}$$
$$=\beta^{2^{r+2}x+2^{r+1}}(1+\beta)^{2^a-2^{r+1}x-3}(1+\beta)$$
$$=\beta^{2^{r+2}x+2^{r+1}}(1+\beta)^{2^a-2^{r+1}x-2}$$

Taking the coefficient of $\beta^{2^a+2^{r+1}x}$,

$$s_{2j,2j,1,\ldots,1}[N^n]=\binom{2^a-2^{r+1}x-2}{2^a-2^{r+1}x-2^{r+1}}=\binom{2^a-2^{r+1}x-2}{2^{r+1}-2}.$$

Now, $2^{r+1}-2=2^r+\cdots+2$ and $2^a-2^{r+1}x-2=2^{a-1}+\cdots+2-2^{r+1}x$ deleting only terms 2^b with $b\geqslant r+1$ from $2^{a-1}+\cdots+2$, so this binomial coefficient is odd. ■

Finally, to complete the proof of the proposition one has:

LEMMA. *If M^n has filtration less than $n-2b$ with $n=2^a+b$, $0\leqslant b<2^a$ and $b=2j-1$, then*

$$\underbrace{s_{2j,2j,1,\ldots,1}}_{n-4j}[M^n]=0.$$

Proof. Since M^n has filtration less than $n-2b=n-4j+2$, all Stiefel-Whitney numbers of M^n involving a class $\overline{w}_i$ with $i\geqslant n-4j+2$ are zero. Then, all classes in the ideal generated by w_i, $i\geqslant n-4j+2$ in $H^*(BO;Z_2)=Z_2[w_i]$ give zero in normal characteristic numbers of M^n. This ideal is, however, the kernel of the homomorphism

$$i^*:H^*(BO;Z_2)\to H^*(BO_{n-4j+1};Z_2)$$

induced by inclusion, and that kernel contains all $s_{j_1,j_2,\ldots,j_s}$ with $s>n-4j+1$ (by the splitting principle $w=\prod_1^{n-4j+1}(1+x_i)$ for BO_{n-4j+1}). Thus $s_{2j,2j,1,\ldots,1}$ with $n-4j$ ones gives a zero normal Stiefel-Whitney number for M^n. ■

The following provides an easy, but crude, bound on filtration.

Observation. For $n=2^r(2s+1)-1$, $r,s>0$, the Dold manifold

$$P(2^r-1,2^rs)$$

has algebraic filtration $n-2b+(2^r-2)$, where $n=2^a+b$, $0\leqslant b<2^a$; i.e., the best possible filtration for an indecomposable M^n lies between

$n - 2b$ and $n - 2b + (2^r - 2)$, where 2^r is the power of 2 dividing $n + 1$.

Proof. Let $n = 2^a + 2^{r+1}x + 2^r - 1$. For $P(2^r - 1, 2^r x + 2^{a-1})$,

$$\overline{w} = (1 + c)(1 + c + d)^{2^{a-1} - 2^r x - 1}$$

so $\overline{w}_i = 0$ for $i > 2^a - 2^{r+1}x - 1 = n - 2b + (2^r - 2)$. Finally, note that

$$\overline{w}_{2^a - 2^{r+1}x - 1}\overline{w}_2^{2^{r+1}x}\overline{w}_{2^r}[P(2^r - 1, 2^r x + 2^{a-1})] \neq 0. \qquad \blacksquare$$

Observation. For $n = 2^{s+3} + 3$, $s \geqslant 0$, the cobordism classes of

$$P(3, 4) + P(1, 2) \times RP^2 \times RP^4 + P(1, 2) \times RP^2 \times RP^2 \times RP^2 \quad (s = 0)$$

and

$$P(3, 2^{s+2}) + P(1, 2^{s+1}) \times RP^2 \times RP^{2^{s+2}} + P(1, 2^s) \times RP^2 \times RP^{2^{s+1}} \times RP^{2^{s+2}} \quad (s > 0)$$

are indecomposables with filtration $n - 2b = n - 6$.

To see this, one computes the characteristic numbers involving $\overline{w}_i$ with $i > n - 2b = 2^{s+3} - 3$. For $P(3, 2^{s+2})$, the only such numbers which are nonzero are

$$\overline{w}_{2^{s+3}-1}\overline{w}_4 = \overline{w}_{2^{s+3}-2}\overline{w}_5 = \overline{w}_{2^{s+3}-2}\overline{w}_3\overline{w}_2 \neq 0.$$

For $P(1, 2^{s+1}) \times RP^2 \times RP^{2^{s+2}}$, the only such numbers which are nonzero are

$$\overline{w}_{2^{s+3}-1}\overline{w}_4 = \overline{w}_{2^{s+3}-2}\overline{w}_4\overline{w}_1 = \overline{w}_{2^{s+3}-2}\overline{w}_3\overline{w}_2 \neq 0.$$

The third summand has nonzero numbers precisely cancelling these, but for $s = 0$, $P(1, 2^s)$ bounds and the third term must be chosen differently. Finally the classes are indecomposable, so there is a nonzero number involving $\overline{w}_{n-2b}$. $\blacksquare$

LEMMA. *Let $\Gamma(M^n) = S^1 \times M^n \times M^n/(-1) \times$ twist.*

(a) *If M^n is indecomposable, so is $\Gamma(M^n)$,*

(b) *If M^n immerses in R^{n+k}, then $\Gamma(M^n)$ immerses in $R^{2n+1+2k}$ for $n + k$ even and in $R^{2n+2+2k}$ for $n + k$ odd.*

(c) *If M^n has algebraic filtration k, then $\Gamma(M^n)$ has algebraic filtration less than or equal to $2k$ if $n + k$ is even or $2k + 1$ if $n + k$ is odd.*

Proof. According to R. L. W. Brown [5], Proposition 4.1, $\Gamma(M^n)$ is indecomposable since

$$\binom{1 + n - 1}{n} = \binom{n}{n} = 1$$

is odd and M^n is indecomposable, giving (a). If

$$M^n \xrightarrow{f} R^{n+k}$$

is an immersion,

$$\Gamma(f) : \Gamma(M^n) = S^1 \times M^n \times M^n/\sim \to S^1 \times R^{n+k} \times R^{n+k}/\sim = \Gamma(R^{n+k})$$

induced by $1 \times f \times f$ is an immersion in the total space of the bundle

$$(n + k)\lambda + (n + k)$$

over $RP^1 = S^1$. Now, $2\lambda \cong 2$, so $E(r\lambda + r)$ imbeds in R^{2r+1} for r even, or in $E((r + 1)\lambda + r)$, hence in R^{2r+2}, for r odd. *Note.* The normal bundle of this immersion is $\Gamma(\nu_f)$ for $n + k$ even or $\lambda \oplus \Gamma(\nu_f)$ for $n + k$ odd, where, for

$$E \xrightarrow{\xi} M$$

a vector bundle,

$$\Gamma(E) \xrightarrow{\Gamma(\xi)} \Gamma(M)$$

is the corresponding bundle, and λ is the line bundle over $\Gamma(M)$ pulled back from RP^1.

If M^n has algebraic filtration k with $\nu : M^n \to BO_j$, the normal bundle of an immersion, cobordant to $\xi : M' \to BO_k$, i.e., to $\xi + (j - k) : M' \to BO_j$, then the bundle $\Gamma(\nu)$ over $\Gamma(M)$ is bordant to

$$\Gamma(\xi \oplus (j - k)) = \Gamma(\xi) \oplus (j - k)\lambda \oplus (j - k)$$

over $\Gamma(M')$. (To see this, note that $E(\nu) \times E(\nu) \to M \times M$ and

$$E(\xi + (j - k)) \times E(\xi + (j - k)) \to M' \times M',$$

with each space given the twist involution, are equivariantly cobordant bundles, since they have cobordant fixed point data. Forming the product with S^1 and dividing out the involution on the cobordism gives the desired cobordism.) With no loss, one may choose j above so that $n + j$ is even, so that the normal bundle of $\Gamma(M)$ is $\Gamma(\nu)$ which is cobordant to $\Gamma(\xi) \oplus (j - k)\lambda + (j - k)$ over $\Gamma(M')$. Since $2\lambda \cong 2$, this bundle reduces to $\Gamma(\xi)$ or $\lambda \oplus \Gamma(\xi)$ as $j - k \equiv n + k \pmod 2$ is even or odd, giving fiber dimension $2k$ or $2k + 1$. ■

Observation. For $n \equiv 3 \bmod 8$ ($n \neq 3$) or for $n = 7 + 2^{s+4}$, there is an indecomposable n-dimensional manifold of algebraic filtration at most $n - 2b + 1$.

Proof. Since $(n - 1)/2$ is congruent to 1 mod 4 and is not of the form $2^s - 1$ or is $3 + 2^{s+3}$, there is an indecomposable manifold $N^{(n-1)/2}$ of algebraic filtration $2^a - (n - 1)/2$, where $2^{a-1} \leqslant (n - 1)/2 < 2^a$ and M^n

$= \Gamma(N^{(n-1)/2})$ has algebraic filtration at most $2(2^a - (n-1)/2) = 2^{a+1} + 1 - n = n - 2b + 1$.

Note. $(n-1)/2 + (2^a - (n-1)/2) = 2^a$ is even, so Γ doubles the filtration. ■

Note. One might hope that the manifold $M^n = \Gamma(N^{(n-1)/2})$ just constructed actually had algebraic filtration $n - 2b$. In fact, it has filtration precisely $n - 2b + 1$ and the hope is forlorn. In fact,

$$\overline{w}_1\overline{w}_{2i_1} \cdots \overline{w}_{2i_r}[M^n] = \overline{w}_{i_1} \cdots \overline{w}_{i_r}[N^{(n-1)/2}].$$

One may iterate this construction to obtain improved bounds for the best possible algebraic filtration of an indecomposable. For example, if $n \equiv 7 \bmod 16$ there is an indecomposable of algebraic filtration $n - 2b + 4$.

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