ILLINOIS JOURNAL OF MATHEMATICS
Volume 27, Number 1, Spring 1983

ON THE VOLUME OF TUBES

BY

ALLEN GORIN

The relationship between the volume of a tubular neighborhood of a submanifold and metric invariants of that submanifold has long been of interest. Steiner examined this problem as long ago as 1840 [12]. The problem was discussed for curves by Hoteling [8], in relation to a statistical problem which motivated H. Weyl [12] to solve the problem for any submanifold of a space of constant curvature. (Actually, he states his results only for Euclidian space and spheres.)

When the ambient manifold is flat, Weyl obtained the surprising result that the volume, as a function of the radius, is a polynomial. Secondly, the coefficients of the polynomial are products of universal constants, depending on dimensions, and intrinsic integral metric invariants of the submanifold. Specifically, these are the integrals of the p^{th} mean curvatures of the submanifold.

Many questions concerning submanifolds can be discussed by restricting one's attention to its tubular neighborhood. Weyl's result suggests restricting one's attention still further to the formula, $V(r)$, for the volume of the tube, and examining the coefficients of its power series expansion. This was done in 1848 for geodesic circles on surfaces by Bertrand and Digret [2].

When the ambient space is symmetric, N. Grossman [7] has obtained order of magnitude estimates for the growth of tubes of large radius, which depend on the rank of the ambient symmetric space. He uses this estimate along with results of Bott to gain information on the Betti numbers of path spaces from the submanifold to a point off it.

In the case of a complex submanifold of complex projective space, Flaherty [6] has given formuli analogous to Weyl's.

When a lower bound is given on the sectional curvatures of the ambient space, V. Dekster [5] has obtained comparison theorems for the volume of tubes which generalize the Rauch Comparison Theorem for the volume of spheres.

The general problem may be stated as follows: Let $M^{(m)}$ and $N^{(m+k)}$ be compact smooth Riemanian manifolds. Let $f: M \to N$ be a smooth isometric imbedding. Consider the tubular neighborhood of radius r about $f(M)$ in

Received April 2, 1981.

© 1983 by the Board of Trustees of the University of Illinois
Manufactured in the United States of America

N. This may be identified with the normal disc bundle of M induced by f. Let $V(r)$ be the volume of the tube as a function of r. This volume in general depends on M, N, r and f. The formalisms for solving this problem can be set up by considering the Jacobi fields induced in the normal bundle by variation of the normal geodesics. See Grossman [7] for an exposition of this. Little has been said about the general case, except that the first approximation to $V(r)$ is

$$V(r) = V_k r^k \operatorname{vol}(M) + O(r^{k+1}),$$

where V_k is the volume of the unit k-disc. When M is flat, Weyl's results apply, and $V(r)$ is a polynomial. The integral invariants which appear in the coefficients are of much interest. The first coefficient is a multiple of the volume, as mentioned above. The last coefficient is the integral of the Lipschitz-Killing curvature of M, and is thus a multiple of the Euler characteristic. The intermediate coefficients are the integrals of Allendoerfer's [1] p^{th} mean curvatures. Thorpe [14] discusses the relation between the constancy of these curvatures and the vanishing of certain Pontryagin classes. These integral curvatures also appear in Chern's kinematic formula [4].

For an arbitrary ambient manifold, to the author's knowledge, the integration for the volume of a tube has not yet been carried out. The purpose of this paper is to make some contributions in this direction for a more general ambient manifold.

Consider a submanifold $M^{(m)}$ of R^{m+k}, and take E to be a subbundle of its normal bundle, NM. We discuss the volume of the tubular neighborhood of M in E and prove some theorems about it. The first states that the volume is a polynomial if E is totally geodesic, and gives a non-polynomial counterexample in case it is not. The second considers E to be the first osculating space of M, and shows that if the volume agrees with the Weyl polynomial to second order, then it must in fact be that polynomial and M must lie in some lower dimensional subspace of R^{m+k}. We then consider one sided tubes, i.e., exponentiating only a quadrant of the normal bundle. The integral of the mean curvature of M appears in this volume formula. These theorems will now be stated in detail.

Let $M^{(m)}$ be a compact smooth submanifold of R^{m+k}. Choose an l-dimensional subbundle, E, of the normal bundle NM. We will discuss the tubular neighborhood of radius r of M in E. Its volume form will be constructed and given in local coordinates, which in certain situations lends itself to interesting invariant formulations. In the case $l = k$, we have $E = NM$, the full tube. This situation was examined in detail by H. Weyl, and was the occasion of the first substantial insight into the general problem.

THEOREM 1 (Weyl [16]). *The tubular neighborhood of radius r of M in R^{m+k} has volume*

$$V(r) = V_l \sum_e \frac{r^{l+e}}{(l+2)(l+4)\dots(l+e)} h_e \quad (e \text{ even}, o \leqslant e \leqslant m),$$

where V_l is the volume of the unit l-disc and the h_e are integral metric invariants intrinsic to M. Specifically, h_e is the integral of the p^{th} mean curvature of M, with h_o the volume. (When m is even, observe that h_m is a multiple of the Euler characteristic.)

We will denote this volume by $W(r)$, calling it the l-dimensional Weyl polynomial of M. It is independent of the isometric imbedding for a fixed codimension. Where the number l is clear, we will merely call it the Weyl polynomial of M.

The volume formula is not a polynomial for all subbundles E of NM. We will show:

THEOREM 2. *If E is a totally geodesic subbundle of NM, then V(r) is a polynomial. It is of the same form as the Weyl polynomial, although the coefficients are not necessarily intrinsic.*

We will then show the following theorems:

THEOREM 3. *Let E contain the image of the vector-valued second fundamental form of M. The volume of the tube of radius r about M in E is*

$$V(r) = r^l V_l \cdot \operatorname{vol}(M) + \frac{r^{l+2}}{l+2}\left[V_l \int h_2 dM + V_m^{-1} \int \left\| \Pi_{E^\perp} \nabla w^e \right\|^2 dS^{l-1}\, dS^{m-1}\, dM \right] + O(r^{l+4}),$$

where the integral is over $UTM \oplus UE$, with $e \in UE$, $w \in UTM$, and $\Pi_{E^\perp}$ denotes projection into $E^\perp$.

This formula reduces to Weyl's formula (Theorem 1) when E is the full tube, (since $E^\perp = 0$). The integral of $\|\Pi_{E^\perp} \nabla e\|^2$ is a measurement of the "twisting" of E. In particular:

COROLLARY. *Consider a non-degenerate curve γ in R^{1+k}. Let E be the subbundle of the normal bundle generated by the first normal vector. (E is also called the first osculating space of γ.) The volume of the tube about γ in E is*

$$V(r) = 2rL + (r^3/6) \int \tau^2\, ds + O(r^5),$$

where L is the length of the curve and τ is the torsion of the curve.

Next consider a generalization of "1-sided" tubes. The integral of the mean curvature appears in the volume formula.

THEOREM 4. *Let $n(1) \cdots n(l)$ be a frame field along M. Consider the region spanned by $m + \Sigma_{p=1}^{l} t_p n(p)_m$, for $t_p \geqslant o$ and $\Sigma t_p^2 \leqslant r^2$. This is a generalization of a 1-sided tubular neighborhood in case $l = 1$. The volume of the '1-sided' tube is*

$$V^+(r) = r^l 2^{-l} V_l \cdot \mathrm{vol}(m) + r^{l+1} 2^{-l} V_{l-1} \int H \cdot n\, dM + O(r^{l+2}),$$

where V_l is the volume of the unit l-disc, 2^l is the number of quadrants in R^l, $n = \Sigma_{i=1}^{l} n(i)$, and H is the mean curvature vector of M.

COROLLARY. *Let γ be a non-degenerate curve in R^{1+k} with normal vector n. Let E be the subbundle of $N\gamma$ generated by n. The volume of the 1-sided tube about γ in E is*

$$V^+(r) = rL + \tfrac{1}{2} r^2 \oint \kappa\, ds + O(r^3),$$

where κ is the curvature of the curve γ, and L is its length.

We remark that Theorems 1–4 also hold when M is a submanifold of any flat space.

The next theorem concerns nicely curved submanifolds, which are a generalization of the concept of a non-degenerate curve.

THEOREM 5. *Let $M^{(m)}$ be a nicely curved submanifold of R^{n+k}. In such a manifold, the range of the second fundamental form has constant dimension l. Denote this subbundle of the normal bundle by $E^{(l)}$. Let $V(r)$ denote the volume of the tube in E. The following are equivalent:*

(1) *M is contained in an $(m + l)$-dimensional linear subspace of R^{n+k}.*
(2) *E is a totally geodesic subbundle of NM.*
(3) *$V(r)$ is the l-dimensional Weyl polynomial of M.*
(4) *The coefficient of r^{l+2} in the power series expansion for $V(r)$ agrees with the corresponding term in the Weyl polynomial.*

Section 1 will develop the volume form for the tube, and in the process we will demonstrate Theorem 2. Section 2 will discuss the integration of this form over the tube and quadrant-tube, which will yield Theorems 3 and 4. Section 3 will prove the equivalences stated in Theorem 5.

1. The Volume Form

We are given $M^{(m)}$ a submanifold of R^{n+k} and $E^{(l)}$ a subbundle of NM. Let (U, x) be a coordinate patch on M. A coordination of the tube in E can be constructed as follows:

Choose $\{n(1) \ldots n(l)\}$ an orthonormal basis section for $E|_U$. Define, for

$t \in R^l$,

$$Y(u, t) = x(u) + \sum_{p=1}^{l} t_p n(p). \tag{1.1}$$

The tube of radius r is defined by the restriction $\Sigma t_p^2 \leq r^2$. We will give the volume form of this coordinatization. Calculations will be done locally. Since local results will be invariant, the global versions will follow immediately.

It will be convenient to use indices in the following ranges:

$$1 \leq i, j, p \leq l, \qquad 1 \leq \alpha, \beta \leq m.$$

Let $\{w_\alpha\}$ be an orthonormal basis for $TM|_U$. Let Y_α and Y_p denote differentiation of Y with respect to w_α and $\partial/\partial t_p$ respectively. It follows from 1.1 that

$$Y_p = n(p), \tag{1.2}$$

$$Y_\alpha = X_\alpha + \sum_p t_p n_\alpha(p). \tag{1.3}$$

We express $n_\alpha(p)$ as a linear combination of the orthonormal vectors x_β and $n(q)$, where $1 \leq \beta \leq m$ and $1 \leq q \leq k$. Extend the basis $\{n(p)\}_{p=1,l}$ of E to an orthonormal basis $\{n(p)\}_{p=1,k}$ of NM. We can then write

$$n_\alpha(p) = \sum_\beta G_\alpha^\beta(p) X_\beta + \sum_{q=1}^{k} L_{\alpha q}(p) n(q), \tag{1.4}$$

where $G_\alpha^\beta(p)$ are the coefficients of the second fundamental form in direction $n(p)$, and $L_{\alpha q}(p) = n_\alpha(q) \cdot n(p)$. We have G symmetric in α and β, and L skew-symmetric in p and q. Combining (1.3) and (1.4), we obtain

$$Y_\alpha = X_\alpha + \sum_\beta \left[\sum_p t_p G_\alpha^\beta(p) \right] X_\beta + \sum_{q=1}^{k} \left[\sum_p L_{\alpha q}(p) t_p \right] n(q). \tag{1.5}$$

Thus, we can write the volume form of E as

$$\det \begin{pmatrix} \delta_{\alpha\beta} + \sum_p t_p G_\alpha^\beta(p) & \sum_p t_p L_{\alpha q}(p) \\ 0 & I_l \quad 0 \end{pmatrix} dM\, dt, \tag{1.6}$$

with $1 \leq \alpha \leq m$, $1 \leq \beta \leq m$, $1 \leq q \leq l$, and $dt = dt_1 \cdots dt_l$. Note that the matrix is $m + l$ by $m + k$. The interpretation of det is to consider the row vectors as in the $m + l$ dimensional subspace which they span.

Weyl considered the case $l = k$, when E is the full normal bundle of M in R^{m+k}. In this situation, I_l fills the lower right hand block of 1.6, thus the volume form becomes, in Weyl's case,

$$\det \left[I + \sum_p t_p G_\alpha^\beta(p) \right] dM\, dt. \tag{1.7}$$

This can be evaluated, and expressed as a polynomial in the t_p. The coefficients of the polynomial are invariant polynomials of the matrices $G(p)$. The volume of the tube of radius r is found by integrating the polynomial over M and the region $\Sigma t_p^2 \leqslant r^2$. Weyl does this using classical invariant theory, and obtains the formula given in Theorem 1.

The block matrix 1.6 can be evaluated by the same technique if $L_{\alpha q}(p) = 0$ for $q \geqslant l + 1$, $p \leqslant l$ and for all α. In this case, the subspace of definition of the determinant is clear, and the volume form is again the form of 1.7. The volume will hence again be a polynomial in r. The condition $L_{\alpha q}(p) = 0$ can be restated as $n_\alpha(p) \cdot n(q) = 0$, or that $\Pi_{E^\perp}\nabla_w e = 0$ for all $w \in TM$ and $e \in E$. We call such a subbundle E totally geodesic. Recall:

DEFINITION. Let E be a subbundle of a vector bundle over M with connection ∇. Then E is a *totally geodesic* subbundle if $\nabla_w e \in E$ for all $e \in E$ and $w \in E$ TM.

Theorem 2 is now proven.

Remark. Although when E is totally geodesic, we obtain a polynomial $V(r)$, the coefficients will only be intrinsic to M if E contains the range of the second fundamental form of M. This follows from Weyl's original proof.

When E is not totally geodesic, we can still obtain an approximation to $V(r)$. We do this by using the standard definition of the volume form as $\sqrt{\det g}\, du\, dt$, where g is the metric tensor expressed in local coordinates. Since we are using an orthonormal frame as a basis for TM, rather than the basis associated to a coordinatization of M, the volume form is actually $\sqrt{\det g}\, dM\, dt$. This form will be integrated over M and over the sphere (or quadrant) of radius r in each fiber.

Referring to equations 1.2 and 1.5, and recalling that the X_i are orthonormal, we obtain the components of g:

For $1 \leqslant \alpha, \eta \leqslant m$,

$$
\begin{aligned}
g_{\alpha\eta} = Y_\alpha \cdot Y_\eta = \delta_{\alpha\eta} &+ 2 \sum_{p=1}^{l} t_p G_\alpha^\eta(p) \\
&+ \sum_\beta \left[\sum_i t_i G_\alpha^\beta(i)\right] \cdot \left[\sum_j t_j G_\eta^\beta(j)\right] \\
&+ \sum_{q=1}^{k} \left[\sum_i t_i L_{\alpha q}(i)\right] \cdot \left[\sum_j t_j L_{\eta q}(j)\right].
\end{aligned} \tag{1.8}
$$

For $1 \leqslant \alpha \leqslant m$ and $1 \leqslant a \leqslant l$,

$$g_{\alpha,(a+m)} = Y_\alpha \cdot n(a) = \sum_i t_i L_{\alpha a}(i). \tag{1.9}$$

For $1 \leqslant a, b \leqslant l$,

$$g_{(a+m),(b+m)} = \delta_{ab}. \tag{1.10}$$

Thus, the matrix g is a second degree polynomial in the variables $\{t_i\}$:

$$(1.11) \qquad g(t) = I + \sum_i A_i + \sum_{i,j} t_i t_i B_{ij},$$

where

$$(1.12) \qquad A_i = \begin{pmatrix} 2G_\alpha^\eta(i) & L_{\alpha p}(i) \\ L_{\alpha p}(i)^{\mathrm{tr}} & 0 \end{pmatrix}, \quad 1 \leqslant \alpha, \eta \leqslant m, \ 1 \leqslant p \leqslant l,$$

$$(1.13) \qquad B_{ij} \begin{pmatrix} \sum_\beta G_\alpha^\beta(i) G_\eta^\beta(j) + \sum_{q=1}^k L_{\alpha q}(i) L_{\alpha q}(j) & 0 \\ 0 & 0 \end{pmatrix}.$$

To find the volume form, we need to calculate $\sqrt{\det g(t)}$. We will first approximate $\det g(t)$ with a Taylor series, then its square root. It will not be necessary for this application to calculate the coefficients of mixed terms, $t_i t_j$, because if $i \neq j$, then $\int t_i t_j \, dS^l$ vanishes.

Let $f(t) = \det g(t)$. Note $f(0) = 1$.

LEMMA. *The first few derivatives of f, at $t = o$ $(t_1 = t_2 = \cdots = t_l = 0)$, are expressed in terms of the A_i and the B_{ij} as follows:*

(1.14) $f_i(0) = \mathrm{tr} A_i$,

(1.15) $f_{ii}(0) = 2[\mathrm{tr}\, B_{ii} + \sigma_2 A_i]$,

where tr *is the trace and σ_2 is the classical second order orthogonally invariant polynomial in the coefficients of a matrix.*

Proof. To evaluate the above derivatives it suffices to restrict f to $\{t | t_a = o \text{ for } a \neq i\}$.

$$\begin{aligned} f(t) &= \det [I + t_i A_i + t_i^2 B_{ii}] \\ &= \det [I + t_i(A_i + t_i B_{ii})] \\ &= 1 + t_i \,\mathrm{tr}(A_i + t_i B_{ii}) + t_i^2 \sigma_2(A_i + t_i B_{ii}) + O(t_i^3) \\ &= 1 + t_i \,\mathrm{tr}\, A_i + t_i^2(\mathrm{tr}\, B_{ii} + \sigma_2 A_i) + O(t_i^3), \end{aligned}$$

which proves the lemma.

LEMMA. *For $h(t) = \sqrt{f(t)} = \sqrt{\det g(t)}$,*

(1.16) $h(0) = 1$,

(1.17) $h_i(0) = \frac{1}{2} \,\mathrm{tr}\, A_i$,

(1.18) $h_{ii}(0) = \mathrm{tr}\, B_{ii} + \sigma_2 A_i - \frac{1}{4}(\mathrm{tr}\, A_i)^2$.

Proof. This follows from the previous lemma by the chain rule.

Letting η_{ij} be the coefficient of $t_i t_j$ in the Taylor series for $h(t)$, we have shown that the coefficient of $dM\, dt$ in the volume is

$$(1.19)\quad 1 + \tfrac{1}{2}\sum_i t_i \operatorname{tr} A_i + \tfrac{1}{2}\sum_i t_i^2\left[\operatorname{tr} B_{ii} + \sigma_2 A_i - \tfrac{1}{4}\operatorname{tr}^2 A_i\right] + \sum_{i\neq j} \eta_{ij} t_i t_j + O(t^3).$$

By $O(t^3)$, we mean $O(\Sigma t_i t_j t_k)$.

We will now calculate the coefficients in terms of the $G_\alpha^\beta(p)$ and the $L_{\alpha p}(q)$.

LEMMA. *The coefficients in equation* 1.19 *are given by*

$$(1.20)\quad \operatorname{tr} A_i = 2\,\Sigma_\alpha G_\alpha^\alpha(i),$$
$$(1.21)\quad \operatorname{tr} B_{ii} = \Sigma_{\alpha,\beta}\, G_\alpha^\beta(i)^2 + \Sigma_\alpha\, \Sigma_{q=1}^k\, L_{\alpha q}(i)^2,$$
$$(1.22)\quad \sigma_2 A_i = 4\sigma_2[G(i)] - \Sigma_\alpha\, \Sigma_{q=1}^l\, L_{\alpha q}(i)^2,$$
$$(1.23)\quad -\tfrac{1}{4}\operatorname{tr}^2 A_i = -\Sigma_{\alpha,\beta}\, G_\beta^\beta(i).$$

Proof. (1.20) and (1.21) follow immediately from equations (1.12) and (1.13); (1.23) follows from (1.20). To show (1.22), we must recall a few facts about the invarient function σ_2.

The first is that for an arbitrary square matrix, $M = (m_{ij})$,

$$\sigma_2 M = \sum_{i<j} m_{ii}m_{jj} - m_{ij}m_{ji}.$$

If the sum is over all i and j, a factor of $\tfrac{1}{2}$ is introduced.

When the matrix M is symmetric and has the special form

$$M = \begin{pmatrix} N & H \\ H^{\mathrm{tr}} & 0 \end{pmatrix},$$

it follows directly from the formula for σ_2 that $\sigma_2 M = \sigma_2 N - \Sigma_{i,j}\, h_{ij}^2$. Noting that A_i indeed has this special form, (see 1.12), and that $\sigma_2(2N) = 4\sigma_2 N$, (1.22) then follows. The lemma is proven.

The coefficient of t_i^2 in (1.19) is thus

$$\tfrac{1}{2}\sum_{\alpha,\beta} G_\alpha^\beta(i)^2 + \tfrac{1}{2}\sum_{q=1}^{k}\sum_\alpha L_{\alpha q}(i)^2 - \tfrac{1}{2}\sum_{\alpha,\beta} G_\alpha^\alpha(i) G_\beta^\beta(i) + 2\sigma_2[G(i)] - \tfrac{1}{2}\sum_{q=1}\sum_\alpha L_{\alpha q}(i)^2,$$

which simplified to (recalling the definition of σ_2)

$$(1.24)\quad \sigma_2[G(i)] + \tfrac{1}{2}\sum_{q>l}\sum_\alpha L_{\alpha q}(i)^2.$$

We have now shown, combining this lemma and (1.19), that the coefficient of $dM\ dt$ in the volume form is

$$
(1.25)\qquad \begin{aligned} 1 &+ \sum_{i,\alpha} t_i G_\alpha^\alpha(i) + \sum_{i<j} \eta_{ij} t_i t_j \\ &+ \sum_i t_i^2 \left\{ \sigma_2[G(i)] + \tfrac{1}{2} \sum_{q>l} \sum_\alpha L_{\alpha q}(i)^2 \right\} + O(t^3). \end{aligned}
$$

Note that the summation of the $L_{\alpha q}(i)$ is only for $q = l + 1$ to k, which represents the orthogonal complement of the subbundle E in NM.

2. The Volume

We will now discuss the integration of the volume form over the tube and quadrant of radius r. First, note that when the volume form, (1.25), is integrated over the fiber (the t variable), we obtain certain integrals as follows.

LEMMA. *Let B be the ball in t-space (the fiber), of radius r. Let Q be the quadrant described by $\{t \in B \mid t_i \geqslant o \text{ for all } i\}$. The integrals of t_i and $t_i t_j$ over B and Q are independent of i and j by symmetry and are given by the following formuli:*

(2.1) $\int_B t_i^2\, dt = r^{l+2} V_l/(l + 2)$,

(2.2) $\int_B t_i t_j\, dt = 0$ *for* $i \neq j$,

(2.3) $\int_B t_i\, dt = 0$,

(2.4) $\int_Q t_i\, dt = r^{l+1} V_{l-1} 2^{-l}$.

Proof. Equations (2.1) through (2.3) are immediate consequences of formula 12 in Weyl's paper. Equation (2.4) follows by straightforward integration.

We can now integrate the volume form, 1.25, over the fiber.

PROPOSITION 2.5. *The volume of the tube, of radius r, about M in the space E is*

$$
(2.5)\qquad \begin{aligned} V(r) &= r^l V_l \int dM \\ &+ \frac{r^{l+2} V_l}{(l + 2)} \int \left\{ \sum_i \sigma_2[G(i)] + \tfrac{1}{2} \sum_{q>l} \sum_\alpha L_{\alpha q}(i)^2 \right\} dM \\ &+ O(r^{l+4}). \end{aligned}
$$

Proof. Integrate (1.25) over the fiber using Fubini's Theorem and equations (2.1) through (2.3). Note $r^l V_l = \int_B 1 \cdot dt$. Also, $\int_B O(t^3)\, dt = O(r^{l+4})$. It is this rather than $O(r^{l+3})$ because $\int_B t_i t_j t_k\, dt = 0$.

We now discuss the geometric significance of the integrands in (2.5). First, we show that the function $\Sigma_{i,\alpha} \Sigma_{q>l} L_{\alpha q}(i)^2$ is actually invariant and measures the torsion of the subbundle E. We can call this torsion because if it vanishes, then $L_{\alpha q}(i) = 0$ for all i, q and α, which was seen in section 1 to be the definition of E being a totally geodesic subbundle of the normal bundle.

PROPOSITION 2.6. *We can express the above function in invariant notation:*

$$\int_B \sum_{i,\alpha} \sum_{q>l} t_i^2 L_{\alpha q}(i)^2 \, dt = \frac{r^{l+2}}{V_m(l+2)} \int \left\| \Pi_{E^\perp} \nabla_w e \right\|^2 dS^{l-1} \, dS^{m-1}, \tag{2.6}$$

where the integration is over $(w, e) \in UTM \oplus UE$.

Proof. Let $n_1 \cdots n_l$ be the already chosen orthonormal basis section for E, and $f_1, \cdots, f_m$ be a local orthonormal frame on M. We proceed via a sequence of lemmas.

LEMMA 2.7.

$$\int_B \sum_{\alpha,i} \sum_{q>l} t_i^2 L_{\alpha q}(i)^2 \, dt = r^{l+2}/(l+2) \int \sum_{\alpha,i} \sum_{q>l} t_i^2 L_{\alpha q}(i)^2 \, dS^{l-1}.$$

Proof. Change to spherical coordinates and note that the integrand is quadratic.

LEMMA 2.8.

$$\int \sum_{\alpha,i} \sum_{q>l} t_i^2 L_{\alpha q}(i)^2 \, dS^{l-1} = \int \sum_{\alpha,q} \left[\sum_i t_i n_\alpha(i) \cdot n(q) \right]^2 dS^{l-1}.$$

Proof. The integrands are equal modulo a function which is a linear combination of terms $t_i t_j$, for $i \neq j$. This function vanishes when integrated over S^{l-1}.

LEMMA 2.9.

$$\int \sum_{\alpha,q} \left[\sum_i t_i n_\alpha(i) \cdot n(q) \right]^2 dS^{l-1} = \int \sum_\alpha \left\| \Pi_{E^\perp} e_\alpha \right\|^2 dS^{l-1}.$$

Proof. Set $e = \Sigma_i \, t_i n(i)$ and then note that the summation is for $q = l + 1$ to k. Then note that $\{n(q)\}_{q>l}$ is an orthonormal basis for $E^\perp$.

LEMMA 2.10.

$$V_m \int \sum_\alpha \left\| \Pi_{E^\perp} e_\alpha \right\|^2 dS^{l-1} = \int \left\| \Pi_{E^\perp} \nabla_w e \right\|^2 dS^{l-1} \, dS^{m-1}$$

Proof. Let $\{f_\alpha\}$ be the orthonormal basis section we have been using for TM. Let $w = \Sigma\, s_\alpha f_\alpha \in UTM$. Note that

$$\left\| \Pi_{E^\perp} \nabla_w e \right\|^2 = \left\| \sum_\alpha s_\alpha \Pi_{E^\perp} e_\alpha \right\|^2 \tag{2.10a}$$
$$= \sum_\alpha s_\alpha^2 \left\| \Pi_{E^\perp} e_\alpha \right\|^2 + m(s),$$

where $m(s)$ is a linear combination of mixed terms, $s_i s_j$, which vanish when integrated over S^{m-1}. When the right hand side of (2.10a) is integrated over S^{m-1}, a factor of $\int s_i^2\, dS^{m-1}$ is introduced. This is equal to V_m, the volume of the unit m-disc. That is a consequence of formula 12 in Weyl's paper.

Proposition 2.6 follows immediately from the above lemmas.

The other integrand in the volume formula, 2.5, is $\Sigma_i\, \sigma_2[G(i)]$. In the situation where E contains the image of the second fundamental form, it follows directly from Weyl's discussion (Section 4), that $\Sigma_i\, \sigma_2 \cdot [G(i)]$ is actually intrinsic to M. Using the fact that the ambient space is Euclidian, he shows that this object is an invariant polynomial in the curvature tensor of M. Specifically, upon examination, it turns out to be scaler curvature. We use his notation and call it h_2.

We are now in position to prove Theorem 3. Combine proposition 2.5 with 2.6 and the above discussion. It follows that the volume of the tubular neighborhood, radius r, about M in E is given locally, and thus globally, by

$$\begin{aligned} V(r) &= r^l V_l \cdot \mathrm{vol}(M) \\ &\quad + \frac{r^{l+2}}{l+2}\left[V_l \int h_2\, dM + V_m^{-1} \int \left\| \Pi_{E^\perp} \nabla_w e \right\|^2 dS^{l-1}\, dS^{m-1}\, dM \right] \\ &\quad + O(r^{l+4}). \end{aligned}$$

Theorem 3 is now proven.

Remark. The deviation of this formula, in the coefficient of r^{l+2}, from Weyl's formula is the square of the L^2 norm of the torsion of E, which measures the non-geodecity of E as a subbundle of NM.

We now prove Theorem 4, which is concerned with the volume of a quadrant tube about M in E. As before, let $n(1), \cdots, n(l)$ be a local orthonormal partial frame field of NM. This time, we consider the region, Q, parameterized by $m + \Sigma_{i=1}^l t_i n(i)$, for $t_i \geqslant o$ and $\Sigma\, t_i^2 \leqslant r^2$. Referring to formula (1.25), we can approximate the volume form by

$$\left[1 + \sum_{i,\alpha} t_i G_\alpha^\alpha(i) + O(t^2) \right] dM\, dt. \tag{2.11}$$

Integrating over the fiber (the t variable), and referring to (2.4) for $\int_Q t_i$

dt, we obtain

$$r^l 2^{-l} V_l \int dM + r^{l+1} 2^{-l} V_{l-1} \int \sum_{i,\alpha} G^\alpha_\alpha(i)\, dM + O(r^{l+2}).$$

Since $\Sigma_\alpha G^\alpha_\alpha(i) = \Sigma_\alpha G^\alpha_\alpha \cdot n(i) = H \cdot n(i)$, where $H = \operatorname{tr} G$ is the mean curvature vector, the volume is

$$r^l 2^{-l} V_l \cdot \operatorname{vol}(M) + r^{l+1} 2^{-l} V_{l-1} \int \left[H \cdot \sum_i n(i) \right] dM + O(r^{l+2}).$$

This proves Theorem 4.

3. A Theorem for Nicely Curved Submanifolds

We now discuss and prove Theorem 5. To define the concept of a nicely curved submanifold, we must first define the osculating spaces of the submanifold. We follow the exposition of Spivak, volume 4, Chapter 7.

Let $M^{(m)}$ be a submanifold of R^{m+k}, and ∇ the covariant deviative on R^{m+k}. If x_1 and x_2 are vector fields on M, $\nabla^N_x Y$ is the normal component of $\nabla_x Y$, known as the vector valued second fundamental form. The range of the second fundamental form, together with TM, comprises the span of $\nabla_{x_1} x_2$ for $x_1, x_2 \in TM$.

Let E_p be the range of the second fundamental form at M_p.

We introduce some notation: $\nabla(x, Y) = \nabla_x Y$, $\nabla(x, Y, Z) = \nabla_x(\nabla_y Z)$, etc.

Definition 3.1. The k^{th} *osculating space* of M at a point p is the span, at p, of

$$x_1,\ \nabla(x_1, x_2),\ \ldots,\ \nabla(x_1, x_2, \ldots x_k) \quad \text{for all } x_i \in TM.$$

Denote this by $\operatorname{Osc}(k, M_p)$.

Note that $\operatorname{Osc}(1, M_p) = TM_p$ and $\operatorname{Osc}(2, M_p) = TM_p \oplus E_p$. We sometimes suppress the M and write $\operatorname{Osc}(k)_p$.

Definition 3.2. M is called *nicely curved* if $\dim \operatorname{Osc}(k)_p$ is the same at all points $p \in M$.

We now consider only nicely curved submanifolds, and we then have a nested sequence of vector bundles $\operatorname{Osc}(k)$, with $\operatorname{Osc}(k) \subseteq \operatorname{Osc}(k+1)$.

Since $\operatorname{Osc}(k) \subseteq NM$ for all k, the sequence must stabilize. We note that if $\operatorname{Osc}(k) = \operatorname{Osc}(k+1)$, then $\operatorname{Osc}(k+1) = \operatorname{Osc}(k+2) = \ldots$. Therefore, there exists a minimum l such that $\operatorname{Osc}(l) = \operatorname{Osc}(l+1)$. The number, $\dim \operatorname{Osc}(l, M)$, is denoted $\#(M)$, and is called the *formal imbedding number* of M. Spivak supplies the following proposition:

Proposition 3.3. *Let M be a nicely curved submanifold of R^{m+k}. Then, M is contained in a $\#(M)$-dimensional linear subspace of R^{m+k}.*

We now prove Theorem 5. We restate it:

THEOREM 5. *Let $E^{(l)}$ be the range of the second fundamental form of a nicely curved submanifold $M^{(m)}$ of R^{m+k}. Let $V(r)$ denote the volume of the tubular neighborhood of radius r about M in E. The following are equivalent:*

(1) *$V(r)$ is the Weyl polynomial of M;*
(2) *$V(r)$ agrees with the Weyl polynomial to order r^{l+2};*
(3) *E is a totally geodesic subbundle of NM;*
(4) *E is contained in a $m + l$ dimensional linear subspace of $R^{m=k}$.*

Proof. We remember that E is totally geodesic if E is closed under the induced connection on the normal bundle of M. In other words, if $\Delta^\perp$ is the induced connection on NM then E is totally geodesic if $\nabla_v^\perp e \in E$ for all $e \in E$ and $v \in TM$. A restatement of this is that $\Pi_{E^\perp} \nabla_v^\perp e = 0$ for all e and v. Since this is the torsion of M, whose integral is the difference between the Weyl coefficient of r^{l+2} and the coefficient of the volume formula (by Theorem 3), we see that (2) and (3) are equivalent. We then recall that $\mathrm{Osc}(1, M) = E \oplus TM$, and that $\mathrm{Osc}(2, M)$ is $\{\nabla_x Y \mid x \in TM, Y \in \mathrm{Osc}(1, M)\}$, where ∇ is the connection on R^{m+k}. Assume E to be totally geodesic. Then $\nabla_x e \in E \ \mathrm{Osc}(1)$ for all $e \in E$. Also, $\nabla_x Y \in E \oplus TM$ for all $Y \in TM$ since the normal component of $\nabla_x Y$ is the vector valued second fundamental form $\mathrm{II}(x, Y)$ which is contained in E. Therefore, we have shown that if E is totally geodesic, then $\mathrm{Osc}(1, M) = \mathrm{Osc}(2, M)$. By Proposition 3.3, M is contained in a linear subspace of R^{n+k} of dimension $\#(M) = \dim \mathrm{Osc}(1, M) = \dim(E \oplus TM) = m + l$. Thus we have shown that (3) implies (4). By Weyl's Theorem, (4) implies (1). Trivially, (1) implies (2). We have then shown the chain of implications (1) $\rightarrow$ (2) $\leftrightarrow$ (3) $\rightarrow$ (4) $\rightarrow$ (1), which proves Theorem 5.

Acknowledgments. This paper contains partial results of the author's thesis, written at the CUNY Graduate Center under the advisement of Professor Edgar Feldman. The author wishes to thank Professors Feldman, Sackstedder and Chavel for many enlightening conversations. The author is currently with the Image Processing Lab at Lockheed Electronics Company.

REFERENCES

1. C. ALLENDOERFER, *Global theorems in Riemanian geometry,* Bull. Amer. Math. Soc., vol. 54 (1948), pp. 249–259.
2. W. BERTRAND and W. DIGUET, J. DE MATH. (Liouville) Series 1, vol. 13 (1848), pp. 80–86.
3. J. CHEEGER and D. EBIN, *Comparison theorems in Riemanian geometry,* North Holland Press, N.Y., 1975.
4. S. S. CHERN, *On the kinematic formula in integral geometry,* J. Math. Mech., vol. 16 (1966), pp. 101–118.
5. B. V. DEKSTER, *The volume of a slightly curved submanifold,* Proc. Amer. Math. Soc., vol. 68 (1978), pp. 203–208.

6. F. J. FLAHERTY, *The volume of a tube in complex projective space,* Illinois J. Math., vol. 16 (1972), pp. 627–637.
7. N. GROSSMAN, *Polynomial estimates for Betti numbers of loop spaces of symmetric and close-to-symmetric Riemanian manifolds,* Indiana Math. J., vol. 20 (1971), pp. 717–730.
8. H. HOTELLING, *Tubes and spheres in n-spaces, and a class of statistical problems,* Amer. J. Math., vol. 61 (1939), pp. 440–460.
9. J. MILNOR, *Kinematic measures, Euler characteristic and angles,* preprint, 1978.
10. H. E. RAUCH, *A contribution to Riemanian geometry in the large,* Ann. Math., vol. 54 (1951), pp. 38–55.
11. M. SPIVAK, *A comprehensive introduction to differential geometry,* vols. 1–5, Publish or Perish, Boston, Mass., 1970–1975.
12. J. STEINER, *Uber Parallele Flachen,* Monatsb. Deutsch, Akad. Wiss. Berlin, 1840, pp. 114–118.
13. H. DONNALEY, *The heat equation and the volume of tubes,* Invent. Math., vol. 29 (1975), pp. 239–244.
14. J. THORPE, *Sectional curvatures and characteristic classes,* Ann. of Math. (2), vol. 80 (1964), pp. 427–443.
15. ———, *On the curvatures of Riemanian manifolds,* Illinois J. Math., vol. 10 (1966), pp. 412–417.
16. H. WEYL, *On the volume of tubes,* Amer. J. Math., vol. 61, (1939), pp. 461–472.

LOCKHEED ELECTRONICS COMPANY
PLAINFIELD, NEW JERSEY