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L^p ESTIMATES FOR THE X-RAY TRANSFORM[1]

BY

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Introduction

Let M_n denote the manifold of lines (1-dimensional affine subspaces) of n-dimensional Euclidean space E_n. In view of [5, Chapter 7, § 2, Theorème 3], one may construct on M_n a positive measure μ invariant under Euclidean motions. Aside from renormalizations, μ is unique with this property. We denote by λ the Lebesgue measure on E_n and for $l \in M_n$, we denote by λ_l the Lebesgue measure on the line l. For a function $f \in C_c(E_n)$, the X-ray transform $Tf \in L^\infty(M_n)$ is defined by

$$Tf(l) = \int f(x) d\lambda_l(x).$$

The reader may consult [6] for a discussion of this transform and its practical applications.

The goal of this article is the following result.

THEOREM. *Let p and q satisfy $1 \leqslant q < n + 1$, $np^{-1} - (n-1)q^{-1} = 1$ (so that $1 \leqslant p < \frac{1}{2}(n+1)$). Then T extends to a bounded operator*

$$T\colon L^p(E_n, \lambda) \to L^q(M_n, \mu).$$

In an analogous way one can define the k-plane transform of f on the manifold of all k-dimensional affine subspaces of E_n. The reader may consult [3] for details. In [2], Stein and Oberlin establish L^p and mixed norm estimates in the case $k = n - 1$ of the so called Radon transform. When $n = 2$, the Radon transform ($k = n - 1$) coincides with the X-ray transform ($k = 1$) and their results contain ours. In fact they prove the above theorem in case $n = 2$, $p = 3/2$ and $q = 3$. The result is open for $p = \frac{1}{2}(n+1)$, $q = n + 1$, $n \geqslant 3$. Neither our methods nor those of Stein and Oberlin seem to yield a good answer to the behaviour of the k-plane transform in case $1 < k < n - 1$.

Note added in proof. The optimal L^p to L^q estimates for the k-plane transform have now been established in case $n \leqslant 2k + 1$ and will be

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presented in a forthcoming article in this journal. In particular the case $n = 3$, $p = 2$, $q = 4$ for the X-ray transform has been settled affirmatively.

The k-plane transform is trivially bounded from L^1 to L^1 and can be bounded from L^p to L^q only if $np^{-1} - (n - k)q^{-1} = k$ and $q \leqslant n + 1$. To see this, say in the case $k = 1$, we need a better description of μ. We may realize M_n as an affine space bundle in which each fibre is a collection of parallel lines. The base of the bundle is essentially projective space which carries a rotation invariant probability measure. It is easy to see that integrating out the ($(n - 1)$-dimensional) Lebesgue measure on each fibre with this probability yields a constant multiple of μ. Now let f be the indicator function of a ball of radius r and let A_r be the subset of M_n of all lines passing within $\frac{1}{2}r$ of the centre. Then $\|f\|_p \leqslant C\, r^{np^{-1}}$ and $Tf > r$ on the subset A_r. Our description of μ shows that $\mu(A_r) \sim r^{n-1}$. Then $\|Tf\|_q \leqslant C\|f\|_p$ yields $r^{1+(n-1)q^{-1}} \leqslant Cr^{np^{-1}}$ for all $r > 0$. Hence $np^{-1} - (n - 1)q^{-1} = 1$. To obtain the other condition, let now f be the indicator function of a box having one side of unit length and the remaining sides of a shorter length δ. Let B_δ be the set of lines meeting both "ends" of the box. Then $\|f\|_p = \delta^{(n-1)p^{-1}}$, $Tf \geqslant 1$ on B_δ and $\mu(B_\delta) \sim \delta^{2(n-1)}$ $(0 < \delta \leqslant 1)$. Together with $\|Tf\|_q \leqslant C\|f\|_p$ this yields $q \leqslant 2p$ which is equivalent to the stated condition.

Methods and Proofs

We denote by μ_x the probability measure on M_n carried by the set of lines passing through the point x and invariant under the stabilizer of x in the Euclidean motion group. One easily verifies the relation

$$d\mu_x(l)d\lambda(x) = d\lambda_l(x)d\mu(l) \tag{1}$$

(as measures on $E_n \times M_n$) for a particular normalization of the measure μ.

Our strategy is to write, in case $q \geqslant 2$,

$$\int (Tf(l))^q d\mu(l) = \int f(x_1)f(x_2)(Tf(l))^{q-2}d\lambda_l(x_1)d\lambda_l(x_2)d\mu(l).$$

Using (1) this expression can further be rewritten as

$$c_n \int f(x_1)f(x_2)(Tf(l(x_1, x_2)))^{q-2}|x_1 - x_2|^{-(n-1)}d\lambda(x_1)d\lambda(x_2)$$

where $l(x_1, x_2)$ denotes the line joining x_1 and x_2. Roughly speaking the idea is now to consider the function

$$(Tf(l(x_1, x_2)))^{q-2}|x_1 - x_2|^{-(n-1)}$$

as a kernel. We shall need the following weak-type estimate. Let us define

$$S_a g(x) = \left\{\int \left|Tg(l)\right|^a d\mu_x(l)\right\}^{1/a}$$

LEMMA 1. *Let* $1 \leqslant a < n$ *and let* $g \in L^a(E_n)$. *Then*

$$\lambda\text{-meas}\{x;\ S_a g(x) > \tau\} \leqslant c_{n,a}(\tau^{-1}\|g\|_a)^b$$

where $b^{-1} = a^{-1} - n^{-1}$.

Proof. For $g \geqslant 0$, $S_1 g = T^*Tg$ and Solmon [3] has shown that T^*T is the Riesz potential of order 1. It is therefore natural to adopt the usual method for controlling Riesz potentials. Clearly

$$S_a g(x) \sim \left\{ \int \left| \int_0^\infty g(x + ry)dr \right|^a d\sigma(y) \right\}^{1/a} \tag{2}$$

where σ is the rotation invariant probability measure on the unit sphere. For $R > 0$ let us define two quantities $S^{(1)}$ and $S^{(2)}$ to be the right hand side of (2) with the range of integration of the inner integral replaced by $[O, R)$ and $[R, \infty)$ respectively. Clearly

$$S_a g(x) \sim S^{(1)} + S^{(2)}. \tag{3}$$

Both $S^{(1)}$ and $S^{(2)}$ are dominated by applying Hölder's inequality to the inner integral. Then for $x \in E_n$ fixed,

$$S^{(1)} \leqslant \left\{ \int_0^R dr \right\}^{1/a'} \left\{ \int_0^R |g(x + ry)|^a dr\, d\sigma(y) \right\}^{1/a} = R^{1/a'} \left\{ |g|^a * \theta_R(x) \right\}^{1/a}$$

where $\theta_R(z) = c_n|z|^{-(n-1)}$ if $|z| < R$ and $\theta_R(z) = 0$ if $|z| \geqslant R$;

$$S^{(2)} \leqslant \left\{ \int_R^\infty r^{-(n-1)a'/a} dr \right\}^{1/a'} \left\{ \int \int_R^\infty |g(x + ry)|^a r^{n-1} dr\, d\sigma(y) \right\}^{1/a}$$

$$\sim R^{-(n-a)/a}\|g\|_a.$$

We choose R such that $R^{-(n-a)/a}\|g\|_a$ is a small multiple of τ. Then, by (3),

$$S_a g(x) > \tau \Rightarrow S^{(1)} > \tfrac{1}{2}\tau.$$

We now use the estimate $\||g|^a * \theta_R\|_1 \leqslant \|g\|_a^a \|\theta_R\|_1 \leqslant C_n R\|g\|_a^a$ and Tchebychev's inequality to verify the statement of the lemma.

PROPOSITION. *Let* $0 < \alpha < 1$ *and let* K *be a symmetric kernel on a measure space* (X, ν) *such that*

$$\operatorname*{ess\,sup}_{x_1} \int_{|K(x_1, x_2)| > t} |K(x_1, x_2)| d\nu(x_2) \leqslant A\, t^{1 - 1/(1-\alpha)} \quad (t > 0) \tag{4}$$

and

$$\operatorname*{ess\,sup}_{x_1} \int_{|K(x_1, x_2)| \leqslant t} |K(x_1, x_2)|^s d\nu(x_2) \tag{5}$$

$$\leqslant c_s A\, t^{s - 1/(1-\alpha)} \quad (t > 0,\ (1 - \alpha)s > 1)$$

Then K is the kernel of a smoothing operator of order α. That is, K is bounded as an operator

$$K\colon L^b(X, \nu) \to L^c(X, \nu) \quad (b^{-1} - \alpha = c^{-1}, b > 1, c < \infty)$$

and the operator norm of K is bounded by $c_b A^{1-\alpha}$.

The proof of the proposition again follows the usual strategy for Riesz potentials—see [4]. Note that (5) asserts that the "lower part of K" is bounded from $L^{s'}$ to L^∞ whereas (4) together with the symmetry condition asserts that the 'upper part of K' is bounded both from L^1 to L^1 and from L^∞ to L^∞ and hence by convexity from $L^{s'}$ to $L^{s'}$. We leave the details of the proof of the proposition to the reader.

LEMMA 2. *Let $2 \leq q < n + 1$. Let $Y \subset E_n$ be a set of finite measure m. Then there is a subset X of Y of measure at least $\frac{1}{2}m$ such that $\|T\mathbf{1}_X\|_q \leq c_{n,q} m^{(q+n-1)/nq}$.*

Proof. Let $g = \mathbf{1}_Y$ and let us define

$$L(x_1, x_2) = (Tg(l(x_1, x_2)))^{q-2}|x_1 - x_2|^{-(n-1)},$$

a symmetric kernel. Routine calculations show that

$$\int_{L>t} L(x_1, x_2) d\lambda(x_2) = c_n A(x_1) t^{1-(1-\alpha)^{-1}} \quad (t > 0), \tag{6}$$

$$\int_{L\leq t} (L(x_1, x_2))^s d\lambda(x_2) = c_{n,s} A(x_1) t^{s-(1-\alpha)^{-1}} \quad (t > 0, (1 - \alpha)s > 1) \tag{7}$$

for $\alpha = n^{-1}$ and where

$$A(x) = \int (Tg(l))^a d\mu_x(l) \quad \text{for } a = (q - 2)n/(n - 1).$$

Since $q < n + 1$, $a < n$ and we may apply Lemma 1 with $\tau = c_{n,a} m^{1/n}$ where $c_{n,a}$ is sufficiently large to ensure that the measure of the exceptional set

$$Z = \{x; S_a g(x) > \tau\}$$

is less than $\frac{1}{2}m$. Let $X = Y \setminus Z$. Then we have

$$A(x) \leq c_{n,a} m^{a/n} \quad \text{a.a.} x \in X.$$

Let ν be the restriction of Lebesgue measure λ to X and let K be the restriction of L to $X \times X$. Then (4) and (5) follow from (6) and (7) respectively with $A = c_{n,a} m^{a/n}$. The proposition now yields

$$\int (T\mathbf{1}_X(l))^2 (T\mathbf{1}_Y(l))^{q-2} d\mu(l) \sim \langle K\mathbf{1}_X, \mathbf{1}_X\rangle \leq c_{n,q} m^{(q+n-1)/n}.$$

The conclusion of the lemma follows since $T\mathbf{1}_X \leq T\mathbf{1}_Y$.

Proof of the theorem. For the range $2 \leq q < n + 1$ we show that T^* is weak type (q', p'). Let $h \in L^{q'}(M_n)$ be an element of unit norm. Let

$t > 0$ and let $Y_1 = \{x; T^*|h|(x) > t\}$. Let Y be an arbitrary subset of Y_1 of finite measure m. Let X be as in Lemma 2. Then we have

$$\tfrac{1}{2}mt \leqslant \langle \mathbf{1}_X, T^*|h|\rangle = \langle T\,\mathbf{1}_X, |h|\rangle \leqslant \|T\,\mathbf{1}_X\|_q \leqslant c_{n,q}m^{(q+n-1)/nq}$$

so that $m \leqslant c_{n,q}t^{-nq/(n-1)(q-1)}$. Since $|T^*h| \leqslant T^*|h|$ we have the required weak type estimate with $p' = nq/(n-1)(q-1)$. Now T^* is clearly a bounded operator from $L^\infty(M_n)$ to $L^\infty(E_n)$. (It suffices to observe that μ_x is a bounded measure for each x.) The general statement of the theorem now follows from the Marcinkiewicz Interpolation Theorem and a duality argument.

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