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INEQUALITIES FOR REPRODUCING KERNEL SPACES

BY

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1. Introduction

In this paper we establish a general result giving a sharp relationship between the reproducing kernel spaces of holomorphic functions in the disk or the entire complex plane and the reproducing kernel space determined by the product of their reproducing kernels. We give several applications of this general result notably in the generalized Hardy and Fischer spaces (cf. [1], [2], [4] and [5]). The latter will constitute an improvement and an extension of a recent result of Saitoh [5] which he obtained by using different methods. The proof given here uses elementary means and the result obtained may be applied to various other situations such as those described in our previous paper [1]. The present result can be also extended to cover the case of several complex variables but we shall not pursue this here (see, however, Burbea [3]).

2. An Inequality

In this paper Δ_ρ, $\rho = 1, \infty$, stands for the unit disk Δ when $\rho = 1$ and the complex plane $\mathbf{C}$ when $\rho = \infty$. The class of all holomorphic functions in Δ_ρ is denoted by $H(\Delta_\rho)$ while $d\sigma(z) = dx\,dy$ is the area Lebesque measure of $\mathbf{C}$. By $P(\Delta_\rho)$ we denote the subclass of $H(\Delta_\rho)$ consisting of all $\phi \in H(\Delta_\rho)$ of the form

$$\phi(z) = \sum_{n=0}^{\infty} c_n z^n; \quad c_n > 0, n \geq 0, z \in \mathbf{\Delta}_\rho, \tag{2.1}$$

and where Δ_ρ is the domain of convergence of the expansion (2.1) of ϕ.

Associated with $\phi \in P(\Delta_\rho)$ is the space

$$H_\phi = \{f \in H(\Delta_\rho)\colon \|f\|_\phi < \infty\}$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (f \in H(\Delta_\rho), z \in \Delta_\rho), \tag{2.2}$$

$$\|f\|_\phi^2 = \sum_{n=0}^{\infty} c_n^{-1} |a_n|^2. \tag{2.3}$$

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Of course, H_ϕ is a Hilbert space with norm $\|f\|_\phi$. Also, for $\zeta \in \Delta_\rho$,

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n = \sum_{n=0}^{\infty} \frac{a_n c_n \zeta^n}{c_n} = (f(z), \phi(z\bar{\zeta}))_\phi$$

and

$$|f(\zeta)|^2 \leqslant \phi(|\zeta|^2)\|f\|_\phi^2 \qquad (f \in H_\phi, \zeta \in \Delta_\rho)$$

which means that

$$k_\phi(z, \bar{\zeta}) = \phi(z\bar{\zeta})$$

is the reproducing kernel of H_ϕ and that $\{e_n\}_{n=0}^{\infty}$ with $e_n(z) = \sqrt{c_n} z^n$ is an orthonormal basis for H_ϕ.

We note that $\phi, \psi \in P(\Delta_\rho)$ implies $\phi\psi \in P(\Delta_\rho)$. With this observation we state our basic theorem:

THEOREM 1. *Let $\phi, \psi \in P(\Delta_\rho)$ and $f \in H_\phi$, $g \in H_\psi$. Then $fg \in H_{\phi\psi}$ and*

$$\|fg\|_{\phi\psi} \leqslant \|f\|_\phi \|g\|_\psi.$$

Equality holds if and only if either $fg = 0$ or f and g are of the forms

$$f(z) = C_1\phi(z\bar{\zeta}), \; g(z) = C_2\psi(z\bar{\zeta}); \quad z \in \Delta_\rho$$

for some $\zeta \in \Delta_\rho$ and some nonzero constants C_1 and C_2.

Proof. We assume that ϕ, f and $\|f\|_\psi$ are as in (2.1)–(2.3). The corresponding quantities for H_ψ will be given by

$$\psi(z) = \sum_{n=0}^{\infty} d_n z^n; \quad d_n > 0, n \geqslant 0, z \in \Delta_\rho,$$

$$\|g\|_\psi^2 = \sum_{n=0}^{\infty} d_n^{-1}|b_n|^2, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, z \in \Delta_\rho.$$

Under these circumstances

$$\phi(z)\psi(z) = \sum_{n=0}^{\infty} B_n z^n, \quad B_n = \sum_{k=0}^{n} c_k d_{n-k},$$

$$f(z)g(z) = \sum_{n=0}^{\infty} A_n z^n, \quad A_n = \sum_{k=0}^{n} a_k b_{n-k}$$

and

$$\|fg\|_{\phi\psi}^2 = \sum_{n=0}^{\infty} B_n^{-1}|A_n|^2.$$

The theorem, therefore, is completely equivalent to the following sharp inequality:

$$\sum_{n=0}^{\infty} B_n^{-1}|A_n|^2 \leqslant \left(\sum_{n=0}^{\infty} c_n^{-1}|a_n|^2\right)\left(\sum_{n=0}^{\infty} d_n^{-1}|b_n|^2\right) \tag{2.4}$$

with equality if and only if either (i) $a_n = 0$ or $b_n = 0$ for all $n \geqslant 0$, or (ii) $a_n = C_1 c_n \bar{\zeta}^n$ and $b_n = C_2 d_n \bar{\zeta}^n$ for all $n \geqslant 0$, for some $\zeta \in \Delta_\rho$ and for some nonzero constants C_1 and C_2. In order to prove this inequality we let $r \in [0, 1]$ and introduce

$$B(r) = \sum_{n=0}^{\infty} B_n^{-1}|A_n|^2 r^n, \quad C(r) = \sum_{n=0}^{\infty} c_n^{-1}|a_n|^2 r^n, \quad D(r) = \sum_{n=0}^{\infty} d_n^{-1}|b_n|^2 r^n.$$

Applying the Cauchy-Schwarz inequality we have

$$\begin{aligned} |A_n|^2 &= \left|\sum_{k=0}^{n} a_k b_{n-k}\right|^2 = \left|\sum_{k=0}^{n} \frac{a_k b_{n-k}}{(c_k d_{n-k})^{1/2}} \left(c_k d_{n-k}\right)^{1/2}\right|^2 \\ &\leqslant \left(\sum_{k=0}^{n} \frac{|a_k|^2}{c_k} \frac{|b_{n-k}|^2}{d_{n-k}}\right)\left(\sum_{k=0}^{n} c_k d_{n-k}\right) \\ &= \left(\sum_{k=0}^{n} \frac{|a_k|^2}{c_k} \frac{|b_{n-k}|^2}{d_{n-k}}\right) B_n \end{aligned}$$

and so

$$B_n^{-1}|A_n|^2 \leqslant \sum_{k=0}^{n} \frac{|a_k|^2}{c_k} \frac{|b_{n-k}|^2}{d_{n-k}}, \quad n \geqslant 0. \tag{2.5}$$

This shows that

$$B(r) \leqslant C(r)D(r), \quad r \in [0, 1). \tag{2.6}$$

Letting $r \to 1 - 0$ in (2.6), inequality (2.4) is obtained. In view of (2.5)–(2.6) and the fact that $B(0) = C(0)D(0)$, equality in (2.4) holds if and only if $B(r) = C(r)D(r)$ for each $r \in [0, 1)$ which is equivalent to having equality in (2.5) for every $n \geqslant 0$. This is, obviously, equivalent to an existence of $\lambda_n \in \mathbf{C}$ so that

$$a_k b_{n-k} = \lambda_n c_k d_{n-k}; \quad k = 0, 1, \dots n, n \geqslant 0. \tag{2.7}$$

Putting $k = 0$ and $k = n$ in (2.7) results in

$$\lambda_n c_o d_n = a_o b_n, \quad \lambda_n c_n d_o = a_n b_o; \quad n \geqslant 0. \tag{2.8}$$

On the other hand summing up (2.7) from $k = 0$ through $k = n$ yields

$$A_n = \lambda_n B_n, \quad n \geqslant 0. \tag{2.9}$$

If $a_o b_o = 0$, then by (2.8), $\lambda_n \geqslant 0$ for all $n \geqslant 0$. Therefore, by (2.9), $A_n = 0$ for all $n \geqslant 0$ which means $a_n = 0$ or $b_n = 0$ for all $n \geqslant 0$. This covers item (i) of the equality statement. We now assume that $a_o b_o \neq 0$. Define

$$C_1 \equiv a_o c_o^{-1}, \quad C_2 \equiv b_o d_o^{-1}, \quad \bar{\zeta} \equiv c_o a_1 (c_1 a_o)^{-1} = d_o b_1 (d_1 b_o)^{-1} \tag{2.10}$$

where (2.8) with $n = 1$ has been used. Clearly, $C_1, C_2 \neq 0$. From (2.8)–(2.10) we have

$$(2.11) \qquad b_n = C_2 C_1^{-1} \frac{a_n}{c_n} d_n, \quad n \geqslant 0,$$

$$a_o b_n + \sum_{k=1}^{n-1} a_k b_{n-k} + a_n b_o = \frac{a_n b_o}{c_n d_o} \sum_{k=0}^{n} c_k d_{n-k}, \quad n \geqslant 1,$$

and, therefore,

$$(2.12) \quad a_n b_o (c_n d_o + c_o d_n) + c_n d_o \sum_{k=1}^{n-1} a_k b_{n-k} = a_n b_o \sum_{k=0}^{n} c_k d_{n-k}, \quad n \geqslant 1.$$

We use induction on n to show that

$$(2.13) \qquad a_n = C_1 c_n \bar{\zeta}^n, \; b_n = C_2 d_n \bar{\zeta}^n; \quad n \geqslant 0.$$

Clearly, by (2.10), (2.13) is true for $n = 0$ and $n = 1$. Assuming (2.13) is true for $k \leqslant n - 1$, $n \geqslant 2$, we find by (2.12) that

$$a_n b_o (c_n d_o + c_o d_n) + c_n d_o C_1 C_2 \bar{\zeta}^n \sum_{k=1}^{n-1} c_k d_{n-k} = a_n b_o \sum_{k=0}^{n} c_k d_{n-k}$$

and so, by (2.10),

$$a_n b_o \sum_{k=1}^{n-1} c_k d_{n-k} = C_1 c_n \bar{\zeta}^n b_o \sum_{k=1}^{n-1} c_k d_{n-k}, \quad n \geqslant 2.$$

This shows that $a_n = C_1 c_n \bar{\zeta}^n$, and also, by (2.11), $b_n = C_2 d_n \bar{\zeta}^n$ and (2.13) is proved. Finally, ζ must be in Δ_ρ because, for example, the value of the first factor on the right side of (2.4) for the solution in (2.13) is $|C_1|^2 \phi(|\zeta|^2)$ and, since the domain of convergence of ϕ is Δ_ρ, $\zeta \in \Delta_\rho$. This concludes the proof.

An immediate consequence of this theorem is the following result.

COROLLARY 1. *Let $\phi_j \in P(\Delta_\rho)$ and $f_j \in H_{\phi_j}$, $j = 1, \ldots, m$. Then*

$$\prod_{j=1}^{m} f_j \in H_{\phi_1 \cdots \phi_m}$$

and

$$\|\prod_{j=1}^{m} f_j\|_{\phi_1 \cdots \phi_m} \leqslant \prod_{j=1}^{m} \|f_j\|_{\phi_j}$$

with equality if and only if either $\Pi_{j=1}^{m} f_j = 0$ or each f_j $(1 \leqslant j \leqslant m)$ is of the form

$$f_j(z) = C_j \phi_j(z\bar{\zeta}) \qquad (z \in \Delta_\rho, j = 1, \ldots, m)$$

for some $\zeta \in \Delta_\rho$ and some nonzero constants C_j $(1 \leqslant j \leqslant m)$.

This general, but simple, result admits many interesting applications by using a suitable choice of the ϕ_j (cf. Burbea [1]). We shall describe several applications for the generalized Hardy and Fischer spaces.

3. Generalized Hardy Spaces

Here we consider holomorphic functions in the unit disk $\Delta_1 = \Delta$. For any $q > 0$ we consider $\phi_q \in P(\Delta)$ given by

$$\phi_q(z) = (1 - z)^{-q}, \quad z \in \Delta.$$

The Hilbert space $H_q \equiv H_{\phi_q}$ determined by this function is called the *generalized q-Hardy space,* and we note that H_1 is the ordinary Hardy space. In this case,

$$H_q = \{f \in H(\Delta) \colon \|f\|_q < \infty\}$$

where

$$\|f\|_q^2 = \sum_{n=0}^{\infty} \frac{n!}{(q)_n} |a_n|^2 \quad \left(f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \Delta\right). \tag{3.1}$$

Here $(q)_o = 1$ and $(q)_n = q(q + 1) \cdots (q + n - 1)$ for $n \geqslant 1$. The reproducing kernel for H_q is, of course, $k_q(z, \bar{\zeta}) = \phi_q(z\bar{\zeta}) = (1 - z\bar{\zeta})^{-q}$. For $q \geqslant 1$ the norm of H_q can be realized as

$$\|f\|_q^2 = \frac{q-1}{\pi} \int_\Delta |f(z)|^2 (1 - |z|^2)^{q-2} d\sigma(z), \quad q > 1,$$

and

$$\|f\|_1^2 = \frac{1}{2\pi} \int_{\partial\Delta} |f(z)|^2 |dz|,$$

where in the last integral, f stands for the nontangential boundary values of the holomorphic function $f(z)$ in Δ. For $0 < q < 1$, the norm of H_q does not admit such a simple integral representation, a fact which is not crucial for we shall only use (3.1) as the norm of $f \in H_q$, $q > 0$.

Under these circumstances, the following corollary is an immediate consequence of Corollary 1:

COROLLARY 2. *Let $q_j > 0$ and $f_j \in H_{q_j}$, $j = 1, \ldots, m$. Then*

$$\prod_{j=1}^{m} f_j \in H_{q_1 + \cdots + q_m}$$

and

$$\|\prod_{j=1}^{m} f_j\|_{q_1 + \cdots + q_m} \leqslant \prod_{j=1}^{m} \|f_j\|_{q_j}$$

with equality if and only if either $\prod_{j=1}^{m} f_j = 0$ *or each* f_j $(1 \leqslant j \leqslant m)$ *is of the form*

$$f_j(z) = C_j(1 - z\bar{\zeta})^{-q_j} \qquad (z \in \Delta, j = 1, \ldots, m)$$

for some $\zeta \in \Delta$ *and some nonzero constants* C_j $(1 \leqslant j \leqslant m)$.

4. Generalized Fischer Spaces

We now consider holomorphic (entire) functions in the plane $\Delta_\infty = \mathbf{C}$. For any α, $\beta > 0$ we consider $\phi_{\alpha,\beta} \in P(\mathbf{C})$ given by

$$\phi_{\alpha,\beta}(z) = {}_1F_1(1; \alpha; \beta z)$$

where ${}_1F_1(1; \alpha; \beta z)$ is a confluent hypergeometric function:

$${}_1F_1(1; \alpha; \beta z) = \sum_{n=0}^{\infty} \frac{1}{(\alpha)_n}(\beta z)^n.$$

In particular,

$$\phi_{1,\beta}(z) = {}_1F_1(1; 1; \beta z) = e^{\beta z} \tag{4.1}$$

and

$$\phi_{2,\beta}(z) = {}_1F_1(1; 2; \beta z) = (\beta z)^{-1}(e^{\beta z} - 1).$$

The Hilbert space $\mathscr{F}_{\alpha,\beta} \equiv H_{\phi_{\alpha,\beta}}$ determined by $\phi_{\alpha,\beta}$ is called the *generalized* (α, β)*-Fischer space*. The space $\mathscr{F}_{1,1}$ is known as the ordinary Fischer space (cf. Newman and Shapiro [4]). In the present case

$$\mathscr{F}_{\alpha,\beta} = \{f \in H(\mathbf{C}) : \|f\|_{\alpha,\beta} < \infty\}$$

where

$$\|f\|_{\alpha,\beta} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{\beta^n} |a_n|^2, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbf{C}.$$

This norm can be realized as

$$\|f\|_{\alpha,\beta}^2 = \frac{\beta^\alpha}{\pi\Gamma(\alpha)} \int_{\mathbf{C}} |f(z)|^2 |z|^{2(\alpha-1)} e^{-\beta|z|^2} d\sigma(z); \quad \alpha, \beta > 0. \tag{4.2}$$

The reproducing kernel for $\mathscr{F}_{\alpha,\beta}$ is, of course,

$$k_{\alpha,\beta}(z, \bar{\zeta}) = \phi_{\alpha,\beta}(z\bar{\zeta}) = {}_1F_1(1; \alpha; \beta z\bar{\zeta}).$$

For α_j, $\beta_j > 0$, $j = 1, \ldots, m$, we write

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m), \quad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$$

and consider $\phi_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in P(\mathbf{C})$ defined by

$$\phi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = \phi_{\alpha_1,\beta_1}(z) \ldots \phi_{\alpha_m,\beta_m}(z).$$

This function determines the Hilbert space

$$\mathscr{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \{g \in H(\mathbf{C}): \|g\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} < \infty\}$$

where

$$\|g\|^2_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{n=0}^{\infty} \{c_n(\boldsymbol{\alpha}, \boldsymbol{\beta})\}^{-1}|b_n|^2 \qquad \left(g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbf{C}\right)$$

with

$$c_n(\boldsymbol{\alpha}, \boldsymbol{\beta}) \equiv \sum_{k_1+\cdots k_m=n} \frac{\beta_1^{k_1}}{(\alpha_1)_{k_1}} \cdots \frac{\beta_m^{k_m}}{(\alpha_m)_{k_m}}, \quad n \geq 0,$$

the sum being over all m-tuples $(k_1, \ldots, k_m)$ of non-negative integers. The reproducing kernel for $\mathscr{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is

$$k_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z, \bar{\zeta}) = \phi_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z\bar{\zeta}) = \prod_{j=1}^{m} {}_1F_1(1; \alpha_j; \beta_j z\bar{\zeta}).$$

We note that for $\boldsymbol{\alpha} = \mathbf{1} = (1, \ldots, 1)$,

$$\mathscr{F}_{\mathbf{1},\boldsymbol{\beta}} = \mathscr{F}_{1,\beta_1+\cdots+\beta_m} \tag{4.3}$$

and

$$\phi_{\mathbf{1},\boldsymbol{\beta}}(z) = \phi_{1,\beta_1+\cdots+\beta_m}(z) = e^{(\beta_1+\cdots+\beta_m)z}. \tag{4.4}$$

Again, under the above circumstances, the following is an immediate corollary of Corollary 1:

COROLLARY 3. *Let α_j, $\beta_j > 0$ and $f_j \in \mathscr{F}_{\alpha_j,\beta_j}$, $j = 1, \ldots, m$. Then $\Pi_{j=1}^m f_j$ belongs to $\mathscr{F}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$, and*

$$\left\|\prod_{j=1}^{m} f_j\right\|_{\alpha,\beta} \leq \prod_{j=1}^{m} \|f_j\|_{\alpha_j,\beta_j}$$

with equality if and only if $\Pi_{j=1}^m f_j = 0$ or each f_j $(1 \leq j \leq m)$ is of the form

$$f_j(z) = C_j {}_1F_1(1; \alpha_j; \beta_j z\bar{\zeta}), \quad z \in \mathbf{C}$$

for some $\zeta \in \mathbf{C}$ and some nonzero constants C_j $(1 \leq j \leq n)$.

An interesting special case of this corollary is obtained by specifying $\boldsymbol{\alpha}$ to be $\mathbf{1}$ and noting (4.1)–(4.4). This gives:

COROLLARY 4. *Let $\beta_j > 0$ and $f_j \in \mathscr{F}_{1,\beta_j}$, $j = 1, \ldots, m$. Then $\Pi_{j=1}^m f_j$ belongs to $\mathscr{F}_{1,\beta_1+\cdots+\beta_m}$ and*

$$\frac{\beta_1 + \cdots + \beta_m}{\pi} \int_{\mathbf{C}} \left|\prod_{j=1}^{m} f_j(z)\right|^2 e^{-(\beta_1+\cdots+\beta_m)|z|^2} \, d\sigma(z) \leq \prod_{j=1}^{m} \left\{\frac{\beta_j}{\pi} \int_{\mathbf{C}} |f_j(z)|^2 e^{-\beta_j|z|^2} d\sigma(z)\right\}$$

with equality if and only if either $\Pi_{j=1}^m f_j = 0$ *or each* f_j $(1 \leqslant j \leqslant m)$ *is of the form*

$$f_j(z) = C_j e^{\beta_j \bar{\zeta} z}, \quad z \in \mathbf{C},$$

for some point $\zeta \in \mathbf{C}$ *and some nonzero constants* C_j $(1 \leqslant j \leqslant m)$.

The last two corollaries constitute an improvement and an extension of a recent result of Saitoh [5]. More specifically, when in the special case result of Corollary 4 we put $\beta_1 = \cdots = \beta_m = 1$, Saitoh's result is obtained. The proof in Saitoh [5] is rather difficult and is based on the theory of tensor products of reproducing kernel spaces of entire functions. Also, it appears that the equality statement in [5] contains an error or a misprint, although the core of the proof is correct. Indeed, according to [5], equality in the above inequality with $\beta_1 = \cdots = \beta_m = 1$ holds if and only if $\Pi_{j=1}^m f_j(z)$ is of the form $Ce^{m\bar{\zeta}z}$ for some $\zeta \in \mathbf{C}$ and some constant C. A counter example for this statement is as follows: We take $m = 2$, $f_1(z) = e^{\lambda\bar{\zeta}z}$ and $f_2(z) = e^{\mu\bar{\zeta}z}$ for some $\zeta \in \mathbf{C}$ and any complex numbers λ and μ with $\lambda + \mu = 2$. Then

$$f_1(z)f_2(z) = e^{2\bar{\zeta}z}$$

and

$$\|f_1f_2\|_{1,2}^2 = e^{2|\zeta|^2}, \quad \|f_1\|_{1,1}^2 = e^{|\lambda|^2|\zeta|^2}, \quad \|f_2\|_{1,1}^2 = e^{|\mu|^2|\zeta|^2}.$$

However, in general, $\|f_1f_2\|_{1,2} < \|f_1\|_{1,1}\,\|f_2\|_{1,1}$, just take $\alpha = 1/2$ and $\mu = 3/2$.

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