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TOWARDS A CLASSICAL KNOT THEORY FOR SURFACES IN $\mathbf{R}^4$

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A tried and true method of knot theory in $\mathbf{R}^3$ is to study a knot K by means of its projection onto some 2-plane. Such a projection is easy to draw, and by keeping track of which crossings (i.e., singular points of the projection) represent under- or over-crossings, we lose no information about K. In fact, the early knot theorists showed how to use projections not only to manipulate knots (e.g., Tietze moves) but also to calculate various knot invariants (e.g., π_1). More recently, J. H. Conway [1] has used the following fact to develop efficient and powerful methods of computing Alexander polynomials:

Fact 1. Any knot projection is the projection of an unknot obtained by changing overcrossings to undercrossings or vice versa as necessary.

In particular, Conway computes the effect on the Alexander polynomial as a crossing is changed, and develops a rich recursive hierarchy useful in several contexts. A careful exposition of these ideas and some applications may be found in [2].

In this paper we ask how far a similar program may be pursued for oriented surfaces knotted in $\mathbf{R}^4$. Specifically, we prove that (after isotopy) any embedded surface may be projected to an immersion in some 3-plane; we give a necessary and sufficient condition that an immersed surface in $\mathbf{R}^3$ lift to an embedding in $\mathbf{R}^4$; and we give a notion of crossing change allowing an analogue of Fact 1. The relevant Alexander invariant is then defined, and we give an example of its calculation using projections.

I would like to thank Robion Kirby for much patient advice and encouragement, John Hughes for many helpful discussions, and the referee for valuable comments and suggestions.

0. Motivation and statement of results

In classical knot theory, the embedding of a circle $f: S^1 \to \mathbf{R}^3$ is studied by means of the immersion f_0 obtained from composition with projection onto a 2-plane:

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$$\begin{array}{ccc} S^1 & \overset{f}{\hookrightarrow} & \mathbf{R}^3 \\ & \searrow^{f_0} & \downarrow \\ & & \mathbf{R}^2 \end{array}$$

For example, the operation of changing a knot crossing can be viewed as the result of first projecting to a 2-plane and then lifting the obtained immersion to a slightly different embedding (Fig. 1).

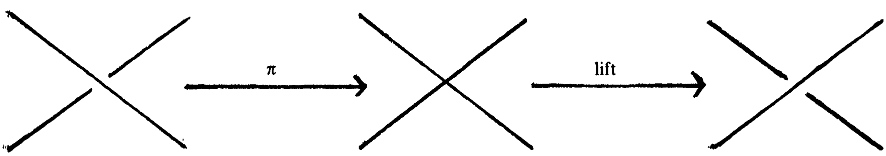


FIG. 1

We wish to carry out an analogous study of knotted surfaces in $\mathbf{R}^4$; i.e., we wish to study embeddings f: $F \to \mathbf{R}^4$ of an oriented surface F by studying the immersion f_0 obtained by composition with the projection:

$$\begin{array}{ccc} F & \overset{f}{\hookrightarrow} & \mathbf{R}^4 \\ & \searrow^{f_0} & \downarrow \\ & & \mathbf{R}^3 \end{array}$$

However, given any embedding f, it is by no means immediate that there is a suitable 3-plane from which to obtain f_0, and conversely given an immersion f_0 it is not clear when there is a corresponding embedding. Accordingly, we later give the following results in the case F is oriented:

Result 1 (Theorem 5). The embedding f may be changed by an isotopy so that projection to some 3-plane gives an immersion.

Result 2 (Section 2). There is an algorithm which determines precisely when any given immersion f_0 lifts an embedding f.

These results will be proved in Sections 2 and 3. The proof of Result 1 is surprisingly difficult.

Using the algorithm of Result 2, in Section 2.2 we give an example of an immersion in $\mathbf{R}^3$ of oriented surfaces of arbitrary genus which do not lift to an embedding in $\mathbf{R}^4$. We also note in Section 3 that Result 1 is false for non-orientable surfaces by showing that no embedding of $\mathbf{RP}^2$ in $\mathbf{R}^4$ projects to an immersion in $\mathbf{R}^3$ (and so no immersion of $\mathbf{RP}^2$ lifts to an embedding in $\mathbf{R}^4$).

Kinoshita in [6] has sketched a computation of $\pi_1(S^4 - f(S^2))$ from f_0 and the subsequent calculation of an Alexander polynomial via the Fox calculus, but must assume without proof that f_0 is in fact an immersion.

Although we now have the machinery with which to study embeddings in $\mathbf{R}^4$ via immersions in $\mathbf{R}^3$, we wish to do more. Recall the basic knot theoretic fact used by Conway that any link can be transformed to the unlink by changing a finite number of crossings, and that abelian invariants can be calculated at each crossing change. We will investigate the analogous situation for embedded surfaces in $\mathbf{R}^4$ by defining a notion of crossing change which will transform any embedding into a trivial one.

Let us first reconsider the case of a knot in $\mathbf{R}^3$. To say that a knot K_1 can be changed to another knot K_2 by changing crossings is to say that there is a regular homotopy h_t from K_1 to K_2. The stages h_t which are not embeddings correspond to crossing changes (Fig. 2).

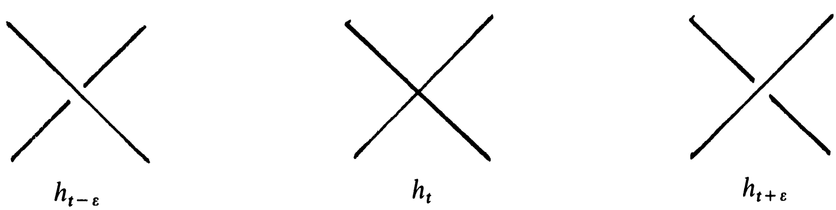


FIG. 2

Now define $H: S^1 \times I \to \mathbf{R}^3 \times I$ by $H(x, t) = (h_t(x), t)$. After making H transverse, we see that $H(S^1 \times I)$ is immersed in $\mathbf{R}^3 \times I$ and hence has only isolated point self-intersections. Each of these self-intersections corresponds to the stage in the homotopy h_t when the knot is passed through itself. Thus Fact 1 can be restated as follows: any knot can be changed to the unknot by a sequence of isotopies (corresponding to the stages of h_t between singular points of H) and passages through singular points of H (corresponding to crossing changes).

In the case of an embedding $f: F^2 \to \mathbf{R}^4$ and a regular homotopy h_t of f, we define $H: F \times I \to \mathbf{R}^4 \times I$ exactly as above. Now, $H(F \times I)$ is an immersed 3-manifold in $\mathbf{R}^4 \times I$ having as self-intersection set an immersed 1-manifold J. Thus, any regular homotopy of f can be accomplished by a series of isotopies of immersions and appearances of the 0-handles, 1-handles, and self-intersections of J.

To state the analogy of Fact 1 in this setting we use the following theorem of Hirsch [3, Theorem 8.2]:

THEOREM. *The regular homotopy classes of immersions $f_0\colon F^{2k} \to \mathbf{R}^{4k}$ are classified by the normal bundle of f_0, i.e., are in one-to-one correspondence with the 2k-plane bundles ν over F such that $\nu \oplus TF = \varepsilon^{4k}$. The correspondence is*

$$f_0 \leftrightarrow \textit{the normal bundle of } f_0(F) \textit{ in } \mathbf{R}^{4k}.$$

However, it is well known that any embedding of F^2 in $\mathbf{R}^4$ has trivial normal bundle, and so we have:

Fact 2. Any embedding of F in $\mathbf{R}^4$ can be transformed to any other by regular homotopy, or equivalently, by a series of isotopies of immersions and appearances of the 0-handles, 1-handles, and self-intersections of J. In particular, any embedding can be so transformed to an embedding in $\mathbf{R}^3 \subseteq \mathbf{R}^4$ with trivial unknotted image.

Accordingly, in Section 4, we define our notion of crossing change and prove Fact 2. A crossing change will not necessarily transform an embedding into an embedding but may introduce self-intersections. This is demanded by the fact that J is a 1-complex.

Thus we have the desired fact, but we wish to go yet further. For knots K in S^3 recall that there are standard ways of constructing the infinite cyclic cover of $S^3 - K$ given a Seifert surface for K and likewise of computing $\Delta_k(t)$. In Section 5 we show that these methods may be applied to embeddings $f\colon F \to \mathbf{R}^4$ in the case that the associated immersion f_0 has no triple points. We give an example of a computation of $\Delta(t)$ of a spun knot, and leave the case of triple points to a future investigation.

1. Preliminary definitions and conventions

This chapter is a hodge-podge of notions which will be needed later.

F will always denote a closed, oriented 2-manifold unless otherwise stated. The symbol $\cong$ will be used to denote a diffeomorphism or isomorphism, whichever is appropriate. We write $f\colon M \looparrowright N$ to indicate that f is an immersion.

DEFINITION. Let G be a graph, i.e., a union of immersed circles s_i. Then the s_i are called the *transverse components of G.*

DEFINITION (Movies). Let $A \subseteq \mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}$ and let $p\colon \mathbf{R}^4 \to \mathbf{R}$ be the projection. Write A_t for $A \cap p^{-1}(t)$. Then a *movie for A* is a sequence of pairs

$$(\mathbf{R}^3, A_{t_1}), (\mathbf{R}^3, A_{s_1}), (\mathbf{R}^3, A_{t_2}), (\mathbf{R}^3, A_{s_2}), \ldots, (\mathbf{R}^3, A_{t_{k-1}}), (\mathbf{R}^3, A_{s_{k-1}}), (\mathbf{R}^3 A_{t_k})$$

such that

(1) $p(A) \subseteq (t_1, t_k)$, and

(2) $A \cap p^{-1}(t_i, t_{i+1}) \cong A_{s_i} \times (0, 1)$, where $s_i \in (t_i, t_{i+1})$.

Thus a movie in a record of successive $\mathbf{R}^3$-sections of A. We will write

$$(\mathbf{R}^3, A_{t_1}) \to (\mathbf{R}^3, A_{s_1}) \to \cdots \to (\mathbf{R}^3, A_{t_k}).$$

Evidently, two spaces A and B are isotopic in $\mathbf{R}^4$ if and only if they have identical movies.

In most cases, our movies will be of handle additions to surfaces in $\mathbf{R}^4$. For example, Fig. 3 is a movie of a torus in $\mathbf{R}^4$.

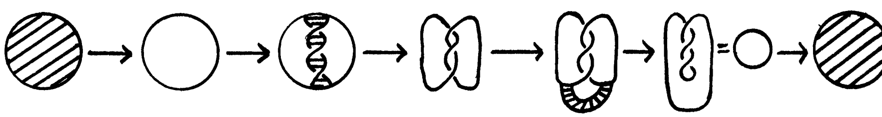


FIG. 3

We sometimes refer to the $\mathbf{R}^3$-sections as *snapshots*.

DEFINITION. Let $f_0: M \looparrowright N$ be an immersion. A homotopy of f, $h: M \times I \to N$, will often be written as f_t, where $f_t(x) = h(x, t)$. A *regular homotopy of* f_0 is such a family of maps f_t, each of which is an immersion, and such that the map $H: M \times I \to N \times I$ defined by $H(x, t) = (f_t(x), t)$ is an immersion. An *isotopy of* f_0 is a regular homotopy of f_0 such that the homeomorphism type of $(N, f_t(M))$ does not change as t varies over I.

DEFINITION. Let $f: F \hookrightarrow \mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}$ be an embedding of F, and let $\pi: \mathbf{R}^4 \to \mathbf{R}^3$ be the projection. If $f_0 = \pi \circ f$ is an immersion we say that f *projects to an immersion* and that f_0 *lifts to an embedding.*

DEFINITION (Pushing). Let A be a simplicial complex embedded in F with regular neighborhood N, and let $f_0: F \looparrowright \mathbf{R}^3$ be an immersion. Let $\gamma: A \to \mathbf{R}$ be any continuous function, and extend γ to F by setting $\gamma(F - N) = 0$ and extending over N such that $\gamma(\partial N) = 0$. Then we say that the map $f: F \to \mathbf{R}^4$ defined by $f(x) = (f_0(x), \gamma(x))$ is the result of *pushing A into $\mathbf{R}^4$ according to γ.* Note that $\gamma(x)$ may be positive, negative, or zero. A *non-zero push of A* (respectively, *positive, negative push*) will be a push in which $\gamma \neq 0$ (respectively, $\gamma > 0$, $\gamma < 0$).

DEFINITION. Given an immersion $f_0: M \looparrowright N$, let

$$S_i = \{x \in M \mid f_0^{-1}(f_0(x)) \text{ contains at least } i \text{ points}\}$$

be the *ith singular set of* f_0, and let $S = \bigcup_{i>1} S_i$ be the *singular set of* f_0. Note that $S_1 = M$. We call $f_0(S_i) = \bar{S}_i$ the *ith singular value set* and $f_0(S) = \bar{S}$ the *singular values of* f_0.

All immersions will be assumed transverse; i.e., each S_i is assumed immersed in M and hence each $\bar{S}_i$ immersed in N. In fact, each S_i is transversally immersed in S_{i-1} (i.e., the inclusion $S_i \hookrightarrow S_{i-1}$ is transverse) and hence $\bar{S}_i$ is transversally immersed in $\bar{S}_{i-1}$ (although not in N). For $f_0(M)$ has transverse self-intersections, and so each $x \in \bar{S}_i$ is contained in a neighborhood $B \subseteq N$ such that

$$(B, \bar{S}_i, x) \cong (B^n, X_i \cap B^n, 0)$$

where B^n is the unit ball in $\mathbf{R}^n$ and X_i is the union of i coordinate planes of dimension m. But then the pullback under f_0 of the pair

$$(\bar{S}_{i-1} \cap B, \bar{S}_i \cap B) = (S_{i-1} \cap f_0^{-1}(B), S_i \cap f_0^{-1}(B))$$

must be homeomorphic to the corresponding pullback for the standard immersion (B^n, X_i), and hence must be a transverse pair. Since any point in S_i lies in some $f_0^{-1}(x)(x \in \bar{S}_i)$, S_i must be transversally immersed in S_{i-1} as claimed.

2. Lifting immersions to embeddings

In this chapter we give an algorithm which determines precisely when a given immersion of an oriented surface in $\mathbf{R}^3$ lifts to an embedding in $\mathbf{R}^4$.

1. *A Lifting Criterion.* Let $f_0: F \looparrowright \mathbf{R}^3$ be an immersion of the closed oriented 2-manifold F with singular set $S \subseteq F$ and singular values $\bar{S} = f_0(S)$. We will assume that S and $\bar{S}$ are immersed closed 1-manifolds in F and $\mathbf{R}^3$ respectively, and that S is transversally immersed. Then each point x in a transverse component $\bar{s}$ of $\bar{S}$ has two distinct preimages (although the preimage of $\bar{s}$ may have only a single component).

Furthermore, each point $x \in \bar{s}$ has one of the neighborhoods $N(x) \subseteq f_0(F)$ immersed in $\mathbf{R}^3$ shown in Fig. 4.

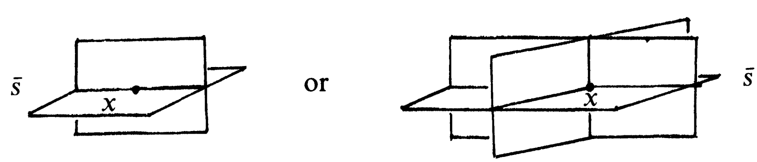


Fig. 4

Thus a neighborhood of $\bar{s}$ in $f_0(F)$ is an immersed copy of

$$[0, 1] \times X/(0, X) \sim_\alpha (1, X),$$

where $X = \{(x, y) \in \mathbf{R}^2 \mid xy = 0 \text{ and } x^2 + y^2 \leq 1\}$ and α is some homeomorphism of X.

For example, if $\bar{s}$ is a figure eight, then a possible $N(\bar{s})$ is given in Fig. 5.

Since F is oriented, it is easy to check that α must be the identity, so that $N(\bar{s})$ is an immersed $S^1 \times X$. Thus $N(\bar{s})$ is the union of two distinct immersed

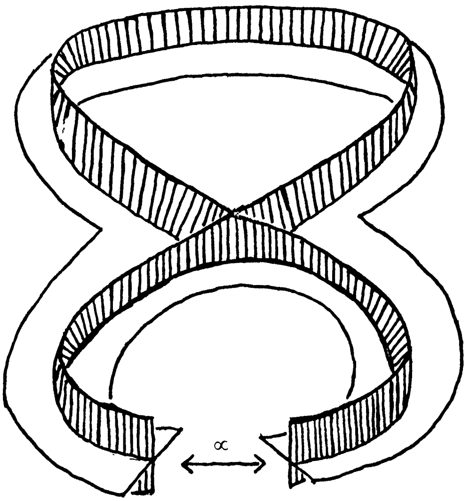


FIG. 5

annuli $A_{\bar{s}}$ and $A'_{\bar{s}}$, and $A_s = f_0^{-1}(A_{\bar{s}})$ and $A_{s'} = f_0^{-1}(A'_{\bar{s}})$ are distinct annuli immersed in F. Let $A = \bigcup_s A_s \cup A'_s$ be the regular neighborhood of S in F. Since s (resp. s') is the core of A_s (resp. A'_s), we have:

COROLLARY 3. *For all transverse components s of S, $s \neq s'$ (i.e., each transverse component of $\bar{S}$ has two distinct preimages in S).* ∎

Now, S is immersed in F and so is a 1-complex with vertices $a_1, \ldots, a_k$ and edges e_{i_j} joining a_i and a_j. Let e'_{ij} be the edge identified to e_{ij} by f_0, and denote the endpoints of e'_{ij} by a'_i and a'_j. Then Corollary 3 shows that $e_{ij} \neq e'_{ij}$ and that e_{ij} and e'_{ij} may intersect only at their endpoints.

Our algorithm is based on the following proposition:

PROPOSITION 4. *There is an embedding $f\colon F \to \mathbf{R}^4$ covering a given immersion f_0 if and only if there is a 1-1 function $\gamma\colon \{a_1, \ldots, a_k\} \to \mathbf{R}$ such that*

$$(1) \qquad \gamma(\partial e_{ij}) \geq \gamma(\partial e'_{ij}) \quad \text{or} \quad \gamma(\partial e_{ij}) \leq \gamma(e'_{ij});$$

i.e., γ is greater on both endpoints of one edge than on either endpoint of the other.

Proof. Define $h\colon S - \{a_1, \ldots, a_k\} \to S - \{a_1, \ldots, a_k\}$ by $h(x) = y$ if and only if $f_0(x) = f_0(y)$. Note that $h(e_{ij}) = e'_{ij}$.

Suppose γ exists. For convenience write e for e_{ij} and e' for e'_{ij}. We will show that γ may be extended to e and e' so that for any $x \in \mathring{e}$, $\gamma(x) > \gamma \circ h(x)$ (resp. for all $x \in \mathring{e}$, $\gamma(x) < \gamma \circ h(x)$).

Now, e and e' are either homeomorphic to intervals or circles and may intersect only at their vertices. If e and e' are disjoint, then (1) ensures that γ

can be extended as desired. The only remaining cases are the four in Fig. 6. But in each case it is easy to see that (1) again ensures that γ exists. In particular, for any $x \in \mathring{e}$, $\gamma(x) \neq \gamma \circ h(x)$. Thus if $x \in e$, $y \in e'$ with $f_0(x) = f_0(y)$, then $\gamma(x) = \gamma(y)$ unless $x = y \in e \cap e'$.

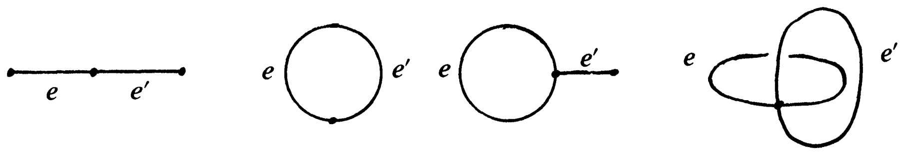


FIG. 6

Now push S into $\mathbf{R}^4$ according to γ, to obtain a map $f = (f_0, \gamma)$ (see Section 1). We claim that f is 1-1. For suppose that x and y are distinct points of F with $f_0(x) = f_0(y)$. If $x \in \mathring{e}$ for some e as above, then $y \in \mathring{e}$, and so by the previous discussion $\gamma(x) \neq \gamma(y)$; thus $f(x) \neq f(y)$. If, however, x is a vertex of S then so is y; let z be the third point in $f_0^{-1}(f_0(x))$. Then locally near x, y, and z, S appears as three crosses identified to $\bar{S}$, shown in Fig. 7, where the edges labeled 1 (resp. 2, 3) are identified by f_0. Clearly there is an e as above with $x \in \partial e$ and with $y \in \partial e'$. But then the preceding argument shows $\gamma(x) \neq \gamma(y)$. In all cases, then, $f(x) \neq f(y)$, and so f is the desired 1-1 embedding covering f_0.

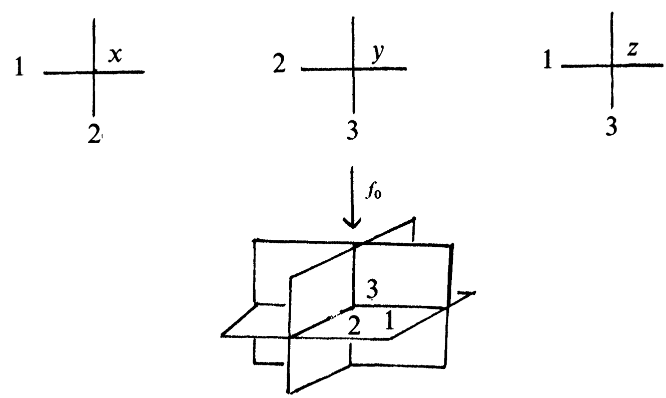


FIG. 7

Remark. γ is nothing more than the restriction to $f(F)$ of the projection $p: \mathbf{R}^4 \to \mathbf{R}$. This part of the proposition shows that specifying a γ on S satisfying the consistency conditions in (1) gives the embedding f.

DEFINITION. An embedding (resp. immersion) arising in this fashion from γ will be said to be a γ-embedding (resp. γ-immersion).

We now prove the remaining implication of Proposition 4. Let $f: F \to \mathbf{R}^4$ be an embedding covering f_0, and define $\gamma: F \to \mathbf{R}$ by $f = (f_0, \gamma)$. Let $\{a_1, \ldots, a_k\}$, e, and e' be as before.

First note that we may perturb γ slightly with the effect of changing f and f_0 by a small isotopy. Thus we may assume that γ is 1-1 on $\{a_1, \ldots, a_k\}$.

Now, e is either homeomorphic to the interval $[0, 1]$ or the circle $\{z \in \mathbf{C}: |z| = 1\}$. Define a function $p: I \to e$ to be this homeomorphism in the first case and to be $e^{2\pi i x}$ otherwise. Similarly define $p': I \to e'$, and arrange that $h \circ p = p'$ on $(0, 1)$. Now let $q = \gamma \circ p$ and $q' = \gamma \circ p'$. Then q and q' are functions on I which agree nowhere on $(0, 1)$ since f is 1-1. Therefore $q(\partial I) \geq q'(\partial I)$ or $q(\partial I) \leq q'(\partial I)$. Furthermore, $\gamma \circ p(\partial I) = \gamma(\partial e)$ and $\gamma \circ p'(\partial I) = \gamma(\partial e')$, and so γ satisfies (1) as claimed. ∎

Remark. The orientability of $N(S)$ in F is essential. For example, $\mathbf{RP}^2$ immerses in $\mathbf{R}^3$ as Boy's surface, and can be shown to have no lifting by noting that $\bar{S}$ is homeomorphic to the curve in Fig. 8, and that $N(\bar{S})$ is an immersed $[0, 1] \times X/(X, 0) \sim_\alpha (X, 1)$, where α rotates $X = +$ by 90°. Then each of the loops in S is double covered by a loop l in S composed of edges e and e' (Fig. 9).

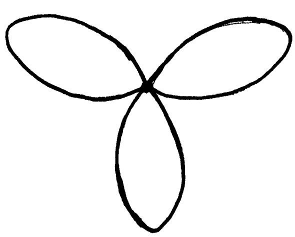


FIG. 8

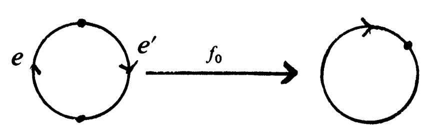


FIG. 9

But $\gamma: l \to \mathbf{R}$ must agree on two antipodal points by the Borsuk-Ulam Theorem, and so $f = (f_0, \gamma)$ cannot be an embedding.

In fact, we show in Section 3 that no immersion of $\mathbf{RR}^2$ in $\mathbf{R}^3$ lifts to an embedding in $\mathbf{R}^4$. However, the standard immersion of the Klein Bottle in $\mathbf{R}^3$ is an example of an immersed non-orientable surface which does lift to an embedding.

2. *The algorithm.* We now sketch an algorithm which determines precisely when a given immersion f_0 lifts to an embedding. Label the double points of S as $a_1, \ldots, a_k$ as usual and pick a transverse component s of S. The identification of s with s' induces an identification of some of the vertices a_i with a_i'. Now choose either $<$ or $>$ and for all such i write $a_i < a_i'$ or $a_i > a_i'$ accordingly. Continue for all such pairs of transverse components of S, choosing $<$ or $>$ at will. To the resulting set of inequalities add $\{a_i \neq a_j$

$| i \neq j\}$. Then a solution to all of these relations is a choice of γ determining a lifting; conversely, if no solution exists for any of the (finite) choices of $<$ and $>$ made along the way, then no γ exists and f_0 does not lift. But the solvability of such a set of inequalities can clearly be accomplished recursively.

Example. Fig. 10 is S for an immersion of the sphere in $\mathbf{R}^3$ [5]. Starting with a_1 and $a_3 = a_1'$, we write

$$a_1 > a_3,\ a_2 > a_4,\ a_2 > a_6,\ a_4 > a_6,\ a_3 > a_5,\ a_1 > a_5,$$

and also $a_i \neq a_j$, all $i, j = 1, \ldots, 6$. But these relations have a solution (e.g., $a_i = 6 - i$) and so this immersion lifts to an embedding.

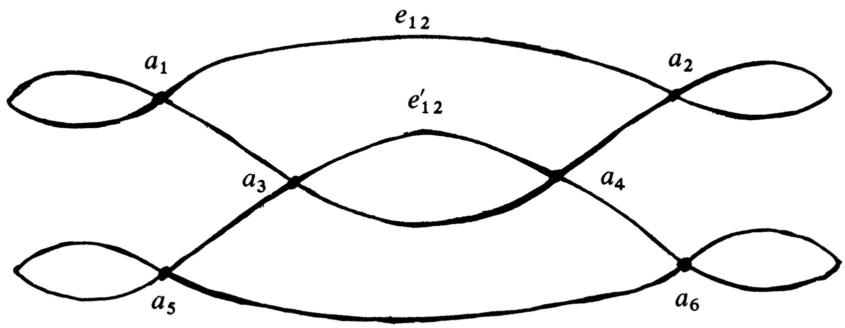


FIG. 10

Remarks. 1. If f_0 is such that each s and s' lie in different components of S, then f_0 lifts; simply define γ to be n on the nth component of S and adjust γ slightly within each component to make γ 1-1 on $\{a_1, \ldots, a_k\}$. In particular, any immersion with no triple points lifts to an embedding.

2. We will see in Section 4.4 that every f_0 lifts to an immersion in $\mathbf{R}^4$ regularly homotopic to an embedding.

Example. We construct an immersion of S^2 in $\mathbf{R}^3$ which does not lift to an embedding of S^2 into $\mathbf{R}^4$ (by forming a connected sum of this example with a standardly embedded genus g surface, we can produce a like example of arbitrary genus).

A well known immersion of $\mathbf{RP}^2$ in $\mathbf{R}^3$ is Boy's surface (for a photograph, see [4], p. 320). Its normal 0-sphere bundle in $\mathbf{R}^3$ is an immersion of its double cover, and hence an immersed S^2; our example is this immersion f_0: $S^2 \looparrowright \mathbf{R}^3$. We may obtain $f_0(S^2)$ from Boy's surface by forming a "double pushoff" (Fig. 11). Thus to draw the singular value set of $f_0(S^2)$, we first draw the singular values of Boy's surface (Fig. 12).

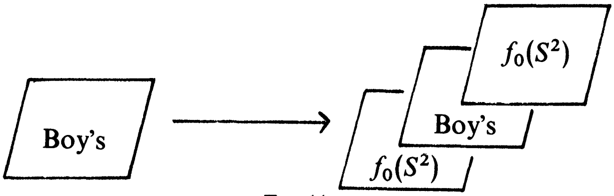
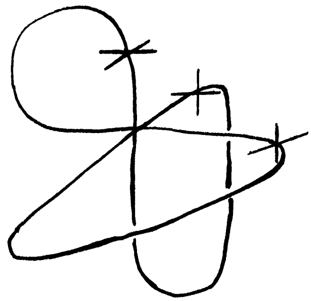


FIG. 11



where

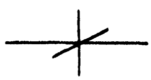
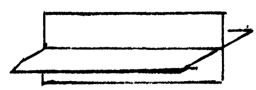
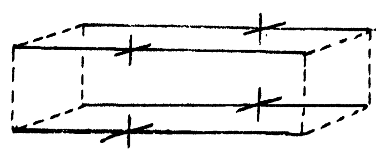
 denotes 

FIG. 12

Forming a double pushoff changes

 to 

Note also that a double pushoff of a neighborhood of the single triple point of Boy's surface yields the eight triple points of $f_0(S^2)$, which we label in Fig. 13.

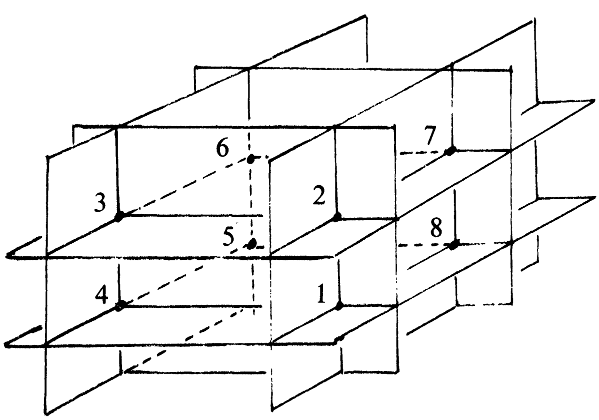


FIG. 13

The reader may now carefully draw the resulting singular set, homeomorphic to Fig. 14.

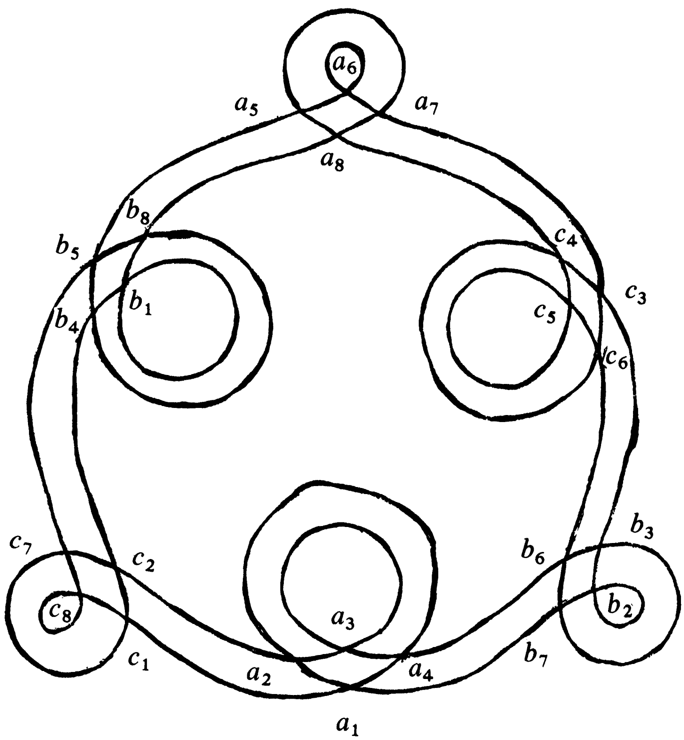


FIG. 14

Here for each i, a_i, b_i, and c_i correspond to the same triple point (viz., the one labelled i as above). Thus segment a_1c_1 is identified by f_0 to segment b_1b_1, c_7b_5 to a_7a_5, b_8b_4 to a_8c_4, etc. A list of all inequalities arising from the algorithm gives, in particular,

$$c_1 > b_8 > a_5 > c_6 > b_3 > a_2 > c_1;$$

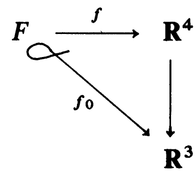
this contradiction then shows that f_0 lifts to no embedding of S^2 into $\mathbf{R}^4$ as claimed.

We close this chapter with a question. Is there an intrinsic characterization of those immersions of oriented surfaces in $\mathbf{R}^3$ which do lift to embeddings in $\mathbf{R}^4$?

3. Projecting embeddings to immersions

In order to understand knotted surfaces in $\mathbf{R}^4$ via immersions in $\mathbf{R}^3$, we must know that any such embedding may be projected to an immersion. More precisely, the object of this chapter is to prove the following theorem:

PROJECTION THEOREM 5. *Let $f: F \to \mathbf{R}^4$ be an embedding of an oriented closed surface F. Then f may be isotoped so that composition with the projection $\Pi: \mathbf{R}^4 \to \mathbf{R}^3$ is an immersion f_0:*



Before giving the proof, we need several lemmas.

LEMMA 6. *Let $f: F \to \mathbf{R}^4$ be an arbitrary embedding (resp. immersion) covering an immersion as above. Then f is isotopic to a γ-embedding (resp. γ-immersion) for some $\gamma: S \to \mathbf{R}$.*

Proof. Define $\gamma_1: F \to \mathbf{R}$ as in Section 2 by setting $f = (f_0, \gamma_1)$. Let A be a regular neighborhood of S in F, and deform γ_1 to $\gamma: F \to \mathbf{R}$ so that

(1) $\gamma|_S = \gamma_1|_S$, and
(2) $\gamma(F - A) = 0$.

Then (f_0, γ) is a γ-embedding (resp. γ-immersion) of F into $\mathbf{R}^4$ covering f_0. Furthermore, since f_0 has no self-intersections away from A, the deformation of γ_1 to γ can be accomplished via an isotopy of f (intuitively we have pushed $F - A$ into $\mathbf{R}^3$). ∎

LEMMA 7. *Let $F_0 \subseteq \mathbf{R}^3 \times [-1, 1]$ be an oriented surface with boundary such that $\partial F_0 \subseteq \partial(\mathbf{R}^3 \times [-1, 1])$. Suppose $\phi_t: \mathbf{R}^3 \times 1 \to \mathbf{R}^3 \times 1$ is an isotopy fixing each point of $F_0 \cap (\mathbf{R}^3 \times 1)$ with ϕ_0 the identity. Then ϕ_t extends to an isotopy*

$$\bar{\phi}_t: \mathbf{R}^3 \times [-1, 1] \to \mathbf{R}^3 \times [-1, 1]$$

fixing F_0 pointwise.

Proof. We may assume $F_0 \cap (\mathbf{R}^3 \times [0, 1]) \simeq (F \cap (\mathbf{R}^3 \times 1)) \times I$, i.e., that F_0 is just a collar on $F_0 \cap (\mathbf{R}^3 \times 1)$ between $s = 0$ and $s = 1$. Define the isotopy $\bar{\phi}_t$ on $\mathbf{R}^3 \times [0, 1]$ by $\bar{\phi}_t(x, s) = (\phi_{ts}(x), s)$. When $s = 0$, $\bar{\phi}_t = \phi_0$ is the identity, so that $\bar{\phi}_t$ extends to $\mathbf{R}^3 \times [-1, 1]$ as desired. ∎

The following geometric lemma allows us to transfer twists between bands in an $\mathbf{R}^3$-slice of a surface in $\mathbf{R}^4$ via an isotopy of the embedding.

TRANSFER LEMMA 8. (1) *Suppose a movie for $f: F \to \mathbf{R}^4$ contains the series of snapshots in Fig. 15. Then f may be isotoped so that this sequence appears as in Fig. 16.*

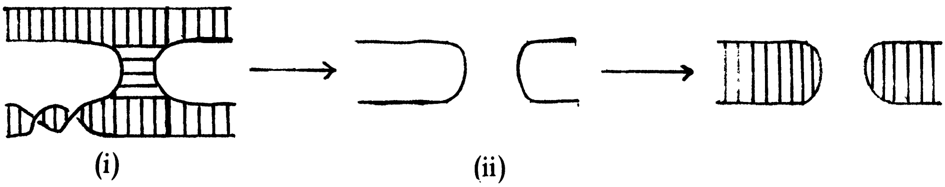


FIG. 15

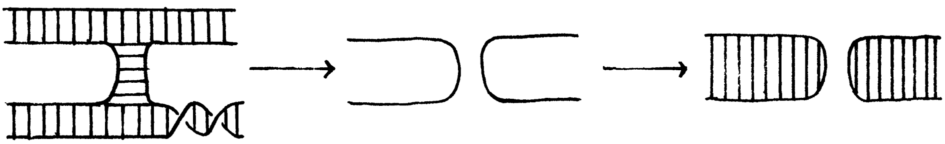


FIG. 16

(2) *Similarly, the sequence in Fig. 17 can be changed to the sequence in Fig. 18.*

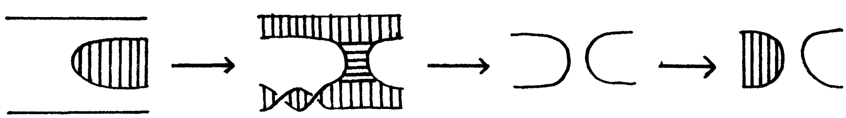


FIG. 17

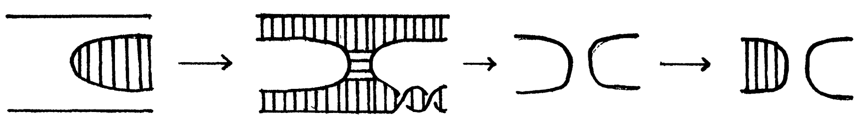


FIG. 18

Proof. We prove only (1), since the construction for (2) is similar. First, isotope f so that the vertical band in level (i) is pushed to (ii). Our movie is then given in Fig. 19.

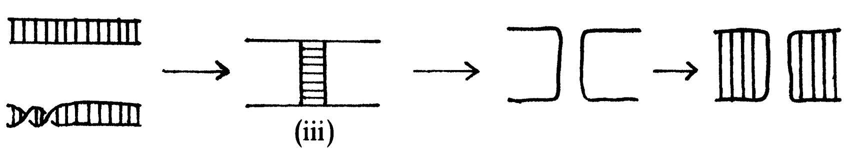
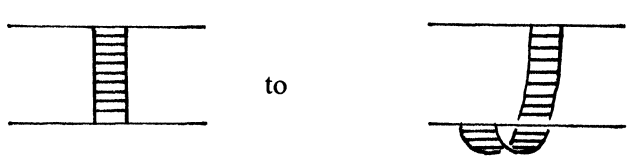


FIG. 19

Let F_0 be the part of F up to and including level (iii). Then the isotopy of $\mathbf{R}^3$ taking



in level (iii) extends by Lemma 7 to $(\mathbf{R}^4, F_0)$. Thus a movie at these levels now looks like Fig. 20.

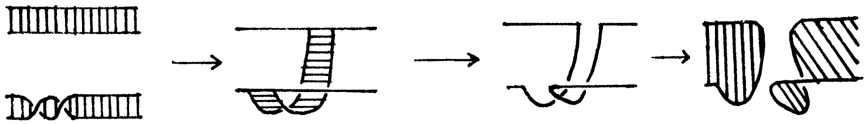


FIG. 20

Pushing the vertical band in (v) back to (iv), we get Fig. 21. Finally, this sequence is isotopic to the sequence in Fig. 22. But this last isotopy can be performed without changing the movie previous to these snapshots (i.e., without moving the lines labeled l), so that Lemma 7 shows we may extend to an isotopy of f. ▮

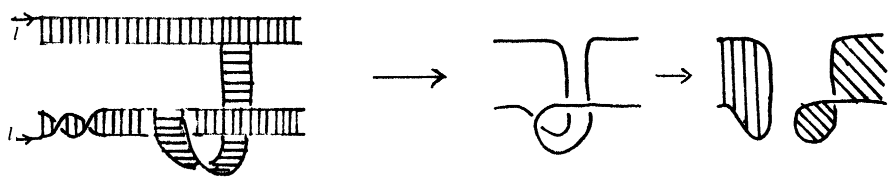


FIG. 21

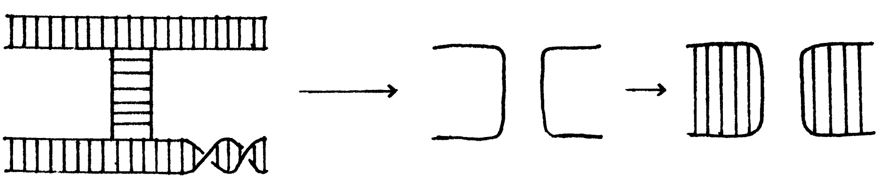


FIG. 22

Proof of Projection Theorem. Let $p: \mathbf{R}^4 \to \mathbf{R}$ be the projection and isotope f so that $\gamma = p \circ f$ is Morse on F. We may assume that $\gamma(F) \subseteq [0, 1]$ with all critical points of index 0 (resp. index 2) lying in $\gamma^{-1}(0)$ (resp. $\gamma^{-1}(1)$). Then the movie of F with respect to γ is a sequence of snapshots in $\mathbf{R}^3$ beginning with the appearance of some disks (0-handles of F), then the appearance of some bands (1-handles) so that the boundary of the result is an unlink, and finally the appearance of some disks D_i (2-handles) capping off this unlink. (Recall the example of a torus in $\mathbf{R}^4$ given in Section 1. Notice that the projection of this torus to $\mathbf{R}^3$ has a standard, unknotted image.)

Clearly $\pi \circ f$ always immerses the 0 and 1-handles of F in $\mathbf{R}^3$, with only ribbon intersections. Let F_0 be this immersed surface, so that ∂F_0 is the unlink c with unknotted components $c_i = \partial D_i$. We may extend $\pi \circ f$ to an immersion over the 2-handles D_i if a collar of c_i in $F_0 \subseteq \mathbf{R}^3$ is untwisted, i.e., if the curve c_i^+ obtained by pushing c_i into F_0 links c_i geometrically zero times (Fig. 23). If the collar is twisted, projecting D_i to $\mathbf{R}^3$ will introduce non-immersed cusps. So, we need to know that all collars are untwisted.

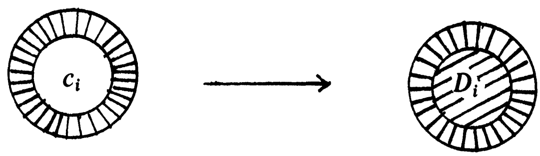


FIG. 23

Case 1. c has one component. Let $l(c, c^+)$ be the linking number of c and c^+. Since F_0 is an oriented simplicial complex in $\mathbf{R}^3$ with boundary c,

$$l(c, c^+) = \#(c^+, F_0)$$

where $\#(c^+, F_0)$ is the signed intersection number of c^+ and F_0.

But F_0 is built by adding bands to disks and allowing only ribbon intersections, and c^+ intersects F_0 only near these intersections (Fig. 24).

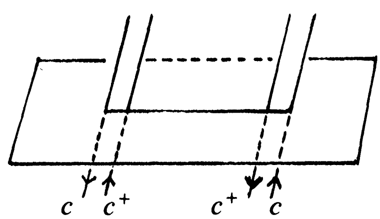


FIG. 24

The net contribution of each ribbon intersection to $\#(c^+, F_0)$ is seen to be zero, and so c^+ is algebraically unlinked from c. But since c^+ is a pushoff of the unknot c, the two knots are geometrically unlinked and the collar is untwisted. Thus we may cap off to an immersion with the 2-handle.

Case 2. c has two components c_1 and c_2. Recall that Lemma 6 shows that f is isotopic to a γ-embedding. Likewise, we may isotope $f|_{F-(D_1 \cup D_2)}$ fixing $\partial(F - (D_1 \cup D_2))$ to arise in the same fashion from some γ; clearly this isotopy extends to one of f. Thus in a movie for f we see $F_0 - A$ appear in the $\mathbf{R}^3 \times 0$ level, followed by a copy of A pushed into $\mathbf{R}^4$ by γ. The boundary of the resulting surface is the unlink c lying in an $\mathbf{R}^3$-level, which is capped off as before by $D_1 \cup D_2$ (Fig. 25).

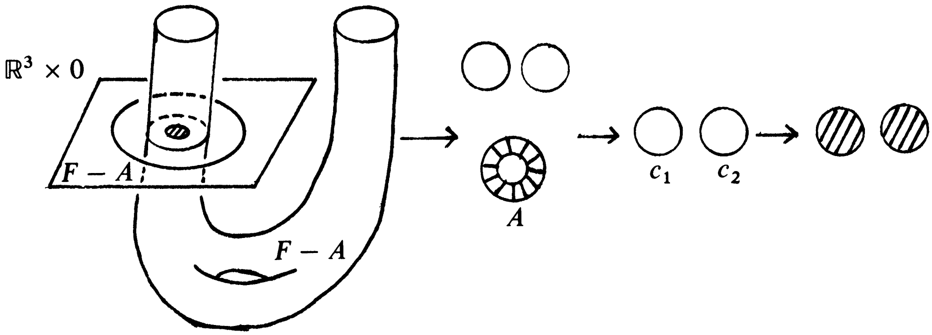


FIG. 25

Let $\omega(t)$ be a path from $p_1 \in c_1$ to $p_2 \in c_2$ on $f(F - (D_1 \cup D_2))$ such that $\pi \circ \omega$ is an embedded arc missing the triple points. Note that $\omega(t)$ lies in $\mathbf{R}^3 \times 0$ except near c_i and near S, where it is pushed into $\mathbf{R}^4$ according to γ (Fig. 26). Let $\varepsilon > 0$ be small and let P be a regular neighborhood in $f(F)$ of $\omega([0, 1 - \varepsilon])$, so that

$$P \simeq \omega([0, 1 - \varepsilon]) \times I.$$

The parts of P near S (i.e., $P \cap F(A)$) are pushed into $\mathbf{R}^4$ by γ; this push can be extended to all of P so that $\mathring{P} \cap (\mathbf{R}^3 \times 0)$ is a disjoint union of bands b_i (Fig. 27).

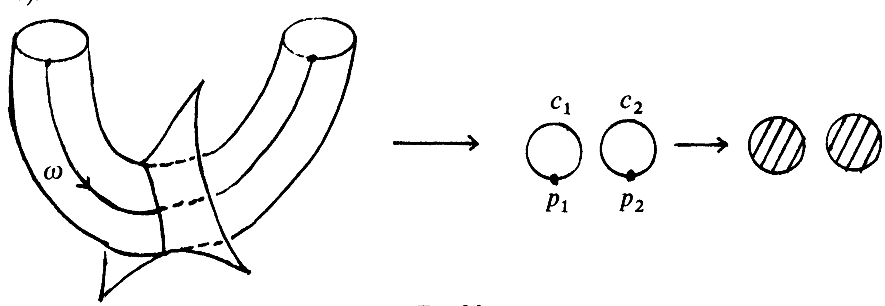


FIG. 26

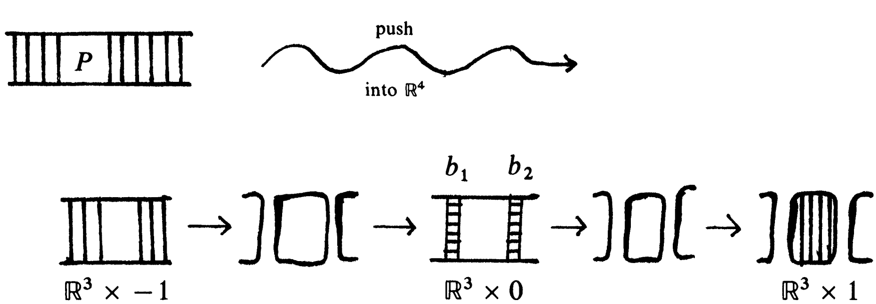
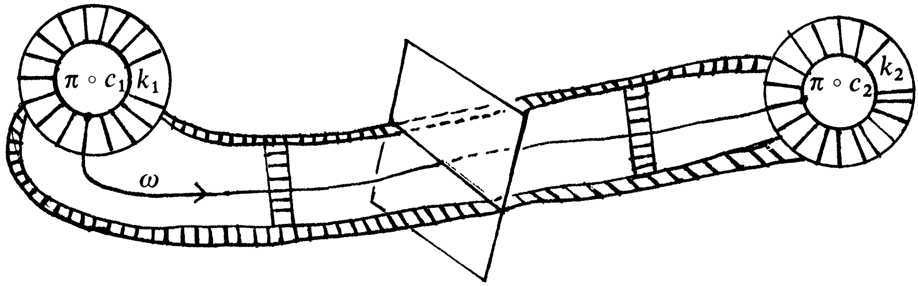


FIG. 27

Thus between $\pi \circ c_1$ and $\pi \circ c_2$ in $\mathbf{R}^3 \times 0$ we have a sequence of gaps and bands along the track of ω. The collars of c_i in F_0 may be twisted (Fig. 28). Furthermore, we may push parts of F into $\mathbf{R}^4$ so that $F \cap (\mathbf{R}^3 \times 0)$ near each band looks like Fig. 29.



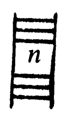
where  denotes n full twists

FIG. 28

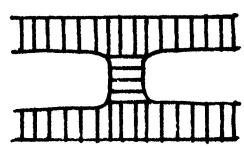


FIG. 29

The Transfer Lemma can now be used to isotope f so that the k_1 twists about c_1 are transferred to the collar about c_2 (Fig. 30).

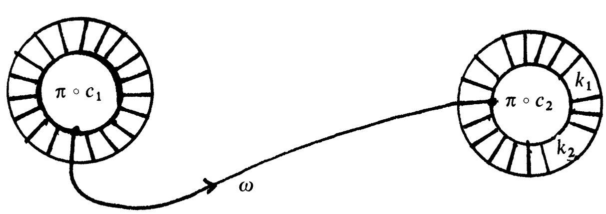


FIG. 30

However, an argument as in Case 1 shows that if c^+ is the link obtained from pushing c into F_0, then $l(c, c^+) = k_1 + k_2 = 0$. But then there is no twisting about either collar and so we may project D_1 and D_2 into $\mathbf{R}^3 \times 0$ to an immersion.

Remark. There are movies for the k-twist spun knot of the trefoil in which c has two components and in which $k_1 = -k_2 = 2k$.

Example. Fig. 31 shows a movie of an S^2 in $\mathbf{R}^4$. The projection into $\mathbf{R}^3$ of this S^2 up to level (i) is given in Fig. 32, so that $k_1 = -k_2 = 1$. Fig. 33 is a sequence of movies showing the required isotopy of f.

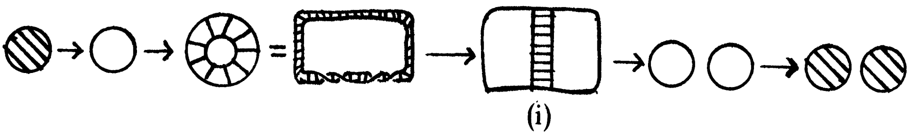


FIG. 31

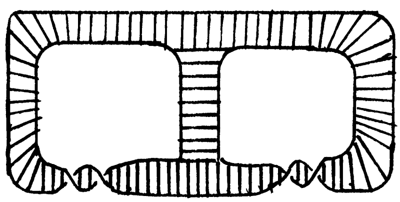


FIG. 32

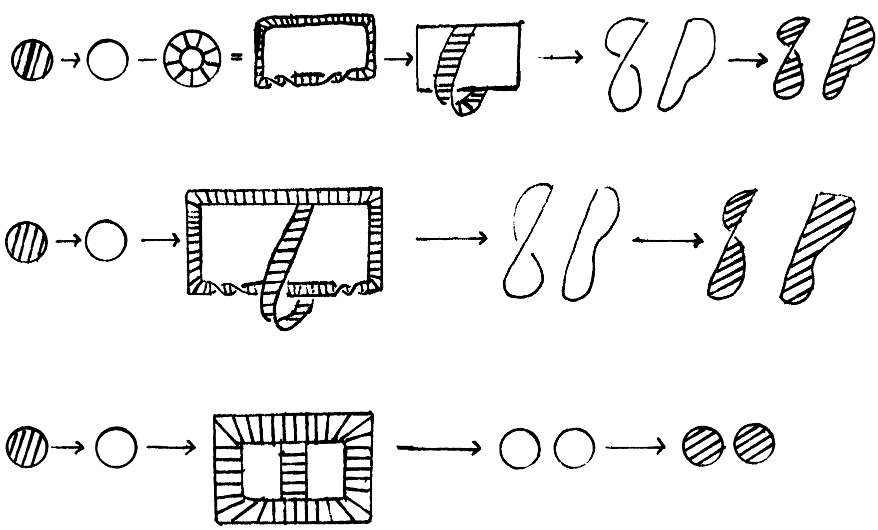


FIG. 33

This new movie projects to an immersion and in fact to an embedded S^2 in $\mathbf{R}^3$. Thus our original S^2 was unknotted.

Case 3. c has more than two components. As in Case 2, $l(c, c^+) = 0$ so that the sum of the twisting numbers is zero. Furthermore, if the twistings about components c_1 and c_2 are k_1 and k_2 respectively, then f may be isotoped as above so that the twistings are 0 and $k_1 + k_2$. Repeating as necessary using the various components of c, we see that we can change all twistings to zero. Thus we may extend $\pi \circ f$ over all the 2-handles to an immersion. ∎

Remark. The Projection Theorem is false for non-orientable surfaces. For example, no embedding f of $\mathbf{RP}^2$ in $\mathbf{R}^4$ projects to an immersion in $\mathbf{R}^3$. To see this, let N be the normal class of f; then formula 8.4 of [11, p. 113] states that

$$N \equiv 2\chi(\mathbf{RP}^2) \pmod 4$$

(Whitney's proof of this is both elegant and geometric). Since the Euler number of $\mathbf{RP}^2$ is $+1$, N must be non-zero. However, if f projected to an immersion f_0, then the normal bundle ν_f of f must split as $\nu_{f_0} \oplus \varepsilon^1$. Then ν_f has a non-zero section, forcing $N = 0$.

In particular, no immersion of $\mathbf{RP}^2$ in $\mathbf{R}^3$ (e.g., Boy's surface) lifts to an embedding in $\mathbf{R}^4$, a fact which was claimed in Section 2.1.

4. The unknotting theorem

As mentioned in Section 0, the Smale–Hirsch Theorem shows that any embedding f of a surface in $\mathbf{R}^4$ is regularly homotopic to a trivial unknotted embedding. In this chapter we define the appropriate notion of crossing change for surfaces and show that in fact any embedded surface can be rendered trivial by a sequence of isotopies and crossing changes.

Our program is as follows. Any immersion of an oriented surface in $\mathbf{R}^3$ is regularly homotopic to one of four simple immersions (Section 4.3) and furthermore any regular homotopy is covered by a sequence of isotopies and our crossing changes (Section 4.2). But if f covers one of these immersions, then f itself is unknotted.

1. *Regular homotopies of codimension one immersions.* In this section we show how the critical points of a certain Morse function enable us to systematically classify the action of any regular homotopy on a given immersion f_0: $F^k \looparrowright \mathbf{R}^{k+1}$.

Let $h: F^k \times I \to \mathbf{R}^{k+1}$ be a regular homotopy of f_0 and define

$$H: F \times I \to \mathbf{R}^{k+1} \times I$$

by $H(x, t) = (h(x, t), t)$. Note that H is then an immersion. Let S_i be the ith singular set of H, so that S_i is the image in $F \times I$ of an immersion

$$\sigma_i: \Sigma_i \to F \times I$$

of a manifold Σ_i. Clearly $S = \bigcup_{i>1} S_i$ may be isotoped within $F \times I$ so that the projection $g\colon F \times I \to I$ is Morse on each S_i, $i > 1$ (i.e., so that $g \circ \sigma_i$ is Morse on Σ_i).

Remark. That S may be so isotoped can be seen by noting that g can be perturbed to be Morse on each S_i by the addition of a linear term $\Sigma\, a_i x_i$ as in [7]. But there is a diffeomorphism α of $F \times I$ such that

$$\begin{array}{ccc} F \times I & \xrightarrow{\ \alpha\ } & F \times I \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle g+\Sigma a_i x_i} & \\ & I & \end{array}$$

commutes, and in fact α is the end result of an isotopy of $F \times I$. Restricting this isotopy to S gives the desired deformation of S within $F \times I$.

The following lemma (with $M = F \times I$, $N = \mathbf{R}^{k+1}$, and $f_0 = H$) shows that this isotopy of S is the result of an isotopy of H.

LEMMA 9. *Let $f_0\colon M \looparrowright N$ be an immersion with singular set $S \subseteq M$, and let $\psi_t\colon S \to M$ be an isotopy of S in M fixing $\partial S \subseteq \partial M$ (i.e., each ψ_t is a homeomorphism). Then there is an isotopy f_t of f_0 such that the singular set of f_t is $\psi_t(S)$ (i.e., ψ_t is the result of isotopy of f_0).*

Proof. S is the image of a transverse immersion and so has a regular neighborhood P in M which is an embedded manifold with boundary. The isotopy ψ_t can be extended to one of P, and by the Isotopy Extension Theorem can be extended to an isotopy $\phi_t\colon M \to M$ of the identity of M. Thus the following diagram commutes:

$$\begin{array}{ccccccccc} S \times 0 & \longrightarrow & P \times 0 & \longrightarrow & M \times 0 & & & & \\ \downarrow & & \downarrow & & \downarrow & & & & \\ S \times I & \longrightarrow & P \times I & \longrightarrow & M \times I & \xrightarrow{\ \Phi\ } & M & \looparrowright & N \\ & & & & \Psi\colon S \times I \to M & & & & \end{array}$$

where $\Phi(x, t) = \phi_t^{-1}(x)$ and $\Psi(x, t) = \psi_t(x)$. Then $f_t = f_0 \circ \phi_t^{-1}$ is the desired isotopy. ∎

Thus assuming $S \subseteq F \times I$ is transverse to $F \times t$ near $\partial(F \times I)$ (which we may do since h is assumed transverse), we may isotope H so that each S_i is transverse to $F \times t$ except at a finite number of t and so that each $g \circ \sigma_i$ is Morse. But then $\bigcup_{i>1}$ (critical values of $g \circ \sigma_i$) is a finite set $t_1, \ldots, t_k$ in I.

PROPOSITION 10. *Let h_s be an immersion $h_s(x) = h(x, s)$. Then after isotopy of H, we have the following.*

(1) *Let $s \in (t_i, t_{i+1})$. Then h: $F \times (t_i, t_{i+1}) \to \mathbf{R}^{k+1}$ is an isotopy of $h_s(F)$ (i.e., $h_s(F)$ has the same homeomorphism type for any such s).*

(2) *Let $s \in (t_i - \varepsilon, t_i + \varepsilon) = I_i$ for ε small. Then there is a ball $D^{k+1} \subset F \times I$ centred at the critical point x_i corresponding to t_i, a ball $\bar{D}^{k+1} \subset \mathbf{R}^{k+1}$ centred at $h_{t_i}(x_i)$, and a ball $U^k \subset F^k$ such that*

$$D \cap (F \times s) = D \cap (U \times s)$$

and

$$(D, (D \cap S) \cup (D \cap (U \times s))) \simeq (\bar{D}, h_s(F) \cap \bar{D}).$$

In other words, the critical point structure of S gives the local action of the homotopy h_t (Fig. 34).

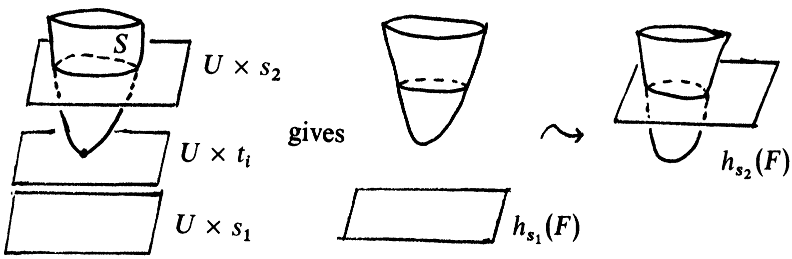


FIG. 34

Proof. (1) No $g \circ \sigma_j$ has any critical values in (t_i, t_{i+1}), and so

$$S_j \cap g^{-1}(t_i, t_{i+1})$$

is a product homeomorphic to (jth singular set of h_s) $\times \mathring{I}$. Thus at no stage of the homotopy h_s is the homeomorphism type of S_j changed.

(2) Note that dim $S = k =$ dim F. Fix $s_0 \in I_i$ and pick $V \subseteq F$ so that $V \times t_i$ contains x_i and so small that

$$h_s(V) \cong h_{s0}(V) \cong B^k = a\ k\text{-ball},$$

for any $s \in I_i$. Since the singular set of $h_s|_{F-V}$ has no critical points on $F \times I$ for $s \in I_i$, we see by (1) that h_s: $(F - V) \times I_i \to \mathbf{R}^{k+1}$ is an isotopy of $h_{s0}(F - V)$. Thus there is a diffeomorphism $h_s(F - V) \to h_{s0}(F - V)$, which can be extended to a diffeomorphism of regular neighborhoods of $h_s(F - V)$ and $h_{s0}(F - V)$ and thence (by the Isotopy Extension Theorem) to a diffeomorphism ϕ_s of $\mathbf{R}^{k+1}$. Then $\phi_t \circ h_s$ $(t \in I)$ defines an isotopy of h_s to a regular homotopy $\phi_s \circ h_s$: $F \to \mathbf{R}^{k+1}$ with

$$\phi_s \circ h_s(x) = \phi_{s0} \circ h_{s0}(x) \quad \text{for } x \in F - V.$$

A partition of unity argument shows we can isotope H without changing $H|_{F\times(I-I_i)}$ so that $H(x, t) = \phi_t \circ h_t$ on some subinterval of I_i containing t_i. For convenience then, we assume this isotopy already done and merely assume that $h_s(x) = h_{s0}(x)$ for $s \in I_i$ and $x \in F - V$.

Now assume that we chose V small enough so that $h_s|_V$ is an embedding for all $s \in I_i$. Then $h_s\colon V \to \mathbf{R}^{k+1}$ describes an isotopy of $V \cong B^k$ fixing ∂V.

For small ε, we can perturb h_s slightly so that for $s_1, s_2 \in I_i$,

$$h_{s1}(V) \cap h_{s2}(V)$$

is a 1-complex $G(s_1; s_2) \subseteq h_{s1}(V)$ (in fact, $h_{s1}(V)$ and $h_{s2}(V)$ are close for small ε and so their intersection is a disjoint union of circles). Furthermore, $x_i \in (V \times t_i) \cap S$ and so

$$h_{ti}(x_i) \in h_{ti}(V) \cap h_{ti}(F - V) = h_{ti}(V) \cap h_{s0}(F - V).$$

Thus h_s (and hence H, as above) may be isotoped so that

$$h_{ti}(V) \cap \bigcup_{s \in I_i} h_s(V) = \bigcup_{s \in I_i} G(t_i; s)$$

misses a neighborhood of $h_{ti}(x_i)$. Note also that for ε small enough,

$$\bigcup_{s \in I_i} G(s_0; s)$$

misses a neighborhood of $h_{s0}(x_i)$.

Now pick a neighborhood $U \subseteq V$ centered at x_i such that

(*a*) $h_{ti}(U) \subset h_{ti}(V) \cong B^k$,
(*b*) $h_{ti}(U)$ misses $\bigcup_{s \in I_i} G(t_i; s)$,
(*c*) $h_s(U)$ contains $h_s(x_i)$ and misses $\bigcup_{s' \in I_i} G(s; s')$.

In particular, for all $s_1, s_2 \in I_i$, $h_{s1}(U)$ and $h_{s2}(U)$ are disjoint.

Let $D = \bigcup_{s \in I_i} U \times s$ and let $\bar{D} = \bigcup_{s \in I_i} h_s(U)$. Then D and $\bar{D}$ are k-balls and we are a homeomorphism $A\colon D \to \bar{D}$ given by $A(U \times s) = h_s(U)$. Therefore,

$$\begin{aligned} S \cap (U \times s) &= (\text{singular set of } h_s) \cap U \\ &= h_s(U) \cap h_s(F - U) \\ &= h_s(U) \cap h_{s0}(F - U). \end{aligned}$$

Taking a union over $s \in I_i$,

$$S \cap D \cong_A h_{s0}(F - U) \cap \bar{D}$$

and

$$F \times s_0 \cap D = (U \times s_0) \cap D \cong_A h_{s0}(U) \cap \bar{D},$$

(where $\cong_A$ is the restriction of the homomorphism A) and so

$$(D \cap S) \cup (D \cap (U \times s_0)) \cong (\bar{D}, [h_{s_0}(F - U) \cup h_{s_0}(U)] \cap \bar{D})$$
$$= (\bar{D}, h_{s_0}(F) \cap \bar{D}). \quad \blacksquare$$

2. Crossing changes. We now define our notion of crossing change and show that any regular homotopy of a surface in $\mathbf{R}^3$ can be covered by a sequence of such changes and isotopies.

Suppose f_0: $F \looparrowright \mathbf{R}^3$ is covered by an immersion f: $F \looparrowright \mathbf{R}^4$. Define γ: $F \to \mathbf{R}$ as previously by $f = (f_0, \gamma)$. Using Lemma 6 we see that f may be taken to be a γ-immersion. Let e and e' be subintervals of $S \subseteq F$ identified by f_0, so that we have a homeomorphism h: $\mathring{e} \to \mathring{e}'$ as before. Suppose further that $\gamma(x) \neq \gamma \circ h(x)$, $x \in \mathring{e}$, so that f is an embedding near e and e'. Then deform $\gamma|_e$ (rel ∂e) to γ_c so that the graphs of γ_c and $\gamma_c \circ h$ intersect at exactly two points (Fig. 35).

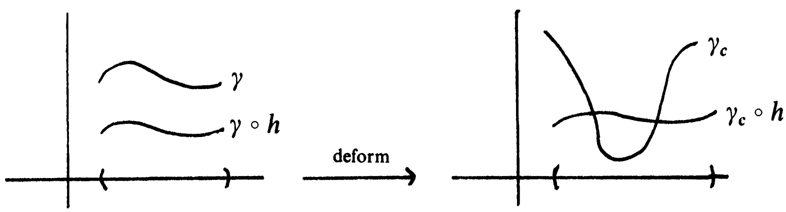


FIG. 35

Now form a new immersion $f_c = (f_0, \gamma_c)$.

DEFINITION. Passing from f to f_c to f will be called *changing a crossing.*

Remarks. 1. Changing a crossing corresponds to the appearance of 0 or 1-handles of the singular set of H: $F \times I \to \mathbf{R}^4 \times I$ as described in Section 0.

2. Suppose s_1 and s_2 are transverse components of S with $\gamma(s_1) > \gamma(s_2)$. As will be seen later, crossing changes can sometimes be used to obtain a new γ with $\gamma(s_1) < \gamma(s_2)$.

THEOREM 11. *Let h: $F \times I \to \mathbf{R}^3$ be a regular homotopy of f_0: $F \looparrowright \mathbf{R}^3$, and let f: $F \looparrowright \mathbf{R}^4$ be an immersion covering f_0. Then there is a sequence of points $t_1, \ldots, t_k \in I$ and maps f_t: $F \to \mathbf{R}^4$, $t \in I - \{t_1, \ldots, t_k\}$ such that*

(1) *f_t covers h_t,*
(2) *f_t describes an isotopy, $t \in (t_i, t_{i+1})$,*
(3) *$f_{t_i+\varepsilon}$ is obtained from $f_{t_i-\varepsilon}$ by a crossing change (ε small).*

That is, h can be covered in $\mathbf{R}^4$ by a sequence of isotopies and crossing changes.

Proof. As in the previous section, h induces an immersion H: $F \times I \looparrowright \mathbf{R}^3 \times I$ with singular set $S \subseteq F \times I$. Let $t_1, \ldots, t_k$ be the critical values of the projections $g \circ \sigma_i$ as before. Then we see $S_2 = S$ built accordingly with 0, 1, and 2-handles, S_3 with 0 and 1-handles, and the discrete set S_4 with 0-handles.

We define f_t inductively. Suppose that $f_{t_i+\varepsilon}$ exists. Then

$$h_t\colon F \times (t_i, t_{i+1}) \to \mathbf{R}^3$$

is an isotopy by Proposition 10.1. Now, if

$$f_{t_i+\varepsilon} = (h_{t_i+\varepsilon}, \gamma_{t_i+\varepsilon}),$$

then, for $t \in (t_i + \varepsilon, t_{i+1})$, set

$$f_t = (h_t, \gamma_t \circ \phi^{-1} \circ h_t),$$

where $\phi\colon h_{t_i+\varepsilon}(F) \to h_t(F)$ is the diffeomorphism induced by the isotopy h_t. This defines an isotopy f_t covering h_t for $t \in (t_i + \varepsilon, t_{i+1})$, and similarly we can define f_t for $t \in (t_i, t_{i+1})$.

The definition of $f_{t_i+\varepsilon}$ from $f_{t_i-\varepsilon}$ will depend on the various types of critical points corresponding to t_i, which we now consider.

0 and 2-handles of S_2. At a 0-handle, S_2 locally is shown in Fig. 36.

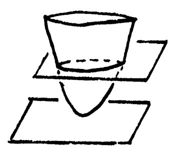


FIG. 36

Thus Proposition 10 shows h as having the effect of pushing the disk $D = h_{t_i-\varepsilon}(U)$ through $h_{s_0}(F - U)$; see Fig. 37.

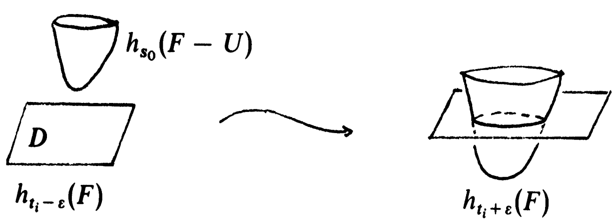


FIG. 37

Let $\bar{W} = \bigcup_{s \in I_i} f_{s_0}(F - U) \cap h_s(U)$ and let $W = h_{t_i-\varepsilon}^{-1}(\bar{W})$. Then

$$W = \bigcup_{s \in I_i} (\text{singular set of } h_s),$$

and in this case $W \cap U$ is a disk. Now, since $h_{t_i-\varepsilon}$ lifts to $f_{t_i-\varepsilon}$, we may define $\gamma_{t_i-\varepsilon}$ by $f_{t_i-\varepsilon} = (h_{t_i-\varepsilon}, \gamma_{t_i-\varepsilon})$. We can clearly deform $\gamma_{t_i-\varepsilon}$ to make

$$\gamma_{t_i-\varepsilon}(W \cap U) > \gamma_{t_i-\varepsilon}(W \cap F - U).$$

Calling this new map $\gamma_{t_i+\varepsilon}$, we define $f'_{t_i+\varepsilon} = (h_{t_i+\varepsilon}, \gamma_{t_i+\varepsilon})$. We have now defined a new immersion of F into $\mathbf{R}^4$ with $W \cap U$ pushed farther into $\mathbf{R}^4$ than $W \cap F - U$. Homotope $f'_{t_i+\varepsilon}$ to a map $f_{t_i+\varepsilon}$ so that $f'_{t_i+\varepsilon}(U)$ is pushed within the $\mathbf{R}^3$-level as in the above picture; because of our choice of $\gamma_{t_i+\varepsilon}$, this homotopy is in fact an isotopy, introducing no new self-intersections.

Thus we have defined $f_{t_i+\varepsilon}$ in the case that t_i corresponds to a 0-handle of S_2. The case for 2-handles is entirely similar: first push $W \cap U$ into $\mathbf{R}^4$ and then push $h_{t_i-\varepsilon}(U)$ across the critical point within the $\mathbf{R}^3$-level.

1-handles of S_2. *S_2 locally is shown in Fig. 38.*

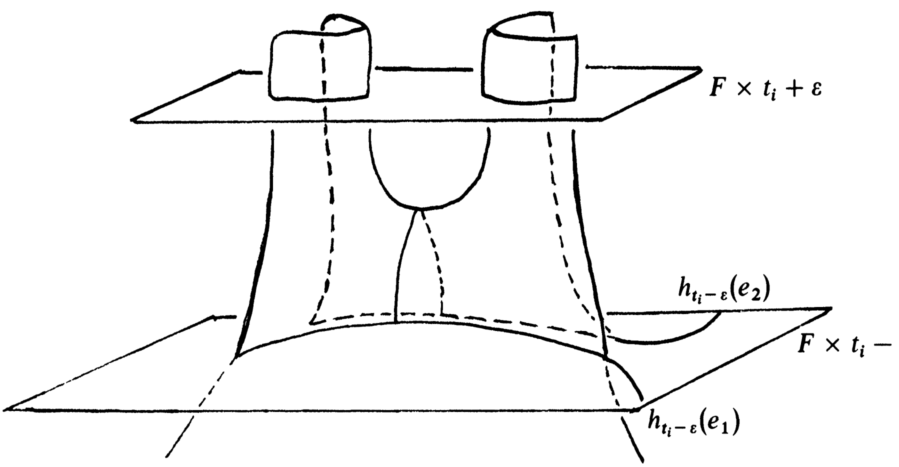


FIG. 38

So h_t has the effect of pushing the disk $h_{t_i-\varepsilon}(U)$ through a pair of (slit) pants. Our procedure is now similar to that of the previous case. Let

$$W = \bigcup_{s \in I_i} (\text{singular set of } h_s).$$

In this case $h_{t_i-\varepsilon}(W) \cap h_{t_i-\varepsilon}(U)$ is again a disk, and we want $h_{t_i+\varepsilon}$ to send $W \cap U$ farther into B^4 than $W \cap F - U$.

Let e_1 and e_2 be two curves in U sent to the curves indicated in the figure by $h_{t_i-\varepsilon}$. Then e'_1 and e'_2 lie in $F - U$ (recall that $h_{t_i-\varepsilon}$ identifies e_i with e'_i). We will obey the following convention in the rest of this section. Since e_i and e'_i are identified, we have an associated homeomorphism $\eta\colon \mathring{e}_i \to \mathring{e}'_i$. If for all $x \in e_i$, $\gamma(x) > \gamma \circ \eta(x)$ (resp. $<$) then we write $\gamma(e_i) > \gamma(e'_i)$ (resp. $<$).

Now, if $\gamma_{t_i-\varepsilon}(e_j) > \gamma_{t_i-\varepsilon}(e'_j)$ (resp. $<$) for $j = 1$ and 2, then we simply deform $h_{t_i-\varepsilon}$ by deforming $\gamma_{t_i-\varepsilon}$ to $\gamma_{t_i+\varepsilon}$ satisfying

$$\gamma_{t_i+\varepsilon}(W \cap U) > \gamma_{t_i+\varepsilon}(W \cap (F - U)) \quad (\text{resp. } <).$$

Then set $f'_{t_i+\varepsilon} = (h_{t_i+\varepsilon}, \gamma_{t_i+\varepsilon})$, and note that the induced deformation from $f_{t_i-\varepsilon}$ to $f'_{t_i+\varepsilon}$ introduces no self-intersections.

If on the other hand $\gamma_{t_i-\varepsilon}(e_1)$ and $\gamma_{t_i-\varepsilon}(e_2)$ are not both greater than (resp. less than) $\gamma_{t_i-\varepsilon}(e'_1)$ and $\gamma_{t_i-\varepsilon}(e'_2)$, then we must perform a crossing change along, say, e_1. This introduces two new self-intersection points which we may assume lie in $F - U$. After the crossing change then,

$$\gamma_{t_i-\varepsilon}(e_j \cap U) > \gamma_{t_i-\varepsilon}(e'_j \cap U) \text{ (resp. } <) \quad \text{for } j = 1 \text{ and } 2.$$

(Note that we do not require that e_1 and e_2 be distinct, since crossing change is a local operation.) Then we proceed as above, defining $f'_{t_i+\varepsilon}$ with this new γ.

Thus we can push $f'_{t_i+\varepsilon}(U)$ within the $\mathbf{R}^3$ level to cover the desired homotopy, and our choice of γ guarantees that this push creates no new self-intersections. We call the resulting map $f_{t_i+\varepsilon}$.

Remark. The immersed surface in Fig. 39 in $\mathbf{R}^3$ is covered by an embedded $S^2 \cup T^2$ in $\mathbf{R}^4$.

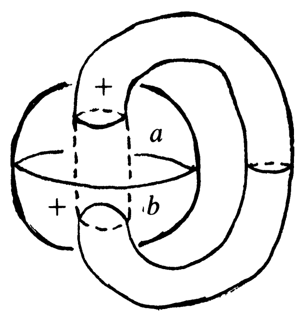


FIG. 39

Let α, β be curves on T^2 sent to a and b respectively by this immersion, and let α', β' be curves on S^2 similarly sent to a and b. If $\gamma(\alpha) > \gamma(\alpha')$ and $\gamma(\beta) < \gamma(\beta')$, then the S^2 and T^2 are linked in $\mathbf{R}^4$. We have indicated the relative magnitude of γ on these curves in the figure by placing a + sign near the component of the singular set pushed farthest into $\mathbf{R}^4$ by γ for each component of $\bar{S}$.

A crossing change can be used to arrange that $\gamma(\alpha) > \gamma(\alpha')$ and $\gamma(\beta) > \gamma(\beta')$. Then the S^2 and T^2 are unlinked in $\mathbf{R}^4$ and can be isotoped apart. This isotopy in fact covers a regular homotopy corresponding to the above case of 1-handles.

0-handles and 1-handles of S_3. At a 0-handle of S_3, S is locally shown in Fig. 40 and h moves the curved sheet in the figure as indicated. (Actually, S_3 should be drawn curved and $h_{t_i-\varepsilon}(U)$ flat as in Fig. 41. We draw Fig. 40 as shown for clarity.)

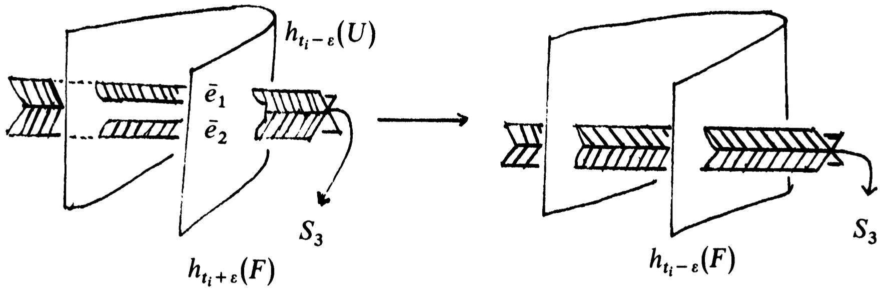


FIG. 40

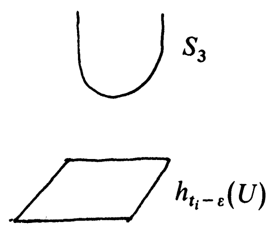


FIG. 41

This case is similar to previous ones. If the $\gamma_{t_i-\varepsilon}(e_i \cap U)$ are greater than (resp. less than) $\gamma_{t_i-\varepsilon}(e_i' \cap U)$, then first pushing $W \cap U$ sufficiently far into $\mathbf{R}^4$ permits the definition of $f_{t_i+\varepsilon}$ as before. Furthermore, by changing crossings if necessary, we can ensure this condition on $\gamma_{t_i-\varepsilon}$ is satisfied.

The 1-handles of S_3 are dealt with entirely similarly, reversing the direction of action in Fig. 40.

0-handles of S_4. Our local picture is given in Fig. 42.
As before, it is easy to see how to use crossing changes to define $f_{t_i+\varepsilon}$. ∎

3. Regular homotopy classes of F^2 *in* $\mathbf{R}^3$. The previous section showed that any regular homotopy of a liftable immersed oriented surface in $\mathbf{R}^3$ can be covered by a sequence of crossing changes and isotopies of a lift in $\mathbf{R}^4$. We now show that every such homotopy class contains an immersion with a standard image.

The Smale–Hirsch Theorem [3] classifies the regular homotopy classes of F^2 in $\mathbf{R}^3$ as being in 1-1 correspondence with the set of homotopy classes of bundle monomorphisms

$$[T(F),\ T(\mathbf{R}^3)] = [T(F),\ \mathbf{R}^3],$$

where T denotes tangent bundle and $T(\mathbf{R}^3)$ is identified with $\mathbf{R}^3$ by parallel translation. In fact, the correspondence assigns to the class containing the

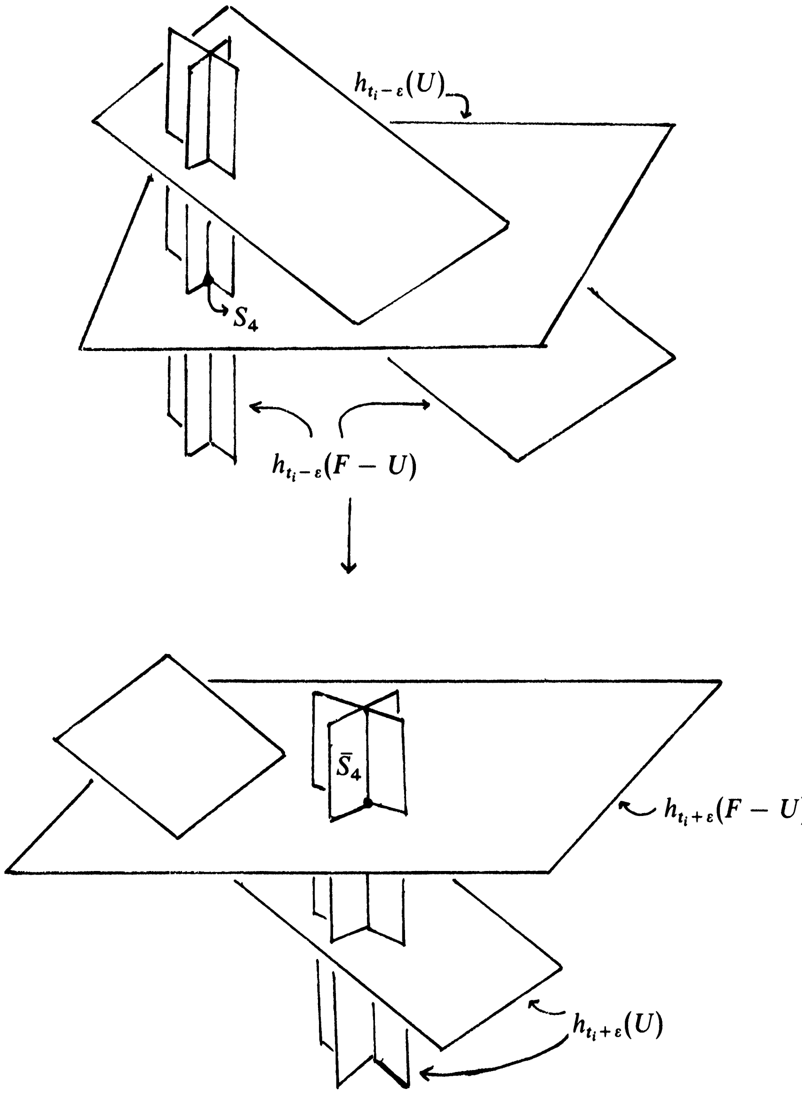


FIG. 42

immersion f_0: $F \looparrowright \mathbf{R}^3$ the homotopy class containing the derivative map Df_0: $T(F) \to T(\mathbf{R}^3)$.

However, F is oriented, so there is a non-zero section of the normal bundle ν to $f_0(F)$ in $\mathbf{R}^3$; thus ν is a trivial bundle and $T(\mathbf{R}^3)|_{f_0(F)}$ splits as $Df_0(T(F)) \oplus \nu$. Furthermore, $f_0^*(\nu)$ is trivial and so

$$T(F) \oplus f_0^*(\nu) \cong T(\mathbf{R}^3)|_F,$$

where we think of F as standardly embedded in $\mathbf{R}^3$. So we see that the homotopy class of Df_0 determines a homotopy class of

$$F \times \mathbf{R}^3 \cong T(\mathbf{R}^3)|_F \cong T(F) \oplus f_0^*(\nu) \xrightarrow{Df_0} T(\mathbf{R}^3)|_{f_0(F)} \cong F \times \mathbf{R}^3,$$

i.e., a homotopy class in $[F, SO(3)]$.

Conversely, given a class in $[F, SO(3)]$, we may use the considerations above to get a map $T(F) \to T(\mathbf{R}^3)$. Thus we have:

LEMMA 12. *Regular homotopy classes of F in $\mathbf{R}^3$ are in 1-1 correspondence with $[F, SO(3)]$. The correspondence takes an immersion f_0 to the map*

$$T(\mathbf{R}^3)|_F \cong T(F) \oplus f_0^*(\nu) \xrightarrow{Df_0} T(f_0(F)) \oplus \nu \cong T(\mathbf{R}^3)|_{f_0(F)}$$

induced by Df_0. ∎

In fact, we need only look at what the map $F \to SO(3)$ does on $\pi_1(F)$:

LEMMA 13. *The homotopy class of a map $h\colon F \to SO(3)$ (and hence the associated regular homotopy class of $F \looparrowright \mathbf{R}^3$) is determined by*

$$h_*\colon \pi_1(F) \to \pi_1(SO(3)) = \mathbf{Z}/2.$$

Proof. If $h_1, h_2\colon F \to SO(3)$ are two maps agreeing on $\pi_1(F)$, then they agree on $\pi_i(F)$, all i (since $\pi_i(F) = 0$, $i \geq 2$). Then h_1 and h_2 are homotopic by Whitehead's Theorem [9, p. 405]. Conversely, if h_1 is homotopic to h_2, then $h_{1*} = h_{2*}\colon \pi_1(F) \to \pi_1(SO(3))$. ∎

We now use this classification to pick an exhaustive set of representatives of regular homotopy classes of F in $\mathbf{R}^3$.

DEFINITION. Suppose $g\colon F \to \mathbf{R}^3$ is an embedding, and let α be a simple closed oriented curve on F with annular neighborhood $A \subseteq F$. Let $D(\alpha)\colon F \to F$ be the diffeomorphism given by performing a Dehn twist about α (see [8]); see Fig. 43. Then $g \circ D\colon F \to \mathbf{R}^3$ is an embedding which we call the result of *performing a Dehn twist to g about α.*

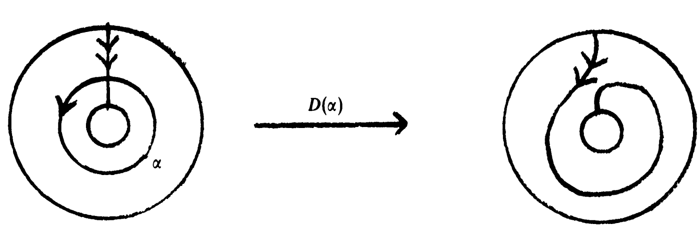


FIG. 43

Now suppose F is a torus, and let α and β be standard generators of $\pi_1(F)$,

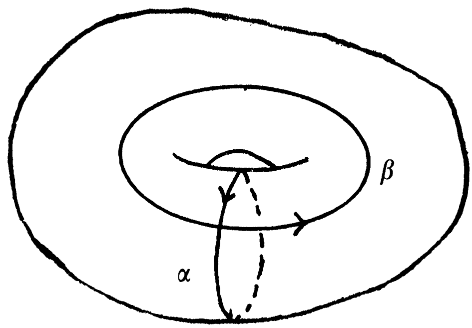


FIG. 44

We consider F as being standardly embedded in $\mathbf{R}^3$ by $i\colon F \hookrightarrow \mathbf{R}^3$ and identify F with $i(F)$.

Let $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ be a pair of 0's and 1's, and let g_ε be the embedding obtained from $i\colon F \hookrightarrow \mathbf{R}^3$ by performing Dehn twists on α (resp. β) if and only if $\varepsilon_1 = 1$ (resp. $\varepsilon_2 = 1$). Notice that the image of any $f_{\bar{\varepsilon}}$ is the (standard) image of i, which we identify with F. So, $f_{\bar{\varepsilon}}$ (viewed as having domain $i(F)$) will map α and β as follows:

$$\alpha \mapsto \alpha + \varepsilon_2 \beta, \qquad \beta \mapsto \varepsilon_1 \alpha + \beta.$$

Now let $f\colon F \looparrowright \mathbf{R}^3$ be any immersion. Pick a framing $\phi\colon F \to SO(3)$ of $T(\mathbf{R}^3)|_F$ so that $\phi(\alpha)$ and $\phi(\beta)$ represent generators of $\pi_1(SO(3))$. Let $\psi\colon F \to SO(3)$ be

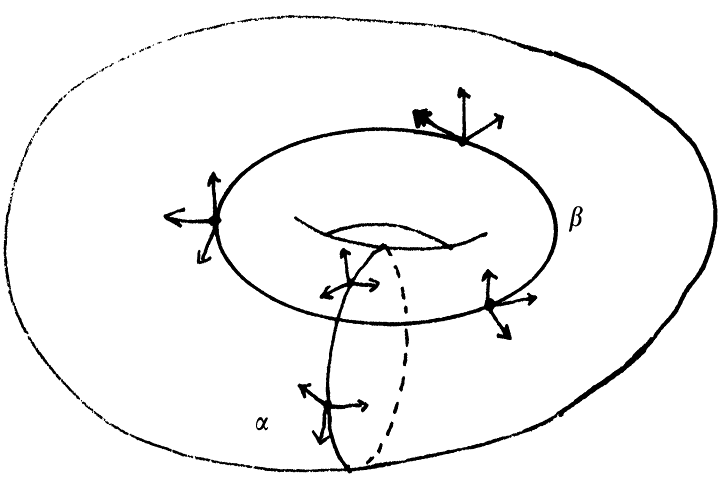


FIG. 45

the restriction of the standard framing of $T(\mathbf{R}^3)|_{f(F)}$. Then Df induces a map h: $F \to SO(3)$ as before corresponding to the regular homotopy class of f. We can define h by comparing ϕ and ψ as follows. $\phi(x)$ is a 3-frame in $T(\mathbf{R}^3)|_x$ and so $Df(\phi(x))$ is a 3-frame in $T(\mathbf{R}^3)|_{f(x)}$. Viewing 3-frames a matrices, let $h(x)$ be the element of $SO(3)$ defined by

$$(x) = h(x) \cdot \phi(x)$$

where $\cdot$ denotes matrix multiplication.

However, Lemma 13 shows that to determine h we need only calculate $h_*(\alpha)$ and $h_*(\beta)$, and that there are at most four regular homotopy classes of the torus F in $\mathbf{R}^3$; we will construct four distinct representatives. The first three are $f_{\bar{\varepsilon}}$ where $\bar{\varepsilon} = (0, 0), (0, 1)$, and $(1, 0)$. We compute h_* in these cases:

Case 1. $\bar{\varepsilon} = (0, 0)$. Then $h_*(\alpha)$ and $h_*(\beta)$ are represented by rotations of $\mathbf{R}^3$ about a single axis, so that both elements represent $1 \in \pi_1(SO(3))$.

Case 2. $\bar{\varepsilon} = (1, 0)$. Then $f_{\bar{\varepsilon}}(\beta) = \alpha + \beta$; so $h_*(\alpha) = 1$ as before and $h_*(\beta)$ is represented by consecutive rotations about two different axes. Thus $h_*(\beta) = 0 \in \pi_1(SO(3))$.

Case 3. $\bar{\varepsilon} = (0, 1)$. As in Case 2, with $h_*(\alpha) = 0$ and $h_*(\beta) = 1$.

To construct the fourth immersion, note that $f_{(1,0)}$: $S^1 \times S^1 \to \mathbf{R}^3$ extends to $S^1 \times B^2$ in the obvious way, where $\alpha = * \times S^1$ and $\beta = S^1 \times *$. Call this extension $\bar{f}$. Let C be a figure-8 embedded in B^2, parametrized by p: $S^1 \to C$. Then

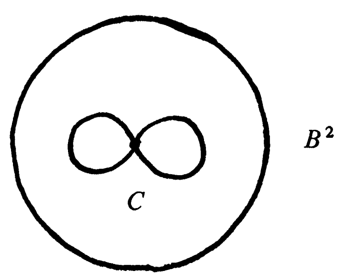


FIG. 46

our fourth immersion f is defined on $S^1 \times S^1$ by

$$f\colon (\theta_1, \theta_2) \mapsto \bar{f}(\theta_1, p(\theta_2)),$$

i.e., f represents moving a figure-8 in a circle in $\mathbf{R}^3$, twisting once.

DEFINITION. The image of f is called a twisted-8 torus.

Note that $f(\alpha)$ is a copy of C, and $f(\beta)$ links the core S^1 once. Thus it is easy to see that in this case $h_*(\alpha) = h_*(\beta) = 0$, and so f is a representative of the fourth homotopy class.

Thus each of $f_{(0,0)}$, $f_{(0,1)}$, $f_{(1,0)}$ and f represent distinct regular homotopy classes of $S^1 \times S^1$; by Lemma 13 there can be at most four such classes and so our list is exhaustive.

To construct a representative for a regular homotopy class of a surface of genus g, simply form the connected sum of images of the f_ε's and f, and define the immersion with image this sum in the obvious manner. An argument as above shows that we can construct representatives of all 4^g classes in this way, and Lemma 13 again guarantees our list to be exhaustive. Thus we have:

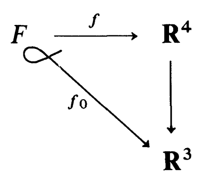
COROLLARY 14. *Any immersion* $f_0 \looparrowright \mathbf{R}^3$ *is regular homotopic to a connected sum of embedded and twisted-8 tori.* ▌

We use the following notion in the next chapter.

DEFINITION. an *unsurface* $F^2 \looparrowright \mathbf{R}^4$ is an embedding of a surface of genus g which is isotopic to an embedding whose image under projection is the standard unknotted surface of genus g in $\mathbf{R}^3$. Thus each of the above f_ε's is an unsurface.

4. The unknotting theorem and another lifting theorem. We can now prove the main theorem of this chapter.

UNKNOTTING THEOREM 15. *Let f be an embedding covering f_0 as before*:



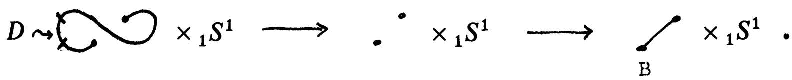
Then f can be changed to an unsurface by a sequence of isotopies and crossing changes.

Proof. In §4.3 we saw that f_0 can be regularly homotoped to an immersion f with image a connected sum of embedded or twisted-8 tori, and Proposition 11 shows that this homotopy can be covered by a sequence of isotopies and crossing changes. Let f_1 denote the resulting immersion of F in $\mathbf{R}^4$.

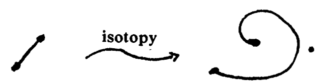
Each twisted-8 torus is immersed with one (unknotted) double curve $\bar{s}$; let s_1 and s_2 be the two preimages of $\bar{s}$ in $f_1(F)$ and let $S = \bigcup_{\bar{s}} s_1$. Let T be a small neighborhood of S in $f_1(F)$. Evidently T is a disjoint union of annuli. Isotope $f_1(F)$ so that $f_1(F) - T$ is pushed into $\mathbf{R}^3$ along lines $\{(0, 0, 0, t) \mid t \in \mathbf{R}\}$ and let f_2 denote this new immersion of F. We can isotope f_2 so that $f_2(F) \cap \mathbf{R}^3$ is a connected sum of a standard genus g surface with a sum of twisted-8 tori-annuli. Now each twisted-8 torus is immersed in $\mathbf{R}^3$ as

$D \rightsquigarrow$ $\times_1 \; S^1$

where $\times_1$ indicates a rotation about $* \times S^1$ in $\mathbf{R}^3$, and the connected sum is built using disks in the annuli $D \times {}_1S^1$. A movie of this piece of $f_2(F)$ is then



Now isotope the interval B in its copy of $\mathbf{R}^3$ as follows:



Then B can be isotoped into $\mathbf{R}^3$ to obtain an embedding:

Note that these isotopies are all rel D, and so can be performed on our connected sum. Thus, f_2 can be isotoped to an unsurface as claimed. ∎

As mentioned previously, this theorem is the analogue of Fact 1 for knots in $\mathbf{R}^3$.

The techniques in this chapter suggest another lifting theorem. First note that any immersion f_0: $F \looparrowright \mathbf{R}^3$ can be lifted to an *immersion* f: $F \looparrowright \mathbf{R}^4$; simply find a γ: $S \to \mathbf{R}$ such that $f(S) = (f_0(S), \gamma(S))$ is transverse to $\mathbf{R}^3 \times 0$ and set $f = (f_0, \gamma)$.

PROPOSITION 16. *In fact, every f_0 lifts to an immersion which is regularly homotopic to an embedding.*

Proof. As already seen, the normal bundle ν_0 of f_0 is trivial. But the normal bundle ν of f satisfies $\nu = \nu_0 \oplus \varepsilon^1$, where f is defined as above and ε^1 is a trivial line bundle. Hence ν is trivial, and by Smale–Hirsch [3], F is regularly homotopic to (any) embedding. ∎

COROLLARY 17. *No immersion of a surface in $\mathbf{R}^4$ with non-trivial normal bundle projects to an immersion in $\mathbf{R}^3$.* ∎

Example. The movie in Fig. 47 of an immersed S^2 in $\mathbf{R}^4$ does not project after any isotopy.

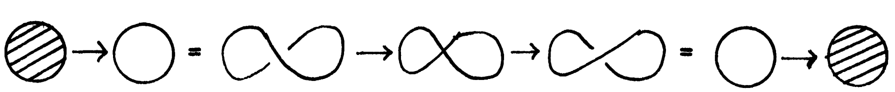


FIG. 47

5. Alexander polynomials of surfaces

Proceeding with our extension of Conway's program to dimension 4, we offer a conjecture of the proper definition of Alexander invariant of a surface in $\mathbf{R}^4$ and in a special case speculate how it might transform under crossing change. These conjectures fail in general, but we give a tantalizing example in which they are adequate and leave full treatment as an open question.

1. *Definition of* $\Delta(t)$. First recall the definition of $\Delta(t)$ for a knot K in S^3; see [8]. We construct the infinite cyclic cover X of $S^3 - K$ by gluing together an infinite number of copies of $S^3 - M$, where M is a Seifert surface for K. Let $\{\alpha_i\}$ be a basis for $H_1(M)$ and denote by α_i^+ (resp. α_i^-) the homology class in $H_1(S^3 - M)$ obtained by pushing a representative of α_i off of F in the positive (resp. negative) normal direction. Then a Mayer-Vietoris sequence shows that $H_1(X)$ is isomorphic as a $\mathbf{Z}[t, t^{-1}]$-module to $H_1(S^3 - M)$ subject to relations $\{\alpha_i^+ = t\alpha_i^-\}$.

Now, Alexander duality gives the following map A:

$$H_1(S^3 - M) \cong H^1(S^3 - M) \cong H_1(M).$$

$$A$$

Then since M is an oriented 2-manifold with boundary, $H_1(S^3 - M)$ is free. Furthermore, A is given by $A(x) = \sum_i l(x, \alpha_i)\alpha_i$ where l denotes linking number.

Thus $H_1(X)$ is isomorphic to $H_1(M)$ subject to relations $A(\alpha_i^+ - t\alpha_i^-) = 0$; but

$$A(\alpha_i^+) = \sum l(\alpha_i^+, \alpha_j)\alpha_j$$

and

$$A(\alpha_i^-) = \sum l(\alpha_i^-, \alpha_j)\alpha_j = \sum l(\alpha_j^+, \alpha_i)\alpha_j.$$

Hence defining a matrix $V = (l(\alpha_i^+, \alpha_j))$ we see that $H_1(X)$ has presentation matrix $V - tV^T$. Some standard algebra [8] now shows that $\Delta(t) = \det(V - tV^T)$ is the well-defined (up to multiplication by $\pm t^n$) Alexander polynomial.

We now develop analogous machinery for a surface F in S^4. Thus suppose M is a 3-manifold embedded in $\mathbf{R}^4$ with $\partial M = F$. We use M as above to construct the infinite cyclic cover X of $S^4 - F$ from copies of $S^4 - M$. In the S^3 case, it was crucial that $H_1(S^3 - M) \cong H_1(M)$ be free, but in our case, $H_1(M) \cong H^2(S^4 - M)$ may not be free. Thus, the algebra requires that we use homology with $\mathbf{Q}$-coefficients; we do so for the remainder of this paper. Note, however, that H_{n-1} (any n-manifold with boundary) is free, and in particular $H_2(M; \mathbf{Z})$ is free.

Aping the arguments given above for knots in S^3, we see that $H_1(X)$ is isomorphic as a $\mathbf{Q}[t, t^{-1}]$-module to $H_1(S^4 - M)$ subject to relations $\alpha_i^+ = t\alpha_i^-$, where $\{\alpha_i\}$ is a basis of $H_1(M)$. Using $A\colon H_1(S^4 - M) \to H_2(M)$ as before, we then see that $H_1(X)$ is $H_2(M)$ subject to relations $A(\alpha_i^+ - t\alpha_i^-) = 0$. Letting $\{\beta_i\}$ be a basis of $H_2(M)$, note that

$$A(\alpha_i^+) = \sum l(\alpha_i^+, \beta_j)\beta_j, \quad A(\alpha_i^-) = \sum l(\alpha_i^+, \beta_j)\beta_j.$$

Defining two matrices

$$V = (l(\alpha_i^+, \beta_j)), \quad V^* = (l(\alpha_i^-, \beta_j)),$$

we see our module has presentation matrix $V - tV^*$. When $F \cong S^2$, this matrix is square and we define $\Delta(t) = \det(V - tV^*)$. We leave the proper definition of $\Delta(t)$ when genus $(F) > 0$ as an open question.

2. *A special case. No triple points.* Now suppose f: $F \to \mathbf{R}^4$ is an embedding such that the corresponding immersion f_0 has no triple points. We show how to construct M and conjecture a Conway identity for the corresponding polynomial.

Let $\bar{S} \subseteq \mathbf{R}^3$ be the singular value set of f_0, so that $f_0(F) - \bar{S}$ is a disjoint union of surfaces with boundary. Join these surfaces together along each arc of $\bar{S}$ in an orientation preserving fashion to obtain a disjoint union of closed surfaces N_i; see Fig. 48.

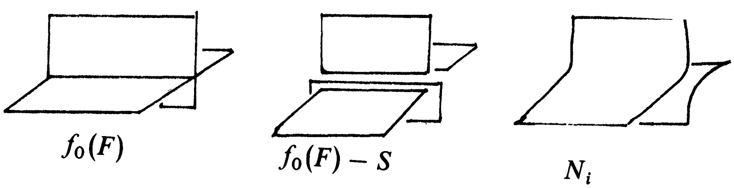


FIG. 48

Now push each N_i into a different 3-plane $\mathbf{R}_i^3$ of $\mathbf{R}^4$, and note that the trace of this push is a collar $N_i \times I$ with $N_i \times 1 \subseteq \mathbf{R}_i^3$. But $N_i \times 1$ bounds an oriented 3-manifold Q_i in $\mathbf{R}_i^3$, and we let $R_i = (N_i \times I) \cup Q_i$; see Fig. 49.

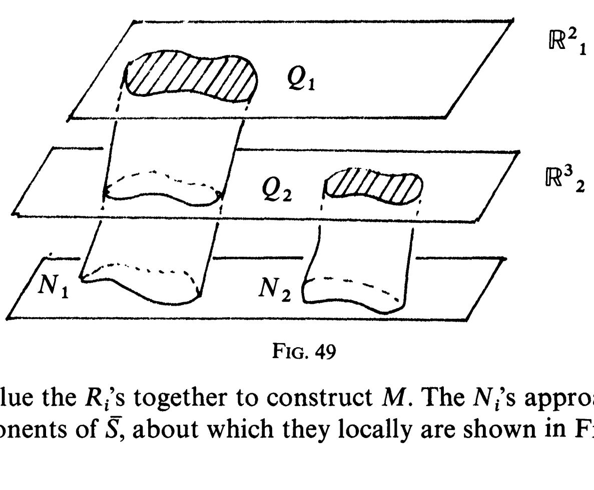


FIG. 49

Now we glue the R_i's together to construct M. The N_i's approach each other along components of $\bar{S}$, about which they locally are shown in Fig. 50.

Let Y denote the first factor in Fig. 49, so that changing each $Y \times \theta$ to $+ \times \theta$ changes $\bigcup_i N_i$ back to $f_0(F)$. However, we can add a band b to each $Y \times \theta$ as in Fig. 51,

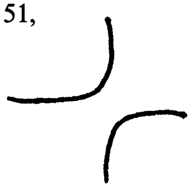

$\times S^1$

FIG. 50

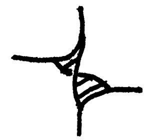

$\times S^1$

FIG. 51

and we can push these bands into $\mathbf{R}^4$ so that no two bands intersect and so that

$$f(F) = \bigcup_b \partial b \cup \bigcup_i N_i.$$

Furthermore, the desired M is $\bigcup_b b \cup \bigcup_i R_i$.

Remark. This construction is merely the analogous construction of Seifert surfaces for knots in S^3.

Thus at a particular component of $\bar{S}$, we have added a band $\times S^1$ to recover the embedding f. But we could alternatively add a band with opposite twisting to obtain a different embedding. We call the move from the first embedding to the second a *proper crossing* change. Notice that if e and e' are two components of S identified to $\bar{e} \subseteq \bar{S}$ by f_0, and if $\gamma(e) > \gamma(e')$, then a proper crossing change at $\bar{e}$ changes γ so that $\gamma(e) < \gamma(e')$. As mentioned in Section 4.2, a proper crossing change can be accomplished by ordinary crossing changes.

Now we proceed to calculate the effect of a proper crossing change on $\Delta(t)$. We consider only the case in which some basis elements of $H_2(M)$ and $H_1(M)$ pass through $\bar{e}$. Then we can rechoose a basis so that only one basis element $a \in H_2(M)$ and $b \in H_1(M)$ pass through e. In the special case that

$$b \in H_1(M)/H_1(\partial M),$$

a proper crossing change changes $l(a^+, b)$ by ± 1, so as in [2] we have a recursion

$$\Delta_l(t) - \Delta_r(t) = (t - 1)\Delta_s(t) \tag{2}$$

where Δ_l and Δ_r are the polynomials of the two embeddings obtained as above, and Δ_s is the polynomial of the embedding before band addition. We may now define a polynomial $\nabla(z)$ associated to an embedding by redefining $\Delta(t)$ with the formula

$$\Delta(t) = \det(t^{1/2}V - t^{-1/2}V^*);$$

then (2) becomes

$$\Delta_l(t) - \Delta_r(t) = (t^{1/2} - t^{-1/2})\Delta_s(t).$$

Setting $z = t^{1/2} - t^{-1/2}$ yields a definition of $\nabla(z)$.

If F_l, F_r, and F_s are the embeddings corresponding to ∇_l, ∇_r and ∇_s, we write $F_l = F_r \oplus F_s$ or $F_r = F_l \ominus F_s$.

Note that we have not proved that $\nabla(z)$ (or even $\Delta(t)$) is well defined in all cases, and we leave that task to a future paper. Furthermore, an example of a surface in $\mathbf{R}^4$ with Alexander polynomial $1 - 2t$ will be given in Section 5.3, showing that the definition of $\Delta(t)$ in general fails.

3. *An example. Spun knots.* A class of examples of knotted 2-spheres in $\mathbf{R}^4$ whose associated immersions f_0 have no triple points is that of spun knots. For example, the spun knot of the trefoil is an embedded S^2 obtained by spinning the knotted arc in Fig. 52 about the plane P in $\mathbf{R}^4$. Then a movie for this knot is given in Fig. 53.

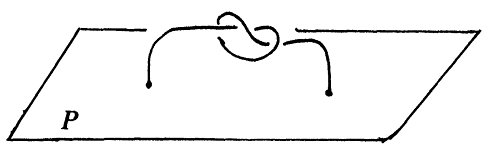


FIG. 52

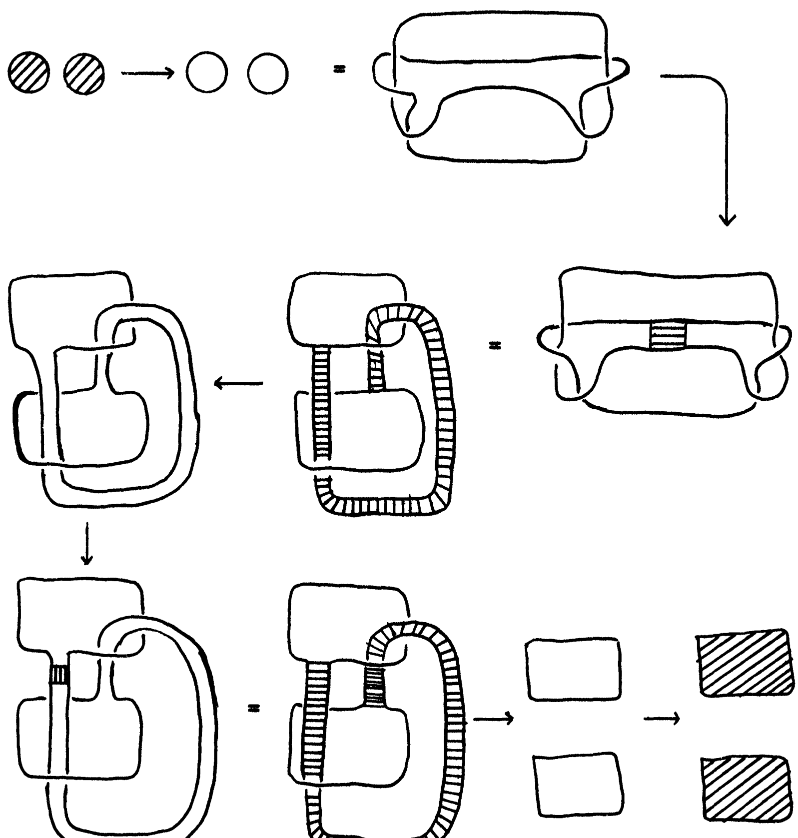


FIG. 53

Then the projection into $\mathbf{R}^3$ is the immersion in Fig. 54.

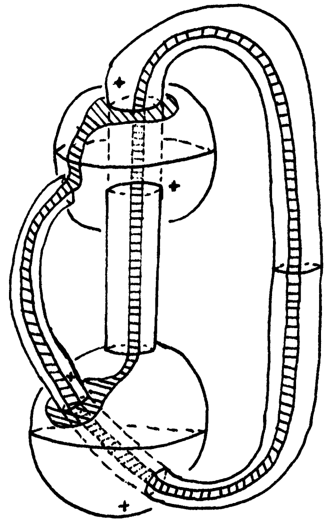


FIG. 54

The shaded region is that contributed by the 0-handles and first 1-handle of the movie. Thus if the sheets near $\bar{S}$ labeled "+" are pushed into $\mathbf{R}^4$ farther than those not so labeled, we recover the embedded surface.

We now show calculations which are for this example correct; as mentioned previously, the general case requires more work. Using notation suggested above, we have the result in Fig. 55.

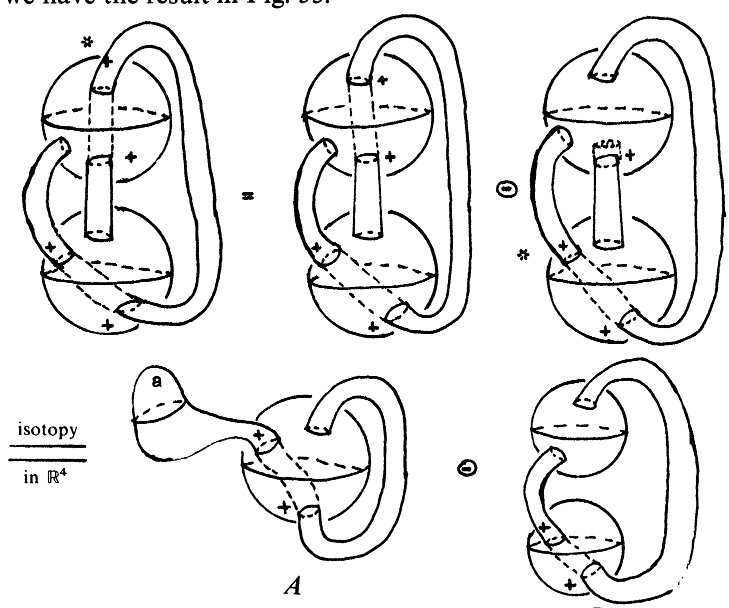


FIG. 55

But the embedding A is isotopic to a trivial embedding (just push the part labeled a through the self-intersections, pushing into $\mathbf{R}^4$ if necessary), while B decomposes as in Fig. 56.

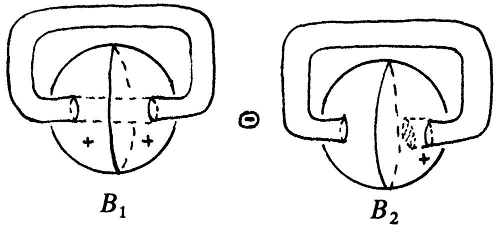
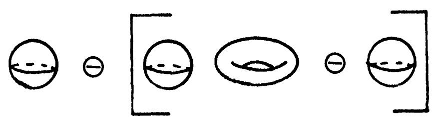
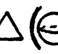


FIG. 56

B_1 is isotopic in $\mathbf{R}^4$ to   and B_2 to . Thus our spun knot decomposes into



Arguments as in [2] show that Δ ( ) $= 0$ and if we take Δ() $= +1$, then the polynomial of this knot becomes $1 - z(-z) = 1 + z^2$.

Remarks. (1) This is the polynomial of the trefoil, which is expected since

$$\pi_1(S^3 - K) \cong \pi_1(S^4 - \hat{K})$$

where $\hat{K}$ is the spun knot corresponding to K (see [12]) and since the polynomial can be calculated from Π_1(see also [6]).

(2) (John Hughes). Changing two crossings of the above knot yields a surface with the asymmetric Alexander polynomial $1 - 2t$; see Fig. 57. But $\Delta(t)$ was defined to be symmetric, and so its definition as given above cannot in general be adequate.

4. *Questions.* We have left many stones unturned. Can $\nabla(z)$ be defined for all links in S^4 as it is for links in S^3? If so, what is the structure of the Skein algebra? What about the case where f_0 has triple points? Can one directly show using these methods that ∇ of a knot in S^3 is the same as ∇ of its spun knot? And finally, the machinery of Section 4.2 applied to regular homotopies of $S^1 \looparrowright \mathbf{R}^2$ gives moves reminiscent of Tietze moves. Are there Tietze moves for $F^2 \hookrightarrow \mathbf{R}^4$?

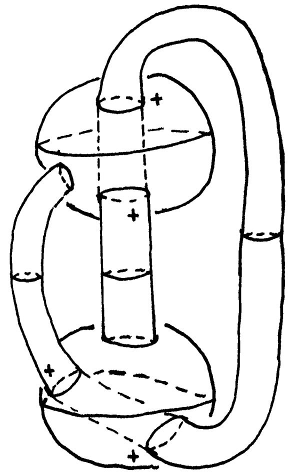


FIG. 57.

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